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Luciano Pandolfi

Stabilization and controllability for a class of control systems

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Analisi funzionale. — *Stabilization and controllability for a class of control systems* (*). Nota di LUCIANO PANDOLFI, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si considera un processo di controllo in uno spazio di Hilbert, descritto da una coppia $(\mathrm{E}(t,s), \mathrm{B}(t))$. $\mathrm{E}(t,s)$ è un operatore di evoluzione invertibile. Si prova che questo processo di controllo è stabilizzabile se e solo se esso è uniformemente controllabile.

1. INTRODUCTION

Let X be an Hilbert space. $\mathrm{E}(t,s)$ is defined to be an evolution operator on X if it is defined when $t \geq s \geq t_0$ for some $t_0 \geq -\infty$ and

i) $\mathrm{E}(t,t) = \mathrm{I} \quad \forall t \geq t_0$ (I is the identity operator on X).

ii) $\mathrm{E}(t,s)$ is a bounded operator on X and $(t,s) \to \mathrm{E}(t,s)$ is strongly continuous for $t \geq s \geq t_0$.

iii) $\mathrm{E}(t,s)\,\mathrm{E}(s,r) = \mathrm{E}(t,r) \qquad t \geq s \geq r \geq t_0$.

We focus our attention on those evolution operators which have the following properties:

a) $\mathrm{E}(t,s)$ is defined for every $t, s \quad t \geq t_0, s \geq t_0$.

b) $\mathrm{E}(t,s)\,\mathrm{E}(s,t) = \mathrm{I} \qquad t \geq t_0, s \geq t_0$.

c) There exist constants $\mathrm{M}, \alpha > 0$ such that

$$\|\mathrm{E}(t,s)\| < \mathrm{M}\exp(\alpha\,|t-s|).$$

Observe that property (*b*) is a consequence of the properties (*a*) and (iii).

In [4] $\mathrm{E}(t,s)$ is called exponentially stable (or simply stable) when we can find $t'_0, \alpha', \mathrm{M}'$ such that

$$\|\mathrm{E}(t,s)\| \leq \mathrm{M}'\exp\{-\alpha'(t-s)\} \qquad t \geq s \geq t'_0$$

and α' is positive.

Let U be another Hilbert space and let $\mathrm{B}(t) \in \mathscr{L}(\mathrm{U}, \mathrm{X})$ (the space of linear continuous operators from U in X), $\mathrm{F}(t) \in \mathscr{L}(\mathrm{X}, \mathrm{U})$ be strongly measurable and bounded on $[t_0, +\infty)$.

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With the evolution operator $E(t, s)$ we associate the evolution operator $E_F(t, s)$ which is the solution of the Volterra equation

$$E_F(t, s) = E(t, s) + \int_s^t E(t, r) B(r) F(r) E_F(r, s) dr \tag{1}$$

$E_F(t, s)$ satisfies conditions (*a*), (*b*), (*c*).

The function $F(t)$ will be called a feedback. We say that the couple $(E(t, s), B(t))$ is *stabilizable* if we can find a feedback $F(t)$ such that $E_F(t, s)$ is stable.

We say that $(E(t, s), B(t))$ is *fully stabilizable* if $(\exp(\mu(t - s)) E(t, s) B(t))$ is stabilizable for each $\mu > 0$.

If P, Q are linear operators on X which are symmetric and positive, we say that $P \geq Q$ when $P - Q$ is (symmetric) and positive ([3]).

From [4] we recall the following result:

THEOREM 1. *Let $t \to Q(t)$ be a function from a half line $[t_0, +\infty)$ in $\mathscr{L}(X) = \mathscr{L}(X, X)$, which is bounded and strongly measurable. Assume that there exists a constant $c > 0$ such that for each $t \geq t_0$, $Q(t) \geq cI$ (I is the identity operator on X). Then $E(t, s)$ is stable if and only if the equation*

$$L(\tau) = E(t, \tau)^* L(t) E(t, \tau) + \int_\tau^t E(s, \tau)^* Q(s) E(s, \tau) ds \tag{2}$$

has a positive symmetric solution $L(t)$ which is bounded for $t \geq t_0$.

The couple $(E(t, s), B(t))$ is stabilizable if and only if the equation

$$\begin{aligned} P(\tau) &= E(t, \tau)^* P(t) E(t, \tau) + \\ &+ \int_\tau^t E(s, \tau)^* [I - P(s) B(s) B^*(s) P(s)] E(s, \tau) ds \end{aligned} \tag{3}$$

has a positive symmetric solution which is bounded for $t \geq t_0$.

Now we want to prove that full stabilization of $(E(t, s), B(t))$ is related to the following property of the pair $(E(t, s), B(t))$:

We say that the pair $(E(t, s), B(t))$ is *uniformly zero controllable* if we can find $\gamma > 0$, $\sigma > 0$ such that

$$\gamma I \leq \int_t^{t+\sigma} E(t, s) B(s) B^*(s) E^*(t, s) ds = H(t, t + \sigma) \tag{4}$$

for each t, at least if $t \geq t_0'$, for some $t_0' \geq -\infty$.

The main result of this paper is the following theorem:

THEOREM 2. *Under the hypothesis a, b, c, the couple $(E(t, s), B(t))$ is fully stabilizable if and only if it is uniformly zero controllable.*

Remark. If X is a finite dimensional space, this result is well known and it has been proved by R. Kalman. A proof may be found in [1]. If $E(t,s) = E(t-s)$ and if $B(t) = B$ (a constant operator), see [6], [7].

The next section is devoted to the proof of Theorem 2.

2. The Proof of Theorem 2

The proof that we are going to give is a generalization of the proof found in [1] for finite dimensional spaces.

We need the following lemmas:

LEMMA 1. *Let* $E(t,s)$ *be an evolution operator which satisfies conditions a, b, c. Let* $P(t)$ *be a positive symmetric solution of Eq.* (3) *for* $t \geq t_0$.

Then $P^{-1}(t)$ *exists in* $\mathscr{L}(X)$ *and* $t \to P^{-1}(t)$ *is bounded on* $[t_0, +\infty)$.

Proof. Let $E_\infty(t,\tau) = E_F(t,\tau)$ with $F(t) = -B^*(t)P(t)$.

Then we can prove that

$$P(\tau) = E_\infty^*(t,\tau)\,P(t)\,E_\infty(t,\tau) + \\ + \int_\tau^t E_\infty^*(s,\tau)\,[P(s)\,B(s)\,B^*(s)\,P(s) + I]\,E_\infty(s,\tau)\,ds$$

([2]), so that, by condition c,

$$\langle x, P(\tau)\,x\rangle \geq \int_\tau^t \|E_\infty(s,\tau)\,x\|^2\,ds \geq \int_\tau^t \|E_\infty(\tau,s)\|^{-2}\,\|x\|^2\,ds\,.$$

Since:

$$\|E_\infty(\tau,s)\| \geq M' \exp[\alpha'(\tau-s)]$$

(it is not restrictive to assume that $\alpha' > 0$), then

$$\langle x, P(\tau)\,x\rangle \geq M' \int_\tau^t \exp[\alpha'(\tau-s)]\,ds\,\|x\|^2 \geq \tfrac{1}{2}(M'/\alpha')\,\|x\|^2\,,$$

at least for large t. This completes the proof of the Lemma.

LEMMA 2. *If* $P(t)$ *is a positive symmetric solution of Eq.* (3), *then* $P^{-1}(t)$ *is a solution of the equation*

$$P^{-1}(\tau) = E(\tau,t)\,P^{-1}(t)\,E^*(\tau,t) + \\ + \int_\tau^t E(\tau,s)\,[B(s)\,B^*(s) - P^{-2}(s)]\,E^*(\tau,s)\,ds\,.$$

Proof. From Eq. (3) it is clear that

$$t \to E^*(t, \tau)\, P(t)\, E(t, \tau)$$

is differentiable. We have

$$\frac{d}{dt} E^*(t, \tau)\, P(t)\, E(t, \tau) = E^*(t, \tau)\, [P(t)\, B(t)\, B^*(t)\, P(t) - I]\, E(t, \tau)$$

so that

$$(E^*(t, \tau)\, P(t)\, E(t, \tau))^{-1} \frac{d}{dt} (E^*(t, \tau)\, P(t)\, E(t, \tau))\, (E^*(t, \tau)\, P(t)\, E(t, \tau))^{-1} =$$
$$= E(\tau, t)\, [B(t)\, B^*(t) - P^{-2}(t)]\, E^*(\tau, t)\,.$$

Equation (6) follows by integrating both members of this equality.

Now we can prove that full stabilizability implies uniform controllability.

We assume that $E(t, \tau) \exp(\mu(t - \tau))$ is stabilizable for every μ.

Then from Theorem 1 and Lemma 2 we can find a solution $Z_\mu(t)$ of the equation

$$- \exp(2\mu(\tau - t))\, E(\tau, t)\, Z_\mu(t)\, E^*(\tau, t) + Z_\mu(\tau) =$$
$$= \int_\tau^t \exp(\mu(\tau - s))\, E(\tau, s)\, [B(s)\, B^*(s) - Z_\mu^2(s)]\, E^*(\tau, s) \exp(\mu(\tau - s))\, ds$$

at least if $t \geq t_0$, for some $t_0 \geq -\infty$ which does not depend on μ. Then $\| Z_\mu(t) \| \leq \gamma_\mu$.

Now we can write

$$- \exp(2\mu(\tau - t))\, E(\tau, t)\, Z_\mu(t)\, E^*(\tau, t) + Z_\mu(\tau) \leq$$
$$\leq \int_\tau^t \exp(\mu(\tau - s))\, E(\tau, s)\, B(s)\, B^*(s)\, E^*(\tau, s) \exp(\mu(\tau - s))\, ds \leq$$
$$\leq \int_\tau^t E(\tau, s)\, B(s)\, B^*(s)\, E^*(\tau, s)\, ds$$

i.e. $\quad Z_\mu(\tau) \leq H(\tau, t) + \exp(2\mu(\tau - t))\, E(\tau, t)\, Z_\mu(t)\, E^*(\tau, t)\,.$

Assume now that $(E(t, s), B(t))$ is not uniformly controllable.

Then for each $\sigma > 0$, $\rho > 0$ and some $\tau, y, y \in X$ (depending on σ, ρ) we have

$$\langle Z_\mu(\tau)\, y, y \rangle \leq \langle H(\tau, \tau + \sigma)\, y, y \rangle +$$
$$+ \exp(-2\mu\sigma)\, \langle E(\tau, \tau + \sigma)\, Z_\mu(\tau + \sigma)\, E^*(\tau, \tau + \sigma)\, y, y \rangle\,.$$

If $M_\mu = \sup \| Z_\mu(t) \|$ and $P_\mu(\tau)\, x = y$ ($\| P_\mu(\tau) \| \leq \overline{M}_\mu$ for some constant $\overline{M}_\mu$) we have

$$\langle x, P_\mu(\tau)\, x \rangle \leq \overline{M}_\mu\, [\rho + M^2 \exp(-2\mu\sigma)\, M_\mu \exp(2\alpha\sigma)]\, \| x \|^2,$$

because $P_\mu(\tau)^{-1} = Z_\mu(\tau)$. Let $\mu > 0$ be fixed. If $\rho \to 0$, $\sigma \to +\infty$ we find that for $\varepsilon > 0$ there is τ_ε such that $\| P_\mu(\tau_\varepsilon) \| < \varepsilon$, which contradicts Lemma 1.

This proves the first part of the Theorem.

Now we prove the sufficiency part. We want to prove that for every $\mu > 0$ we can stabilize $(\exp(\mu(t-s)) E(t,s), B(t))$.

We begin to show that if $(E(t,s), B(t))$ is uniformly zero controllable, then also $(\exp(\mu(t-s)) E(t,s), B(t))$ has the same property. This is shown by the following inequality:

$$\int_t^{t+\sigma} \exp(\mu(t-s)) E(t,s) B(s) B^*(s) E^*(t,s) \exp(\mu(t-s))\, ds \geq$$
$$\geq \exp(-2\mu\sigma) H(t, t+\sigma) \geq \exp(-2\mu\sigma)\gamma I .$$

Now we define the operator $H_\mu(t, t+\sigma)$ by:

$$H_\mu(t, t+\sigma) = \int_t^{t+\sigma} \exp(2\mu(t-s)) E(t,s) B(s) B^*(s) E^*(t,s) \exp(2\mu(t-s))\, ds .$$

Then for some $\sigma > 0$ we can find $h > 0$ which satisfies $hI < H_\mu(t, t+\sigma)$ $t \geq t_0$.

Now we consider the function $L(t) = H^{-1}(t, t+\sigma)$. Clearly

$$\| L(t) \| \leq 1/h$$

for each t.

We prove that $L(t)$ is a bounded solution of Eq. (2) written for $(\exp(\mu(t-s)) E(t,s))_{-B^*L}$ with

$$Q(t) = E(s, s+\sigma) B(s+\sigma) B^*(s+\sigma) E^*(s, s+\sigma) \exp(-2\mu\sigma) +$$
$$+ 2\mu H_\mu(s, s+\sigma) + B(s) B^*(s) \geq 2\mu H_\mu(s, s+\sigma) .$$

For simplicity we call $E_0(t,s)$ the evolution operator $\exp(\mu(t-s)) E(t,s)$. So we can write

$$[E_0^*(t,\tau) L(t) E(t,\tau)]^{-1} =$$
$$E(\tau, t) \exp(\mu(\tau - t)) \int_t^{t+\sigma} \exp(2\mu(t-s)) E(t,s) B(s) B^*(s) E^*(t,s) \cdot$$
$$\cdot \exp(2\mu(t-s))\, ds\, E^*(\tau, t) \exp(\mu(\tau - t)) =$$
$$= \int_t^{t+\sigma} \exp(\mu(\tau - s)) E(\tau, s) B(s) B^*(s) E(\tau, s) \cdot$$
$$\cdot \exp(\mu(\tau - s)) \exp(2\mu(t-s))\, ds .$$

Taking the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}\,[\mathrm{E}_0^*(t,\tau)\,\mathrm{L}(t)\,\mathrm{E}_0(t,\tau)]^{-1} =$$

$$\mathrm{E}(\tau,t+\sigma)\exp(2\mu(\tau-t-\sigma))\,\mathrm{B}(t+\sigma)\,\mathrm{B}^*(t+\sigma)\,\mathrm{E}^*(\tau,t+\sigma)\cdot$$

$$\cdot\exp(-2\mu\sigma)-\mathrm{E}(\tau,t)\exp(2\mu(\tau-t))\,\mathrm{B}(t)\,\mathrm{B}^*(t)\,\mathrm{E}^*(\tau,t)+$$

$$+2\mu\int_t^{t+\sigma}\mathrm{E}(\tau,s)\exp(\mu(\tau-s))\,\mathrm{B}(s)\,\mathrm{B}^*(s)\,\mathrm{E}^*(\tau,s)\cdot$$

$$\cdot\exp(\mu(\tau-s))\exp(2\mu(t-s))\,\mathrm{d}s$$

so that, as in the proof of Lemma 1,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,[\mathrm{E}_0^*(t,\tau)\,\mathrm{L}(t)\,\mathrm{E}_0(t,\tau)] = -[\mathrm{E}_0^*(t,\tau)\,\mathrm{L}(t)\,\mathrm{E}_0(t,\tau)]\cdot$$

$$\cdot\{\exp(2\mu(\tau-t-\sigma)\,\mathrm{E}(\tau,t+\sigma)\,\mathrm{B}(t+\sigma)\,\mathrm{B}^*(t+\sigma)\,\mathrm{E}^*(\tau,t+\sigma)\cdot$$

$$\cdot\exp(-2\mu\sigma)-\mathrm{E}(\tau,t)\,\mathrm{B}(t)\,\mathrm{B}^*(t)\,\mathrm{E}^*(\tau,t)\exp(2\mu(\tau-t))+$$

$$+2\mu\int_t^{t+\sigma}\mathrm{E}_0(\tau,s)\,\mathrm{B}(s)\,\mathrm{B}^*(s)\,\mathrm{E}_0^*(\tau,s)\exp(2\mu(t-s))\,\mathrm{d}s\cdot$$

$$\mathrm{E}_0^*(t,\tau)\,\mathrm{L}(t)\,\mathrm{E}_0(t,\tau),$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\,[\mathrm{E}_0^*(t,\tau)\,\mathrm{L}(t)\,\mathrm{E}_0(t,\tau)] = \mathrm{E}_0^*(t,\tau)\,\mathrm{L}(t)\,\{\mathrm{B}(t)\,\mathrm{B}^*(t)-$$

$$-\exp(-4\mu\sigma)\,\mathrm{E}_0(t,t+\sigma)\,\mathrm{B}(t+\sigma)\,\mathrm{B}^*(t+\sigma)\,\mathrm{E}^*(t,t+\sigma)-$$

$$-\mathrm{L}^{-1}(t)\}\,\mathrm{L}(t)\,\mathrm{E}_0(t,\tau).$$

Taking the primitives we have

$$\begin{aligned}&\mathrm{L}(\tau)=\mathrm{E}_0^*(t,\tau)\,\mathrm{L}(t)\,\mathrm{E}_0(t,\tau)+\int_\tau^t\mathrm{E}_0(s,\tau)\,\mathrm{L}(s)\cdot\\ &\cdot\{-\exp(-4\mu\sigma)\,\mathrm{E}_0(s,s+\sigma)\,\mathrm{B}(s+\sigma)\,\mathrm{B}^*(s+\sigma)\,\mathrm{E}_0^*(s,s+\sigma)+\\ &+\mathrm{B}(s)\,\mathrm{B}^*(s)-2\mu\,\mathrm{L}^{-1}(s)\}\,\mathrm{L}(s)\,\mathrm{E}_0(s,\tau)\,\mathrm{d}s.\end{aligned}\tag{7}$$

Now it is easy to prove that $\overline{\mathrm{E}}(t,s)$, the evolution operator solution of the Volterra equation

$$\overline{\mathrm{E}}(t,s)=\mathrm{E}_0(t,s)+\int_s^t\mathrm{E}_0(t,r)\,[-\mathrm{B}(r)\,\mathrm{B}^*(r)\,\mathrm{L}(r)]\,\overline{\mathrm{E}}(r,s)\,\mathrm{d}s$$

is stable.

We observe that (7) may be written as

$$\mathrm{L}(\tau) = \overline{\mathrm{E}}^*(t, \tau)\,\mathrm{L}(t)\,\overline{\mathrm{E}}(t, \tau) + \int_{\tau}^{t} \overline{\mathrm{E}}(s, \tau)^*\,\mathrm{L}(s)\,\{\mathrm{B}(s)\,\mathrm{B}^*(s) +$$

$$+ 2\,\mathrm{L}^{-1}(s) + \exp(-4\mu\sigma)\,\mathrm{E}_0(s, s+\sigma)\,\mathrm{B}(s+\sigma)\,\mathrm{B}^*(s+\sigma)\,\mathrm{E}_0^*(s, s+\sigma)\}\,\cdot$$

$$\cdot\,\mathrm{L}(s)\,\overline{\mathrm{E}}(s, \tau)\,\mathrm{d}s$$

as a consequence of Lemma 1 in [2] (see also [5]).

This concludes the proof of Theorem 2.

Remark: Observe that the sufficiency part of Theorem 2 has been proved without using hypothesis *c*.

REFERENCES

[1] R. CONTI (1977) – *Linear Differential Equations and Control, Institutiones Mathematicae*, Vol. 1, Academic Press, New York.

[2] R. CURTAIN (1975) – *The Infinite Dimensional Riccati Equation with Application to Affine Hereditary Differential Systems* « SIAM J. Control », *13*, 48–88.

[3] A. N. KOLMOGOROF and S. V. FOMIN (1974) – *Elements de la Théorie des fonctions et de l'Analyse Fonctionelles*, Editions MIR, Moscow.

[4] L. PANDOLFI – *Stabilization of Control Processes in Hilbert Spaces*, To appear on « Roy. Math. Soc. of Edinburg », ser. A.

[5] L. PANDOLFI – *Non Autonomous Regulator Problem in Hilbert Spaces*, To appear, « J. Opt. Theory Appl. ».

[6] J. ZABCZYK (1976) – *Complete Stabilizability Implies Exact Controllability*, *Seminarul de Ecuatii Functionale*, Universitatea din Timisoara, Romania.

[7] M. SLEMROD (1974) – *A note ou complete controllability and stabilizability for linear control systems in Hilbert space*, « SIAM J. Control », *12*, 500–508.