Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

P.J. McKenna, Howard Shaw

The structure of the solution set of some nonlinear problems

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **65** (1978), n.6, p. 239–243.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1978_8_65_6_239_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1978.

Matematica. — *The structure of the solution set of some nonlinear problems.* Nota di P. J. McKENNA e HOWARD SHAW, presentata (*) dal Socio D. GRAFFI.

RIASSUNTO. — Per equazioni operazionali $Lu + Nu = h$, L ed N operatori in uno spazio di Hilbert reale X, L lineare, N non lineare, e sotto moderate ipotesi su L ed N, l'insieme delle soluzioni è, generalmente, una varietà di dimensione uguale all'indice di Fredholm di L. Precisamente, questo accade effettivamente se la proiezione di h su un opportuno sottospazio E di dimensione finita in X non cade su un certo insieme Z di E, di misura zero oppure di prima categoria.

1. INTRODUCTION

In recent years many papers have studied the range set of non-linear operators of the type L + N, where usually L is a linear differential operator and N is a nonlinear operator which in some sense is small or bounded compared with L. For example, in [12], [18], [20] the operator L was uniformly elliptic with kernel and N was a bounded Niemytsky operator on L^2.

In these papers, various necessary and sufficient conditions were given for a function to be in the range of L + N. In [1], [2], [7], [15], the multiplicity of solutions of the equation

$$Lu + Nu = h \tag{1}$$

was studied. In all these papers, a key requirement for the existence of solutions was some condition involving the projection of h onto a finite dimensional subspace.

We summarize here results on the structure of the set of elements u which are solutions of (1). The main result (section 2) shows the existence of a finite dimensional subspace depending on L and N such that if the orthogonal projection of h does not belong to a small subset of this space (either of measure zero, or of first category), then the solution set is actually a manifold of the same dimension as the Fredholm index of the operator L.

The hypotheses of this theorem are sufficiently general to cover a wide variety of situations including elliptic, parabolic and hyperbolic operators L, and the theorem itself gives new information even on the widely studied self-adjoint case.

In section 3, we give examples of many different situations in which our general theorem extends the results of recent papers. Proofs and details will appear in [14].

(*) Nella seduta del 16 dicembre 1978.

2. Assumptions and Statement of Results.

Let Ω be a connected open set in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$; in fact most of the following generalizes to smooth Riemannian manifolds with or without boundary. We seek information about the set of solutions $u = u(x)$ to the nonlinear equation (1) where h is a given function in $L^2(\Omega)$ and L and N are respectively linear and nonlinear operators in L^2, possibly only densely defined.

The operator L (which is usually a differential operator with or without boundary conditions) is assumed to be closed and to have finite dimensional kernel and a closed range of finite codimension, with index $i = \dim(\ker L)$ — codim (range L). N is a Niemytsky operator of the form $Nu = f(u)$, where for $r = 1 + \max(0, i)$, f is a C^r real valued function on $\mathbf{R}$ and $\sup |f'| = M < \infty$.

We make an additional hypothesis on the spectrum of LL^*, where L^* is the adjoint of L. In the next section, we give several examples in which this hypothesis is satisfied. LL^* is a positive self adjoint operator by a theorem of Von Neumann [21], and so the spectrum $\sigma(LL^*)$ is real and non-negative. We assume that for some $M' > M^2$, $\sigma(LL^*) \cap [0, M']$ consists of isolated points, each of finite multiplicity.

One might hope that for any $h \in L^2(\Omega)$, the set of solutions might be a differentiable manifold of dimension i; in fact this statement is generically true in the following sense:

THEOREM. *Given the above hypotheses on* L *and* N, *there exists a certain finite dimensional subspace* S_1 *of* L^2 *and its orthogonal complement* S_2 *with the following property: for any* $h_2 \in S_2$ *there exists a set* $A \subseteq S_1$ *of measure zero so that if* $h_1 \in S_1$, $h_1 \notin A$, *then*

$$Lu + Nu = h_1 + h_2$$

has an i-dimensional manifold as its solution set.

The space S_1 can be realized as the linear span of the first few eigenvectors of LL^* (ordered by size of eigenvalue). As a finite dimensional linear space, S_1 inherits a notion of measure zero from Lebesgue measure on $\mathbf{R}^n$, and it is in this sense that it has measure zero. Alternatively, the bad set A may be taken to be small in the sense of being first Baire category.

In the important case in which $i = 0$, the theorem states that, generically, the solution set is discrete. Since the solution set is closed, it must consist of isolated points. This i-manifold turns out to be a submanifold of a finite dimensional subspace of L^2, although it may also be viewed as a submanifold of L^2. If from other considerations, we know that the solutions obey an L^2 a priori bound, this will imply that the solution set is actually finite, generically.

If $i < 0$, the conclusion is that *solutions do not exist*, in general, since the empty set is the only i-manifold for $i < 0$. Analogously, if $i \geq 0$, the solution set generically is either an i–dimensional manifold, or the empty set, and in order to ensure that this trivial case does not arise, one must add to the very general hypotheses listed here, some additional hypotheses, which are widespread in the literature.

The methods of proof will appear in [14] together with generalizations to N of the form $Nu = f(x, u, Du)$, where Du represents derivatives of order generally (though not necessarily) less than the order of L.

Use is made of the alternative method of Cesari [4], [5], [6], and Hale [8] in which the concepts of functional analysis are injected into the classical scheme of Lyapunov-Schmidt. The equation is split into two coupled parts, a contraction mapping equation on the infinite-dimensional part, and a finite dimensional part which can then be analyzed by differential topology.

These methods apply in similar problems of structure, as in the following. Consider the 1-parameter family of Neumann problems

$$\Delta u + \lambda u + \arctan u = h \qquad \text{in } \Omega ,$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega ,$$

with $|\int h| < (\pi/2)$ meas (Ω). For each fixed real λ, the Landesman-Lazer theorem [12] guarantees the existence of solutions $u_\lambda(x)$. We prove that for generic h, the solution set $\{(\lambda, u_\lambda(x)\}$ will be a smooth 1-manifold in $\mathbf{R} \oplus L^2$.

3. Examples and Applications.

First we consider the case studied in [12]. Let L be a strongly elliptic self-adjoint differential operator on a region Ω with smooth (C^∞) boundary $\partial\Omega$ and coercive boundary conditions. Assume in addition that f is a bounded function with limits at $\pm\infty$ satisfying $f(-\infty) < f(s) < f(+\infty)$ for all s, $-\infty < s < +\infty$. Then a necessary and sufficient condition that the equation

$$Lu + f(u) = h(x) \qquad \text{in } L^2(\Omega) \tag{2}$$

have solutions is that

$$f(+\infty)\int_{\theta>0}\theta + f(-\infty)\int_{\theta<0}\theta > \int h\theta > f(-\infty)\int_{\theta>0}\theta + f(+\infty)\int_{\theta<0}\theta , \tag{3}$$

where θ spans the kernel of L.

The proof implies in addition that any solution $u(x)$ admits of an a priori bound. Since the index of the self-adjoint operator L is zero, our theorem shows that there exists a finite dimensional subspace S_1 of L^2 such that for any given $h_2 \perp S_1$ the equation

$$Lu + Nu = h_1 + h_2$$

admits of only a finite number of solutions for almost all $h_1 \in S_1$.

We now turn our attention to a genuinely non self-adjoint problem. Let $\Omega \subseteq \mathbf{R}^2$ be any bounded connected region with smooth boundary $\partial\Omega$, let L be the biharmonic operator Δ^2 with boundary conditions

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial n} \Delta u = 0 \qquad \text{on} \quad \partial\Omega \,,$$

and let us consider the problem $\Delta^2 u + f(u) = h(x, y)$. The operator L has as kernel the space spanned by $\{1, y, y^2\}$ and as cokernel the space spanned by $\{1\}$. An analysis similar to that of [12] shows that if the function f is as in the first example, then a necessary and sufficient condition for the existence of solutions is (3). Our theorem shows that in fact generically the solution set is a two-manifold in $L^2(\Omega)$.

If we consider the equation $L^* u + f(u) = h(x, y)$ with L defined above, then the Fredholm index is negative and our theorem shows that generically, no solution exists.

The theorem applies equally well to some hyperbolic problems. Consider the equation

$$\begin{aligned} & u_{tt} - u_{xx} - cu + g(u) = h(x, t) \,, \qquad c > 0 \,, \\ & u(0, t) = u(\pi, t) = 0 \,, \\ & u(x, t) = u(x, t + 2\pi) \,. \end{aligned}$$

Then as in [16], [17], solutions exist if $g' < c$ and if h satisfies a set of inequalities resembling (3). Our theorem again shows that generically there are only a finite number of solutions.

In these, it is easy to construct a right hand side where the solution set is actually a continuum but our theorem shows that this is generically not true. Finally, we consider an example where the nonlinear operator is not of "slow (meaning sublinear) growth".

It is known (see [11]) that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ are the eigenvalues of a linear self-adjoint operator L with $|\lambda_n| \to \infty$ as $n \to +\infty$, then the equation

$$Lu + f(u) = h(x) \tag{4}$$

has solutions for all $h \in L^2(\Omega)$ if $\lambda_n < f'(-\infty) < f'(+\infty) < \lambda_{n+1}$. Since for a self-adjoint linear operator the Fredholm index is zero, and since a priori estimates exist for any solution, we may conclude that the equation admits of only a finite number of solutions except for a measure zero set of projections of h onto a finite dimensional subspace.

REFERENCES

[1] AMBROSETTI A. and PRODI G. (1972) – *On the inversion of some differentiable mappings with singularities between Banach spaces*, « Ann. di Mat. Pura Appl. », IV, *93*, 231-247.

[2] BERGER M. and PODOLAK E. (1975) – *On the solutions of a nonlinear Dirichlet problem*, « Indiana J. Math. », *24*, 837–846.

[3] BREZIS H. and NIRENBERG L. (1978) – *Characterisations of the ranges of some nonlinear operators and applications to boundary value problems*, « Ann. Scuola Norm. Sup. Pisa », *4*, 225–326.

[4] CESARI L. (1976) – *Functional analysis, nonlinear differential equations, and the alternative method*, in « Nonlinear Functional Analysis and Differential Equations » (L. Cesari, R. Kannan and J. D. Schuur, Eds.) Dekker, New York, 1–197.

[5] CESARI L. (1963) – *Functional analysis and periodic solutions of nonlinear differential equations*. « Contributions to Differential Equations, John Willey », *1*, 149–187.

[6] CESARI L. (1964) – *Functional analysis and Galerkin's method*, « Michigan Math. Journal », *11*, 385–414.

[7] FUCIK S. (1976) – *Boundary value problems with jumping nonlinearities*, « Cas Prestovani Mat. », *101*, 69–87.

[8] HALE J. K. (1971) – *Applications of alternative problems*, Lecture notes of the Lefschetz Center for Dynamical Systems, Brown University.

[9] HESS P. and RUF B. – *On a superelliptic boundary value problem*, to appear.

[10] HESS P. (1974) – *On semicoercive problems*, « Indiana Univ. Math. J. », *23*, 645–654.

[11] KAZDAN J. L. and WARNER F.W. (1975) – *Remarks on some quasilinear elliptic equations*, « Comm. Pure Appl. Math. », *28*, 567–597.

[12] LANDESMAN E.M. and LAZER A. C. (1970) – *Nonlinear perturbations of elliptic boundary value problems at resonance*, « J. Math. Mech. », *19*, 609–623.

[13] LERAY J. and SCHAUDER J. (1934) – *Topologie et equations fonctionnelles*, « Ann. Sci. École Norm. Sup. », *51*, 45–78.

[14] McKENNA P. J. and SHAW HOWARD – *On the structure of the set of solutions to some nonlinear boundary value problems*, « J. Diff. Equations », to appear.

[15] McKENNA P. J. – *On a superlinear elliptic boundary value problem at resonance*, « Proc. Amer. Math. Soc. », to appear.

[16] McKENNA P. J. – *On the reduction of a semilinear hyperbolic problem to a Landesman-Lazer type problem*, to appear in « Houston Journal of Math. ».

[17] RABINOWITZ P. H. (1978) – *Some minimax theorems and applications to nonlinear partial differential equations*, in « Nonlinear Analysis », a volume in honor of E. H. Rothe, Academic Press, 161–177.

[18] SHAW HOWARD (1977) – *A nonlinear elliptic boundary value problem at resonance*, « J. Diff. Equations », *26*, 335–346.

[19] STRAUSS W. (1971) – *On weak solutions of semilinear hyperbolic equations*, « An. Acad. Brasil, Ci. », *42*, 645–671.

[20] WILLIAMS S. A. (1970) – *A sharp sufficient conditions for solution of a nonlinear elliptic boundary value problem*, « J. Diff. Equations », *8*, 580–588.

[21] YOSIDA KOSAKU (1965) – *Functional Analysis*, Springer Verlag, Berlin.