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On periodic meromorphic functions on $\mathbf{C}^n$

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta dell'11 marzo 1978

Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On periodic meromorphic functions on* $\mathbf{C}^n$. Nota di ABRAMO HEFEZ, presentata (*) dal Corrisp. A. ANDREOTTI.

RIASSUNTO. — Si dimostra che per una varietà quasi abeliana il corpo delle funzioni meromorfe ha grado di trascendenza infinito.

Let G be a discrete subgroup of $\mathbf{C}^n$ with rank $G = r$ ($\leq 2n$). We are concerned here with the field of meromorphic functions on $\mathbf{C}^n$ having G as group of periods. It is a classical result that if $r = 2n$, then this field is an algebraic field of functions of transcendence degree over $\mathbf{C}$ less than or equal to n. It is equal to n if and only if $\mathbf{C}^n/G$ is an Abelian manifold. In this paper we show that if $r < 2n$ and $\mathbf{C}^n/G$ is a quasi-Abelian manifold, then the transcendence degree is infinite. This fact has been conjectured by A. Andreotti to whom I am indebted for very useful suggestions.

1. INTRODUCTION

Let G be a discrete subgroup of $\mathbf{C}^n$ of real rank r and complex rank s. $\mathbf{C}^n/G$ has a natural structure of an n-dimensional complex Lie group. From real analytic point of view, these quotients depend only on n and r, more precisely, $\mathbf{C}^n/G$ is isomorphic to $(S^1)^r \times \mathbf{R}^{2n-r}$. The field of G-periodic meromorphic functions on $\mathbf{C}^n$ is naturally isomorphic to the field of meromorphic functions on $\mathbf{C}^n/G$. It is easy to verify the following facts:

(*) Nella seduta dell'11 marzo 1978.

(*a*) If $s < n$, then $\mathbf{C}^n/G \cong \mathbf{C}^s/G' \times \mathbf{C}^{n-s}$ (as complex Lie groups), where G' is a discrete subgroup of $\mathbf{C}^s$.

(*b*) If $s = n = r$, then $\mathbf{C}^n/G \cong (\mathbf{C}^*)^n$.

(*c*) If $s = n < r$, then with a change of coordinates in $\mathbf{C}^n$, we can suppose that G is generated over $\mathbf{Z}$ by vectors $e_1, \cdots, e_n, v_1, \cdots, v_m$, which are linearly independent over $\mathbf{R}$, here $\{e_1, \cdots, e_n\}$ is the canonical basis of $\mathbf{C}^n$ and $r = n + m$.

So in cases (*a*) and (*b*), the field of G-periodic meromorphic functions is obviously of infinite transcendence degree.

Let G be as in (*c*), then G generates over $\mathbf{R}$ a real vector space of dimension $n + m$ which we denote by $\mathbf{R}_G^{n+m}$. It is easy to see that $\mathbf{R}_G^{n+m}$ contains a complex vector subspace F of dimension m (and not bigger than m). Consider now the following two conditions due to Gherardelli and Andreotti [2].

There exists a Hermitian form H on $\mathbf{C}^n \times \mathbf{C}^n$ such that:

(A) $H_{/F \times F}$ is positive defined

(B) Im H has integral values on $G \times G$.

Note that these conditions, when $r = 2n$, are exactly Riemann's conditions for the lattice G, since in this case $F = \mathbf{C}^n$.

DEFINITION (Gherardelli-Andreotti) if conditions (A) and (B) hold for G, then $\mathbf{C}^n/G$ is called a quasi-Abelian manifold.

In the same paper, Gherardelli and Andreotti prove the following

THEOREM. *Let* $\mathbf{C}^n/G$ *be a quasi-Abelian manifold, then there exists a discrete subgroup* G' *of* $\mathbf{C}^n$, *such that* $\mathbf{C}^n = \mathbf{R}_G^{n+m} \oplus \mathbf{R}_{G'}^{n-m}$ *and* $\mathbf{C}^n/\Gamma$ *is an Abelian manifold where* $\Gamma = G \oplus G'$.

We suppose that G *is generated over* $\mathbf{Z}$ *by the vectors* $e_1, \cdots, e_n, v_1, \cdots, v_m$, *we may suppose that* Γ *is generated by* $e_1, \cdots, e_n, v_1, \cdots, v_m, v_{m+1}, \cdots, v_n$. *Put these vectors as columns of a* $n \times 2n$ *matrix* Ω. Ω *is then a Riemann matrix. Conditions* (A) *and* (B) *for* Γ *are equivalent to the existence of an invertible* $2n \times 2n$ *skew symmetric integral matrix* A *such that*

$$i\bar{\Omega} A^{-1\,t}\Omega > 0 \quad \text{and} \quad \Omega A^{-1\,t}\Omega = 0 . \tag{1}$$

The relations between the matrix A *and the Hermitian form* H *are given by*

$$H^{-1} = i\bar{\Omega} A^{-1\,t}\Omega \quad \text{and} \quad A = {}^t\Omega H \bar{\Omega} - {}^t\bar{\Omega}^{t} H \Omega . \tag{2}$$

For the proof of these classical facts see for example Siegel [4].

2. THE TRASCENDENCE DEGREE

Take $B = -H\bar{\Omega}$, then from the second formula of (2) we get ${}^tB\Omega - {}^t\Omega B = A$. From this, (1) and the existence theorem of Jacobian functions ([4], Theorem 12) we can guarantee the existence of a non zero entire function h such that

$$\begin{cases} h(z+e_j) = e^{2\pi i[\langle b_j, z\rangle + \mu_j]}\, h(z) \\ h(z+v_j) = e^{2\pi i[\langle b_{n+j}, z\rangle + \mu_{n+j}]}\, h(z) \end{cases} \qquad j = 1, \cdots, n,$$

where b_k is the k-th column of the matrix B, $\langle b_k, z\rangle = \sum_{i=1}^{n} b_{ik} z_i$ and the μ_k are any complex numbers. The above mentioned Jacobian functions are called of type $\{\Omega ; B ; \mu_1, \cdots, \mu_{2n}\}$.

Take a sequence $\{\lambda_i\}_{i\in\mathbf{N}}$ of complex numbers such that the set $\{1, \lambda_0, \lambda_1, \cdots\}$ is linearly independent over $\mathbf{Z}$. For each $i \in \mathbf{N}$, choose a Jacobian function h_{λ_i} of type $\{\Omega ; B ; 0, \cdots, 0, \lambda_i\}$ and take h of type $\{\Omega, B ; 0, \cdots, 0\}$.

Define $f_{\lambda_i} = \dfrac{h_{\lambda_i}}{h} \cdot f_{\lambda_i}$ is a meromorphic function periodic with respect to $e_1, \cdots, e_n, v_1, \cdots, v_{n-1}$ and satisfies trivially the functional equation

$$f_{\lambda_i}(z + kv_n) = e^{2\pi\sqrt{-1}k\lambda_i} f_{\lambda_i}(z) \qquad \text{for every} \quad k \in \mathbf{N}. \tag{3}$$

THEOREM. *The functions* f_{λ_i}, $i \in \mathbf{N}$, *are algebraically independent.*

Proof. Suppose by absurdity that $f_{\lambda_0}, \cdots, f_{\lambda_l}$ are algebraically related, then there exists a non zero polynomial

$$P(x) = \sum_{|\alpha|\leq\beta} a_\alpha x_0^{\alpha_0} \cdots x_l^{\alpha_l}$$

such that

$$P(f_{\lambda_0}, \cdots, f_{\lambda_l}) = 0. \tag{4}$$

From (4) we get, where it is defined, that

$$\begin{aligned} 0 &= P(f_{\lambda_0}(z+kv_n), \cdots, f_{\lambda_l}(z+kv_n)) = \\ &= \sum_{|\alpha|\leq\beta} a_\alpha e^{2\pi\sqrt{-1}k\langle\alpha,\lambda\rangle} f_{\lambda_0}^{\alpha_0}(z) \cdots f_{\lambda_l}^{\alpha_l}(z) \end{aligned} \tag{5}$$

here $\lambda = (\lambda_0, \cdots, \lambda_l)$ and as usual, $\langle\alpha, \lambda\rangle = \sum_{j=0} \alpha_j \lambda_j$.

Fix now an order function

$$\tau : \{1, 2, \cdots, N\} \to \{\alpha \in \mathbf{N}^{l+1} / |\alpha| \leq \beta\}$$

and define

$$\begin{aligned} w_t(z) &= a_{\tau(t)} f_{\lambda_0}^{\tau(t)_0}(z) \cdots f_{\lambda_l}^{\tau(t)_l}(z) \\ \Sigma_t &= \langle\tau(t), \lambda\rangle. \end{aligned} \qquad t = 1, \cdots, N$$

From (5), taking $k = 0, \cdots, N-1$, we get for each $z \in \mathbf{C}^n$ where everything is defined, a system of linear equations with coefficients not depending on z

$$\begin{cases} \sum_{t=1}^{N} w_t(z) = 0 \\ \sum_{t=1}^{N} w_t(z)\, e^{2\pi i \Sigma_t} = 0 \\ \cdots \\ \sum_{t=1}^{N} w_t(z)\, e^{2\pi i (N-1)\Sigma_t} = 0 \end{cases}$$

but for some $z_0 \in \mathbf{C}^n$, this system clearly admits a non trivial solution $(w_1(z_0), \cdots, w_N(z_0)) \neq 0$, so the determinant of the coefficients' matrix M is zero.

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{2\pi i \Sigma_1} & e^{2\pi i \Sigma_2} & \cdots & e^{2\pi i \Sigma_N} \\ e^{4\pi i \Sigma_1} & e^{4\pi i \Sigma_2} & \cdots & e^{4\pi i \Sigma_N} \\ e^{2(N-1)\pi i \Sigma_1} & e^{2(N-1)\pi i \Sigma_2} & \cdots & e^{2(N-1)\pi i \Sigma_N} \end{bmatrix}$$

is a Vandermonde's matrix, so

$$\det M = (-1)^{\frac{N(N-1)}{2}} \prod_{\substack{j,k=1,\cdots,N \\ j<k}} (e^{2\pi i \Sigma_j} - e^{2\pi i \Sigma_k}) = 0 .$$

Then for some $j \neq k$, $e^{2\pi i \Sigma_j} = e^{2\pi i \Sigma_k}$, this implies that $\Sigma_j - \Sigma_k \in \mathbf{Z}$ which gives us a non trivial relation with integer coefficients between $1, \lambda_0, \cdots, \lambda_l$.

COROLLARY 1. *If $\mathbf{C}^n/G$ is a quasi-Abelian manifold, then the field of G-periodic meromorphic functions on $\mathbf{C}^n$ has infinite transcendence degree over $\mathbf{C}$.*

COROLLARY 2. *If G is a discrete subgroup of $\mathbf{C}^2$ such that $rk\, G < 4$, then the field of G-periodic meromorphic functions on $\mathbf{C}^2$ has infinite transcendence degree.*

Proof. If G satisfies condition (*a*) or (*b*) (of the introduction), then the result holds. We now suppose that G is generated over Z by $e_1, e_2, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. The conditions (A) and (B) are always satisfied, so apply Corollary 1 to conclude the proof.

COMMENT

Let G be a discrete subgroup of rank 3 of $\mathbf{C}^2$, suppose G generated by $e_1, e_2, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. The following conditions are equivalent:

(i) The only holomorphic functions on $X = \mathbf{C}^2/G$ are the constants

(ii) $1, v_1, v_2$ are linearly independent over $\mathbf{Z}$ (or $\mathbf{Q}$).

This can be found in Morimoto [3].

So we can construct a manifold X satisfying the above conditions.

Denoting by K (X) the field of quotients of global holomorphic functions on X and by $\mathcal{M}$ (X) the field of meromorphic functions on X, we have $td_{K(X)} \mathcal{M}(X) = td_{\mathbf{C}} \mathcal{M}(X) = \infty$. Compare with [1], p. 79–80.

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