Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Delfina Roux, Clemente Zanco

Kannan maps in normed spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **65** (1978), n.6, p. 252–258.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1978_8_65_6_252_0>

Analisi funzionale. — *Kannan maps in normed spaces* (*). Nota di DELFINA ROUX e CLEMENTE ZANCO (**), presentata (***) dal Socio L. AMERIO.

RIASSUNTO. — Si introduce la nozione di « struttura quasi-normale relativa » per i sottoinsiemi chiusi (non necessariamente convessi) di uno spazio normato. Si prova quindi che ogni mappa di Kannan (generalizzata) che muta in sè un sottoinsieme dotato di tale struttura e debolmente compatto di uno spazio normato ha un punto fisso. Analogo risultato vale per i sottoinsiemi dotati di tale struttura e debolmente* chiusi di uno spazio duale; in particolare ogni mappa di Kannan che muta in sè l^1 o L^p, $1 < p \leq \infty$, (o una sua bolla chiusa) ha un punto fisso.

1. INTRODUCTION

Let X be a normed space and K a weakly compact convex subset of X.

It is known ([5], theorem II) that, if K has quasi-normal structure, any Kannan map which leaves K invariant has one (and only one!) fixed point. In this Paper we prove that the quoted result holds, more generally, for the weakly compact (not necessarily convex) subsets K of X satisfying a weaker condition than the one of quasi-normal structure, which here will be called " quasi-normal relative structure ".

Moreover, if X is a dual space, we prove that a Kannan map, which leaves a weak* closed, with quasi-normal relative structure, subset of X invariant, has one fixed point.

In particular, the results stating that a Kannan self-mapping of $L^p(X, S, \mu)$ (or of a closed ball in $L^p(X, S, \mu)$), if $1 < p < \infty$, has necessarily a fixed point (1), are thus extended to l^1 and $L^\infty(X, S, \mu)$ (μ σ-finite measure).

2. QUASI-NORMAL RELATIVE STRUCTURE

Here and in what follows, let X be a normed space, K a nonempty closed subset of X, $T : K \to K$ a Kannan map, i.e. a map satisfying the following condition.

For every $x, y \in K$

$$\| Tx - Ty \| \leq b(x, y) \| x - Tx \| + b(y, x) \| y - Ty \|$$

(*) Lavoro eseguito nell'ambito del G.N.A.F.A. (C.N.R.).

(**) Indirizzo degli Autori: Istituto Matematico, Università, via C. Saldini 50, 20133 Milano, Italia.

(***) Nella seduta del 16 dicembre 1978.

(1) See [5], Cor. II.

with $b : K \times K \to [0, 1)$ such that $b(x, y) + b(y, x) \leq 1$ and $b(x, y) \to 1 \Rightarrow \Rightarrow \mathrm{Max}(\|x - Tx\|, \|y - Ty\|) \to 0$ or ∞.

The following proposition holds

PROPOSITION 1. *The following condition on* K:

(C) *for every bounded non singleton subset* $B \neq \varnothing$ *of* K

$$\exists z \in K : \|z - x\| < \operatorname{diam} B \qquad \forall x \in B ,$$

is a necessary condition to ensure that every Kannan map (2) $T : K \to K$ *has a fixed point.*

It is obvious that spaces (and sets) with normal, quasi-normal, normal relative (3) structure satisfy this condition.

There exist, however, some spaces, even reflexive, which don't satisfy it (4); in those spaces (and in their closed balls) some Kannan self-mappings without fixed points must exist.

The validity of (C), however, is not enough, in the general case, to ensure the existence of fixed points, as the following example shows.

Let K be the subset of R^2 defined as follows:

$$K = \{x \in R^2 ,\, x \equiv (u, v) : u \geq 0 ,\, v = \pm \tfrac{1}{2} u\}.$$

It is easy to see that K satisfies (C).

Let $T : K \to K$ defined as follows:

$$Tx = T(u, v) = \begin{cases} (1, \frac{1}{2}) & \text{if } x = (0, 0) \\ (1, -\frac{1}{2}\operatorname{sgn} v) & \text{if } x \neq (0, 0)\ x \notin S \\ \left(4 + \dfrac{1}{1-u}, -\left[2 + \dfrac{1}{2(1-u)}\right]\operatorname{sgn} v\right) & \text{if } x \in S \end{cases}$$

where S is the intersection of the strip $(1/5) < u < 1$ and K.

It is obvious that T has no fixed point and T can be verified to be a Kannan map; indeed we can assume (supposing $\|x - Tx\| \geq \|y - Ty\|$) $b(x, y) = \dfrac{\|Tx - Ty\|}{\|x - Tx\|}$ if $\|x - Tx\| > > \|y - Ty\|$ and x or $y \in S$, $b(x, y) = \frac{1}{2}$ otherwise.

The possibility to construct a counterexample depends on the existence of a non singleton subset B of K every point of which is diametral and such that for every $z \in K$, satisfying (C), there exists a ball with radius greater than the diameter of B containing B but not containing z (5).

This leads to the following definition.

We say that K has *quasi-normal relative structure* (in what follows q.n.r. structure) if for every bounded non singleton subset $B \neq \varnothing$ of K there

(2) Also restricting to the class of Kannan maps with $b(x, y) \equiv \frac{1}{2}$.

(3) See [2], § 4.

(4) Such is, for example, the space Y of [5] pag. 107, as easily verified.

(5) In the above example, the subset B consists of the two points $(1, \pm \frac{1}{2})$.

exists $z_B \in K$ such that

i) $\| z_B - x \| < r_B = \text{diam } B \qquad \forall x \in B$

ii) if $y \in K$ and $\rho > r_B$, then

$$\| y - x \| < \rho \qquad \forall x \in B \Rightarrow \| y - z_B \| \le \rho \text{ (6)} .$$

Remarks:

1) A subset with q.n.r. structure is not necessarily convex. It is enough to consider in R^2 the set of the points of the form $(0, x)$ or $(x, 0)$, $0 \le x \le 1$.

2) The validity if i) does not imply that of ii) (see the example preceding the definition).

3) The following spaces have q.n.r. structure:

a) spaces with quasi-normal structure (in particular uniformly convex Banach spaces (7) and separable Banach spaces (8);

b) the dual spaces of (complex) AL-spaces (9).

Hence $L^p(X, S, \mu)$ spaces, $1 \le p \le \infty$, (separable for $p = 1$ and with μ σ-finite measure for $p = \infty$) have q.n.r. structure.

4) Any closed ball of the above spaces has q.n.r. structure.

5) The closed and convex subsets of the spaces with normal or quasi-normal structure have q.n.r. structure. This in general is not true for closed convex subsets of the dual spaces of AL-spaces. Indeed the following proposition holds.

PROPOSITION 2. *If* $L^\infty(X, S, \mu)$ (10) *is not separable, there exists a weak* compact convex subset* K *such that*

$$\forall x \in K \qquad \exists z \in K : \| x - z \| = \text{diam } K .$$

COROLLARY 1. *If* $L^\infty(X, S, \mu)$ (10) *is not separable, there exists a closed convex subset which has not q.n.r. structure.*

COROLLARY 2. $L^\infty(X, S, \mu)$ (10) *has not quasi-normal structure, unless it is separable.*

Also this remark seems to be new.

3. FIXED POINT RESULTS

THEOREM 1. *If* K *is a weakly compact subset with q.n.r. structure of a normed space* X, *every Kannan map* $T : K \to K$ *has one fixed point.*

(6) Which is equivalent to

$$y \in K, \rho \ge r_B \wedge \| y - x \| \le \rho \quad \forall x \in B \Rightarrow \| y - z_B \| \le \rho .$$

(7) See [1], th. 4.1.

(8) See [5], Lemma I.

(9) See [6], Lemma and related remark and [4].

(10) The measure in $L^\infty(X, S, \mu)$ is supposed to be σ-finite.

Remarks:

1) Theorem 1 contains theorem II of [5] and extends it in various directions.

2) If X is a dual space, the weak compactness assumption on K may be replaced by weak* compactness hypothesis.

3) Theorem 1 can be applied to the closed balls of l^1 and $L^p(X, S, \mu)$, $1 < p \leq \infty$ [10] (see § 2 remark 4).

4) Theorem 1 can be applied to weak (weak*) compact convex subsets of $L^p(X, S, \mu)$, $1 \leq p < \infty$ (with L^1 separable). In the case $p = \infty$, the result is not, generally, true. Indeed from propositions 1 and 2 we have

PROPOSITION 3. *If* $L^\infty(X, S, \mu)$ [10] *is not separable, there exists a weak* compact convex subset* K *with a Kannan self-mapping without fixed points.*

If X is a dual space, the assumption on weak* compactness (see remark 2) of K can be weakened. Indeed the more general following theorem holds.

THEOREM 2. *If* K *is a weak* closed subset with q.n.r. structure of a dual space* X, *every Kannan map* $T : K \to K$ *has one fixed point.*

This theorem contains some known results ([5], cor. I and II). Moreover, after remark 3 in § 2, the following corollaries can be obtained.

COROLLARY 3. *If* X *is a separable dual space, every Kannan map* $T : X \to X$ *has one fixed point.*

COROLLARY 4. *If* X *is the dual space of a (complex)* AL-*space, every Kannan map* $T : X \to X$ *has one fixed point.*

In particular

COROLLARY 5. *If* $X = l^1$ *or* $X = L^p(X, S, \mu)$ $(1 < p \leq \infty)$ [10], *every Kannan map* $T : X \to X$ *has one fixed point.*

4. PROOFS

PROPOSITION 1. Suppose (C) not be satisfied. Then there exists a bounded non singleton subset $B^* \neq \varnothing$ of K such that

$$\forall x \in K \quad \exists z_x \in B^* : \| x - z_x \| \geq \operatorname{diam} B^* > 0 .$$

Let us consider the self-mapping of K defined by

$$T : x \to z_x .$$

For every $x, y \in K$ we have

$$\| Tx - Ty \| = \| z_x - z_y \| \leq \operatorname{diam} B^* \leq \tfrac{1}{2} \| x - Tx \| + \tfrac{1}{2} \| y - Ty \| .$$

Hence T is a Kannan self-mapping of K without fixed points.

PROPOSITION 2. If $L^\infty(X, S, \mu)$ is not separable, it is always possible to find an infinite sequence $\{E_n\}_{n=1,2,\cdots}$ of disjoint subsets of X with positive finite measure [11].

Let H be the (uncountable) subset of $L^\infty(X, S, \mu)$ represented by functions $x = x(t)$ such that

$$x(t) = \begin{cases} 1 & \text{a.e. in } E_{n_h} \quad (h = 1, 2, \cdots) \\ 0 & \text{a.e. in } X \setminus \bigcup_h E_{n_h} \end{cases}$$

where $\{n_h\}_{h=1,2\cdots}$ is a sequence of positive integers such that

$$n_{h+1} > 2\, n_h \qquad \forall h .$$

Let K be the weak* closure of the convex hull of H. We shall prove that K satisfies the required conditions.

Remark that K is convex [12], weak* compact [13] and that

$$\operatorname{diam} K = \operatorname{diam} H = 1 .$$

If $x \in K$, we have

$$x(t) = \begin{cases} x_n \geq 0 & \text{a.e. in } E_n \, (n = 1, 2, \cdots) \\ 0 & \text{a.e. in } X \setminus \bigcup_n E_n . \end{cases}$$

Now we shall prove that

$$\inf_n x_n = 0 . \tag{4.1}$$

Indeed, if not so, a positive integer N should exist such that $x_n > 1/N \;\forall n$. Then $y \in \mathrm{coH}$ [14] should exist,

$$y = \sum_1^P \lambda_k z^{(k)} , \quad \text{with } \lambda_k \geq 0 , z^{(k)} \in H \quad k = 1, \cdots, P \quad \text{and } \sum_1^P \lambda_k = 1 ,$$

such that

$$y(t) = \sum_1^P \lambda_k z^{(k)}(t) > 1/N \qquad \text{a.e. in } \bigcup_N^{2N} E_n .$$

(11) If not so, we might write $X = \bigcup_{n=1}^{N} E_n$, E_n such that, for every n, does not contain subsets of positive measure less than the measure of E_n, and we should have $L^\infty(X, S, \mu) \simeq \simeq L^\infty(x_1, x_2, \cdots, x_N)$.

(12) Indeed, weak* topology is compatible with the linear structure.

(13) Indeed, K is bounded.

(14) coH convex hull of H.

But this is absurd, because for every K there exists at most a set E_n, with $N \leq n \leq 2N$, a.e. in which $z^{(k)}(t) > 0$.

Hence for every $x \in K$ (4.1) holds. Then it is certainly possible to exhibit $z \in H$ such that $\| x - z \| = 1$ and proposition 2 is proved.

The proofs of theorems 1 and 2 are based on the Lemma of [3] and on the following Lemma.

LEMMA. *For every* $r \in R^+$ *let* $A_r = \{x \in K : \| x - Tx \| \leq r\}$. *Let* B_r *be the weak (weak*) closure of* TA_r. *If* B_r *is nonempty and non singleton, then*

$$\operatorname{diam} B_r \leq r, \tag{4.2}$$

$$\| z_{B_r} - Tz_{B_r} \| \leq r. \tag{4.3}$$

Indeed for every $x, y \in B_r$ and for every $\varepsilon > 0$ there exist $u, v \in A_r$ such that

$$\| x - y \| - \varepsilon \leq \| Tu - Tv \| \leq b(u, v) r + b(v, u) r \leq r,$$

whence we have (4.2).

Now let us suppose, by contradiction, $\| z_{B_r} - Tz_{B_r} \| = \rho > r$

Let $\tilde{b} = \operatorname*{Sup}_{x \in B_r} b(z_{B_r}, x)$. B_r being bounded, we have $\tilde{b} < 1$.

For every $x \in B_r$ and for every $\varepsilon > 0$ there exists $u \in A_r$ such that

$$\| Tz_{B_r} - x \| - \varepsilon \leq \| Tz_{B_r} - Tu \| \leq \tilde{b}\rho + (1 - \tilde{b}) r < \rho,$$

whence, from ii),

$$\| z_{B_r} - Tz_{B_r} \| \leq \tilde{b}\rho + (1 - \tilde{b}) r < \rho$$

which is absurd. (4.3) follows.

THEOREM 1. Let $s = \operatorname*{Inf}_{x \in K} \| x - Tx \|$. By the Lemma of [3] we have $B_s \neq \varnothing$. If we had $s > 0$, from (4.3) $\| z_{B_s} - Tz_{B_s} \| = s$ would follow, whence $Tz_{B_s} \in B_s$ [15] which is absurd by i) and (4.2). Hence $s = 0$ and Theorem 1 is proved.

THEOREM 2. Let r such that $A_r \neq \varnothing$; by (4.2) B_r is weak* compact. Moreover $TB_r \subset B_r$ [16] and $s = \operatorname*{Inf}_{x \in K} \| x - Tx \| = \operatorname*{Inf}_{x \in B_r} \| x - Tx \|$. From the Lemma of [3] applied to B_r it follows $B_s \neq \varnothing$ and the proof is to be completed with the same reasoning as in Theorem 1.

(15) Observe that $\| Tx - T^2 x \| \leq \| x - Tx \| \quad \forall x \in K$.

(16) Indeed $B_r \subset A_r$: see proof of the Lemma of [3].

REFERENCES

[1] L. P. BELLUCE and W. A. KIRK (1967) - *Nonexpansive mappings and fixed points in Banach spaces*, « Illinois J. Math. », *11*, 474-479.

[2] W. A. KIRK (1969) - *Nonexpansive mappings and the weak closure of sequences of iterates*, « Duke Math. J. », *36*, 639-645.

[3] D. ROUX and E. MALUTA (1979) - *Contractive Kannan maps in compact spaces*, « Riv. Mat. Univ. Parma », (4) *5*.

[4] H. H. SCHAEFER (1974) - *Banach lattices and positive operators*, Springer Verlag, Berlin, Heidelberg, New York.

[5] P.M. SOARDI (1972) - *Struttura quasi-normale e teoremi di punto unito*, « Rend. Ist. Mat. Univ. Trieste », *4*, 105-114.

[6] P.M. SOARDI (1979) - *Existence of fixed points of nonexpansive mappings in certain Banach lattices*, « Proc. Amer. Math. Soc. », *73*, 25-29.