Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Giovanni Prouse

On the non linear vibrating rod equation. Nota II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **67** (1979), n.1-2, p. 1–7.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_67_1-2_1_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1979.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Ferie 1979 (Luglio-Agosto)

(Ogni Nota porta a pie' di pagina la data di arrivo o di presentazione)

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *On the non linear vibrating rod equation.* Nota II di GIOVANNI PROUSE, presentata (*) dal Socio L. AMERIO.

RIASSUNTO. — Si dà la dimostrazione del teorema enunciato nella nota I e di un teorema ausiliario.

4. – Let us begin by proving the following auxiliary theorem.

THEOREM 2. – *If the assumptions made in Theorem 1 are verified, there exists,* $\forall$ *fixed* $\varepsilon > 0$, *a function* $u(t)$ *such that*

a$_1'$) $u(t) \in L^2(0, T; H_0^2 \cap H^3)$, $u'(t) \in \tilde{K}_0$, $u(0) = z_0$;

b$_1'$) $u(t)$ *satisfies,* $\forall \varphi(t) \in K_0$ *with* $\varphi'(t) \in L^2(0, T; H_0^2 \cap H^3)$, $\varphi''(t) \in L^2(0, T; L^2)$, *the inequality*

$$(4.1) \qquad -\frac{1}{2}\|u'(t)\|^2_{L^2} - (H(Du(t)), 1)_{L^2} - \frac{1}{2}\|G(Du(t))\|^2_{H_0^1} -$$

$$-\frac{\varepsilon}{2}\|u(t)\|^2_{H^3 \cap H_0^2} + (u'(t), \varphi'(t))_{L^2} +$$

$$+\int_0^t [\varepsilon(u, \varphi')_{H^3 \cap H_0^2} + a(u, \varphi') - (u', \varphi'')_{L^2} - (f, \varphi' - u')_{L^2}]\, d\eta \geq$$

(*) Nella seduta del 14 giugno 1979.

$$\geq -\tfrac{1}{2}\|z_1\|^2_{L^2} - (H(Dz_0), 1)_{L^2} - \tfrac{1}{2}\|G(Dz_0)\|^2_{H^1_0} -$$

$$-\frac{\varepsilon}{2}\|z_0\|^2_{H^3\cap H^2_0} + (z_1, \varphi'(0))_{L^2}$$

a.e. in $[0, T]$.

Let $\{g_j\}$ be a basis in $H^2_0 \cap H^4$ and set

$$u_n(x, t) = \sum_{j=1}^{n} \alpha_{jn}(t) g_j(x); \tag{4.2}$$

let, moreover, ν be a penalisation operator associated to $\tilde{K}_0$. By the definition given we can set (see, for instance, [1] ch. 3, § 5)

$$\nu(v) = v - Pv, \tag{4.3}$$

where P is the projection operator from $L^2(Q)$ to $\tilde{K}_0$. The operator ν is monotone and hemicontinuous from $L^2(Q)$ to itself and

$$\tilde{K}_0 = \{v : v \in L^2(Q), \nu(v) = 0\}. \tag{4.4}$$

Consider now the system of ordinary differential equations in the unknown functions $\alpha_{jn}(t)$

$$(u_n''(t) + n\nu(u_n'(t)) - f(t), g_j)_{L^2} + \varepsilon(u_n(t), g_j)_{H^3\cap H^2_0} + a(u_n(t), g_j) = 0 \tag{4.5}$$
$$(j = 1, \cdots, n$$

satisfying the initial conditions

$$u_n(0) = \Pi_n z_0 = z_{0,n}, \qquad u_n'(0) = \Pi_n z_1 = z_{1,n}, \tag{4.6}$$

where Π_n denotes the projection operator on the subspace defined by $g_1, \cdots, g_n$. As it can be immediately verified, (4.5), (4.6) admit a solution in a conveniently small neighbourhood of $t = 0$; we shall show, by means of a priori estimates, that this solution exists in $[0, T]$.

Multiplying (4.5) by α_{jn}', adding and integrating between 0 and $t \in [0, T]$, we obtain, in exactly the same way as (3.8) and bearing in mind that $(\nu(u_n'), u_n')_{L^2} \geq 0$,

$$\tfrac{1}{2}\|u_n'(t)\|^2_{L^2} + (H(Du_n(t)), 1)_{L^2} + \tfrac{1}{2}\|G(Du_n(t))\|^2_{H^1_0} + \tag{4.7}$$

$$+\frac{\varepsilon}{2}\|u_n(t)\|^2_{H^3\cap H^2_0} \leq \int_0^t (f, u_n')\,d\eta + \tfrac{1}{2}\|z_{1,n}\|^2_{L^2} + (H(Dz_{0,n}), 1)^2_{L^2} +$$

$$+\tfrac{1}{2}\|G(Dz_{0,n})\|^2_{H^1_0} + \frac{\varepsilon}{2}\|z_{0,n}\|^2_{H^3\cap H^2_0}.$$

From (4.7) it follows, by the assumptions made, that, $\forall t \in [0, T]$,

$$\| u_n'(t) \|_{L^2} \leq M_1 \quad , \quad \sqrt{\varepsilon} \, \| u_n(t) \|_{H^3 \cap H_0^2} \leq M_2 \tag{4.8}$$

with M_i independent of n and ε; consequently, denoting, as we shall always do in the sequel, again by $\{u_n\}$ an appropriate subsequence selected from $\{u_n\}$,

$$u_n'(t) \to u'(t) \qquad u_n(t) \to u(t) \tag{4.9}$$

respectively in the weak-star topologies of $L^\infty(0, T; L^2)$ and of $L^\infty(0, T; H^3 \cap H_0^2)$. Moreover, since

$$L^2(0, T; H^3) \cap H^1(0, T; L^2) \subset H^{1/6}(0, T; H^{5/2}) \tag{4.10}$$

and the embedding of this space in $L^2(0, T; H^2)$ is completely continuous, it follows from (4.9) that

$$\frac{\partial u_n}{\partial x} \to \frac{\partial u}{\partial x} \quad , \quad \frac{\partial^2 u_n}{\partial x^2} \to \frac{\partial^2 u}{\partial x^2} \tag{4.11}$$

in the strong topology of $L^2(Q)$, i.e. $\dfrac{\partial u_n(x, t)}{\partial x} \to \dfrac{\partial u(x, t)}{\partial x}$, $\dfrac{\partial^2 u_n(x, t)}{\partial x^2} \to$ $\to \dfrac{\partial^2 u(x, t)}{\partial x^2}$ a.e. in Q.

The a priori estimates obtained assure us that a solution of (4.5), (4.6) exists on the whole of $[0, T]$; we must now prove that, when $n \to \infty$, the limit function $u(t)$ satisfies a_1'), b_1').

Observe that, by (4.7), (4.8),

$$n \int_0^T (\nu(u_n'(t), u_n'(t))_{L^2} \, dt \leq M_3 \tag{4.12}$$

and, consequently,

$$\lim_{n \to \infty} \int_0^T (\nu u_n', u_n')_{L^2} \, dt = 0 . \tag{4.13}$$

Using a standard procedure (see, for instace, [2], ch. 3, § 5) (1) we can then prove that $u'(t) \in \tilde{K}_0$; hence, by (4.6), (4.9), $u(t)$ satisfies condition a_1').

Denoting by $\mathring{K}_0$ the set $\left\{ v(t) \in H^1(0, T; L^2) : \left| \dfrac{\partial v}{\partial x} \right| < N_1'', \left| \dfrac{\partial^2 v}{\partial x^2} \right| < N_2'', \right.$ $\left. \left| \dfrac{\partial^3 v}{\partial x^3} \right| < N_3'' \text{ a.e. in } Q \right\}$, let $\varphi(t)$ be a function $\in \mathring{K}_0 \cap H^1(0, T; H_0^2 \cap H^4)$

(1) The bibliography is given at the end of note I.

with $\varphi''(t) \in L^2(0, T; L^2)$ and set

$$(4.14) \qquad \varphi(x, t) = \sum_{j=1}^{\infty} \gamma_j(t) g_j(x) \quad , \quad \varphi_p(x, t) = \sum_{j=1}^{\infty} \tilde{\gamma}_j(t) \varphi_j(x)$$

$$\text{with} \quad \tilde{\gamma}_j = \begin{cases} \gamma_j, & j \le p \\ 0, & j > p. \end{cases}$$

Assuming that $n > p$, multiplying (4.5) by $\tilde{\gamma}'_j - \alpha'_{jn}$ adding and integrating between 0 and $t \in [0, T]$, we obtain

$$(4.15) \qquad \int_0^t [(u''_n + n\nu(u'_n) - f, \varphi'_p - u'_n)_{L^2} + \varepsilon(u_n, \varphi'_p - u'_n)_{H^3 \cap H^2_0} + a(u_n, \varphi'_p - u'_n)]\, d\eta = 0 .$$

Integrating (4.15) by parts and bearing in mind that, since $\nu(\varphi'_p) = 0$,[2]

$$(4.16) \qquad \int_0^t (\nu(u'_n), \varphi'_p - u'_n)_{L^2}\, d\eta = \int_0^t (\nu(u'_n) - \nu(\varphi'_p), \varphi'_p - u'_n)_{L^2}\, d\eta = 0 ,$$

we have, $\forall t \in [0, T]$,

$$(4.17) \qquad -\tfrac{1}{2} \| u'_n(t) \|^2_{L^2} - (H(Du_n(t)), 1)_{L^2} - \tfrac{1}{2} \| G(Du_n(t)) \|^2_{H^1_0} - \frac{\varepsilon}{2} \| u_n(t) \|^2_{H^3 \cap H^2_0} + (u'_n(t), \varphi_p(t))_{L^2} +$$
$$+ \int_0^t [a(u_n, \varphi'_p) - (u'_n, \varphi''_p)_{L^2} + \varepsilon(u_n, \varphi'_p)_{H^3 \cap H^2_0} - (f, \varphi'_p - u'_n)_{L^2}]\, d\eta \ge$$
$$- \tfrac{1}{2} \| z_{1,n} \|^2_{L^2} - (H(Dz_{0,n}), 1)_{L^2} - \tfrac{1}{2} \| G(Dz_{0,n}) \|^2_{H^1_0} - \frac{\varepsilon}{2} \| z_{0,n} \|^2_{H^3 \cap H^2_0} + (z_{1,n}, \varphi'_p(0))_{L^2} .$$

Observe that, by (4.9), (4.11), $\forall t \in [0, T]$,

$$(4.18) \qquad \lim_{n \to \infty} \int_0^t a(u_n, \varphi'_p)\, d\eta = \lim_{n \to \infty} \int_0^t (g(Du_n) D^2 u_n, D(g(Du_n) D\varphi'_p))_{L^2}\, d\eta =$$

(2) Since $\varphi_p(t) \xrightarrow[C^0(0,T;H^4)]{} \varphi(t)$ implies $\varphi_p(t) \xrightarrow[C^0(0,T;C^3)]{} \varphi(t)$, it is obvious that, for sufficiently large p, $\varphi_p(t) \in K_0$, with $\varphi_p(0) = \Pi_p \varphi(0)$. Hence by observation I in note I, $\varphi'_p(t) \in \tilde{K}_0 \Rightarrow \nu(\varphi'_p) = 0$.

$$= \int_0^t (g(Du)\, D^2 u \,,\, D(g(Du)\, D\varphi_p'))_{L^2}\, d\eta = \int_0^t a(u \,,\, \varphi_p')\, d\eta \,,$$

$$(4.19) \quad \lim_{n\to\infty} \int_0^t (u_n \,,\, \varphi_p')_{H^3 \cap H_0^2}\, d\eta = \int_0^t (u \,,\, \varphi_p')_{H^3\cap H_0^2}\, d\eta \,,$$

$$\lim_{n\to\infty} \int_0^t [(u_n' \,,\, \varphi_p'')_{L^2} - (f \,,\, \varphi_p' - u_n')_{L^2}]\, d\eta = \int_0^t [(u', \varphi_p'')_{L^2} - (f \,,\, \varphi_p' - u')]\, d\eta \,.$$

Moreover, if $\psi(t)$ is an arbitrary function $\in C^0[0, T]$ with $\psi(t) \geq 0$, we have, again by (4.9), (4.11),

$$(4.20) \quad \lim_{n\to\infty} \int_0^T \psi(t)\, [(H(Du_n) \,,\, 1)_{L^2} + \tfrac{1}{2} \| G(Du_n) \|_{H_0^1}^2 - (u_n' \,,\, \varphi_p)_{L^2}]\, dt =$$

$$= \int_0^T \psi(t)\, [(H(Du) \,,\, 1)_{L^2} + \tfrac{1}{2} \| G(Du) \|_{H_0^1}^2 - (u', \varphi_p)_{L^2}]\, dt$$

and also, by well known properties of the weak limit,

$$(4.21) \quad \liminf_{n\to\infty} \int_0^T \psi(t) \left[\tfrac{1}{2} \| u_n'(t)) \|_{L^2}^2 + \frac{\varepsilon}{2} \| u_n(t) \|_{H^3\cap H_0^2}^2 \right] dt \geq$$

$$\geq \int_0^T \psi(t) \left[\tfrac{1}{2} \| u'(t) \|_{L^2}^2 + \frac{\varepsilon}{2} \| u(t) \|_{H^3 \cap H_0^2}^2 \right] dt \,.$$

Letting therefore $n \to \infty$ in (4.17) and taking the inferior limits, we obtain, by (4.18), (4.19), (4.20), (4.21), (4.6),

$$(4.22) \quad \int_0^T \psi(t) \Big\{ -\tfrac{1}{2} \| u'(t) \|_{L^2}^2 - (H(Du(t)) \,,\, 1)_{L^2} - \tfrac{1}{2} \| G(Du(t) \|_{H_0^1}^2 -$$

$$- \frac{\varepsilon}{2} \| u(t) \|_{H^3\cap H_0^2}^2 + (u'(t) \,,\, \varphi_p(t))_{L^2} + \int_0^t [a(u \,,\, \varphi_p') -$$

$$- (u', \varphi_p'')_{L^2} + \varepsilon (u \,,\, \varphi_p')_{H^3 \cap H_0^2} - (f \,,\, \varphi_p' - u')_{L^2}]\, d\eta \Big\}\, dt \geq$$

$$\geq \int_0^T \psi(t) \Big\{ -\tfrac{1}{2} \| z_1 \|_{L^2}^2 - (H(Dz_0) \,,\, 1)_{L^2} -$$

$$- \tfrac{1}{2} \| G(Dz_0) \|_{H_0^1}^2 - \frac{\varepsilon}{2} \| z_0 \|_{H^3 \cap H_0^2}^2 + (z_1 \,,\, \varphi_p(0))_{L^2} \Big\}\, dt$$

Hence, $u(t)$ satisfies $b_1')$ a.e. in $[0, T]$ $\forall$ test function φ_p defined by the second of (4.14) and also, letting $p \to \infty$, $\forall \varphi$ given by the first of (4.14).

Since, on the other hand, the space $\overset{\circ}{K}_0 \cap H^1(0, T; H_0^2 \cap H^4) \cap H^2(0, T; L^2)$ is dense in $K_0 \cap H^1(0, T; H_0^2 \cap H^3) \cap H^2(0, T; L^2)$, relation (4.1) holds a.e. in $[0, T]$. This completes the proof of Theorem 2.

5. – We now give the proof of Theorem 1.

Let $u_\varepsilon(t)$ be a solution of $a_1')$, $b_1')$ corresponding to the coefficient ε. Setting in (4.1) $\varphi = 0$, we obtain, by the assumptions made,

$$\| u_\varepsilon'(t) \|_{L^2} \leq M_4 \quad \text{a.e. in } [0, T] \tag{5.1}$$

and also, since $u_\varepsilon(t) \in K_0$,

$$\left| \frac{\partial u_\varepsilon}{\partial x} \right| \leq N_1'' \quad , \quad \left| \frac{\partial^2 u_\varepsilon}{\partial x^2} \right| \leq N_2'' \quad , \quad \left| \frac{\partial^3 u_\varepsilon}{\partial x^3} \right| \leq N_3'' \tag{5.2}$$

a.e. in Q. Since the embedding on $L^2(0, T; H^3) \cap H^1(0, T; L^2)$ in $L^2(0, T; H^2)$ is completely continuous, we have, by (5.1), (5.2),

$$\lim_{\varepsilon \to 0} u_\varepsilon'(t) = u'(t) \tag{5.3}$$

in the weak-star topology of $L^\infty(0, T; L^2)$ and

$$\lim_{\varepsilon \to 0} \frac{\partial u_\varepsilon}{\partial x} = \frac{\partial u}{\partial x} \quad , \quad \lim_{\varepsilon \to 0} \frac{\partial^2 u_\varepsilon}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \tag{5.4}$$

a.e. in Q.

Since $u_\varepsilon(0) = z_0$ and $u_\varepsilon(t) \in L^2(0, T; H_0^2)$, it follows from (5.2), (5.3), (5.4) that $u(t)$ satisfies condition a').

By (5.3), (5.4) we have also, $\forall \varphi(t) \in K_0 \cap H^1(0, T; H_0^2 \cap H^3) \cap$ $\cap H^2(0, T; L^2)$,

$$\lim_{\varepsilon \to 0} \int_0^t a(u_\varepsilon, \varphi)\, d\eta = \int_0^t a(u, \varphi)\, d\eta \tag{5.5}$$

and, denoting by $\psi(t)$ an arbitrary function $\in C^0[0, T]$ with $\psi(t) \geq 0$,

$$\lim_{\varepsilon \to 0} \int_0^T \psi(t) \left[(H(Du_\varepsilon), 1)_{L^2} + \tfrac{1}{2} \| G(Du_\varepsilon) \|_{H_0^1}^2 - (u_\varepsilon', \varphi)_{L^2} \right] dt =$$
$$= \int_0^T \psi(t) \left[(H(Du), 1)_{L^2} + \tfrac{1}{2} \| G(Du) \|_{H_0^1}^2 - (u', \varphi)_{L^2} \right] dt, \tag{5.6}$$

$$\liminf_{\varepsilon\to 0} \int_0^T \psi(t) \| u'_\varepsilon(t) \|^2_{L^2} \, dt \geq \int \psi(t) \| u'(t) \|^2_{L^2} \, dt . \tag{5.7}$$

Moreover, by (5.2),

$$\lim_{\varepsilon\to 0} \int_0^T \psi(t) \left\{ -\frac{\varepsilon}{2} \| u_\varepsilon(t) \|^2_{H^3 \cap H^2_0} + \int_0^t \varepsilon \, (u_\varepsilon , \varphi')_{H^3 \cap H^2_0} \, d\eta \right\} dt = 0 . \tag{5.8}$$

Letting $\varepsilon \to 0$ in (4.1) we then obtain

$$\begin{aligned} & -\tfrac{1}{2} \| u'(t) \|^2_{L^2} - (H(Du(t)), 1)_{L^2} - \tfrac{1}{2} \| G(Du(t)) \|^2_{H^1_0} + \\ & + (u'(t), \varphi'(t))_{L^2} + \\ & + \int_0^t [a(u, \varphi') - (u', \varphi'')_{L^2} - (f, \varphi' - u')_{L^2}] \, d\eta \geq \\ & \geq -\tfrac{1}{2} \| z_1 \|^2_{L^2} - (H(Dz_0), 1)_{L^2} - \tfrac{1}{2} \| G(Dz_0) \|^2_{H^1_0} + (z_1, \varphi'(0))_{L^2} \end{aligned} \tag{5.9}$$

a.e. in $[0, T]$, $\forall \varphi(t) \in K_0 \cap H^1(0, T; H^2_0 \cap H^3) \cap H^2(0, T; L^2)$.

Since the space of such test functions is dense in $K_0 \cap H^1(0, T; H^2_0) \cap \cap H^2(0, T; L^2)$, condition b′) is satisfied and the theorem is proved.