Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Ivan Erdelyi

A class of weakly decomposable unbounded operators

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **64** (1978), n.2, p. 112–117.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1978_8_64_2_112_0>

Analisi matematica. — *A class of weakly decomposable unbounded operators.* Nota di IVAN ERDELYI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si estendono dei risultati ottenuti in una precedente Nota [4] sulla decomposizione spettrale di una classe di operatori non-limitati. Si ottiene una caratterizzazione per certi sottospazi invarianti e per un operatore con lo spettro compatto risulta una decomposizione spettrale di tipo debole.

The class of decomposable operators, introduced by Foias [5] and developed in the monograph [2], possesses a rich spectral theory. In a previous paper [4], the class of decomposable operators has been extended to the unbounded case. We shall draw heavily from [4] to obtain some further spectral properties for that class of unbounded operators.

For a Banach space X, $\mathbf{S}(X)$ denotes the collection of all subspaces (closed linear manifolds) of X. For an operator T, by which is meant a closed linear mapping with domain $D_T \subset X$ into X, $\sigma(T)$ expresses the spectrum and the restriction of T to a subspace $Y \subset D_T$ is written as $T \mid Y$. For the definitions of the local spectrum $\sigma_T(x)$ and local resolvent set $\rho_T(x)$ of an operator T having the single valued extension property, see e.g. [3, XV]. As referred to the complex plane π, $\mathbf{F}$ and $\mathbf{K}$ represent the family of all closed and of all compact subsets of π, respectively.

We recall the definitions and the basic properties which are needed to explain the new results.

1. DEFINITION [4]. *A strong spectral capacity in* X *is an application*

$$E : \mathbf{F} \to \mathbf{S}(X)$$

that satisfies the following conditions:

(i) $E(\emptyset) = \{0\}$, $E(\pi) = X$;

(ii) $E\left(\bigcap_{n=1}^{\infty} F_n\right) = \bigcap_{n=1}^{\infty} E(F_n)$, *for every* $\{F_n\} \subset \mathbf{F}$;

(iii) $E(F) = \sum_{i=1}^{n} E(F \cap \overline{G}_i)$, *for every* $F \in \mathbf{F}$ *and every finite open cover* $\{G_i\}_1^n$ of F;

(iv) *for every* $F \in \mathbf{F}$, *the linear manifold*

$$E_0(F) = \{x \in E(K) : K \in \mathbf{K} \ \textit{and} \ K \subset F\}$$

is dense in E (F).

(*) Nella seduta dell'11 febbraio 1978.

For $F = \pi$, (iii) becomes

(iii′) $X = \sum_{i=1}^{n} E(\overline{G}_i)$,

where $\{G_i\}_1^n$ is an arbitrary finite open cover of π.

Also, for $F = \pi$, (iv) asserts that

(iv′) $E_0(\pi) = \{x \in E(K) : K \in \mathbf{K}\}$

is dense in X.

2. DEFINITION [4]. *A closed linear operator* $T : D_T (\subset X) \to X$ *with a nonvoid resolvent set is said to have a strong spectral capacity* E *if the following conditions hold*:

(v) $E(K) \subset D_T$, *for all* $K \in \mathbf{K}$;

(vi) $T[E(F) \cap D_T] \subset E(F)$, *for all* $F \in \mathbf{F}$

(vii) *the restriction* $T_F = T \mid E(F) \cap D_T$ *has the spectrum*

$$\sigma(T_F) \subset F, \quad \textit{for each} \quad F \in \mathbf{F}.$$

The following two definitions introduce some special type of invariant subspaces and a related spectral decomposition.

3. DEFINITION [5]. *A subspace* $Y \subset D_T$ *invariant under* T *is said to be spectral maximal for* T *if any invariant subspace* Z *with* $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ *is contained in* Y.

T *is said to be decomposable on* X *if for every finite open cover* $\{G_i\}_1^n$ *of* $\sigma(T)$, *there is a system* $\{Y_i\}_1^n$ *of spectral maximal spaces of* T *which perform the spectral decomposition*

$$X = \sum_{i=1}^{n} Y_i; \tag{1}$$

$$\sigma(T \mid Y_i) \subset G_i, (\text{or}, \ \sigma(T \mid Y_i) \subset \overline{G}_i), \qquad i = 1, 2, \cdots, n. \tag{2}$$

If (1) is weakened by

$$X = \overline{\sum_{i=1}^{n} Y_i}$$

i.e. if every vector in X is a norm-limit of sums of vectors from the spectral maximal spaces Y_i then T is referred to as a *weakly decomposable* operator on X.

4. DEFINITION [7]. *A subspace* $Y \subset D_T$, *invariant under* T, *is called analytically invariant under* T *if for every function* $f : D_f \to D_T$ *analytic on some open* $D_f \subset \pi$, *the condition*

$$(\lambda - T) f(\lambda) \in Y \quad \text{on } D_f$$

implies that $f(\lambda) \in Y$ *on* D_f.

For a bounded linear operator T which has the single valued extension property, every spectral maximal space is analytically invariant [7]. The opposite implication does not hold in general.

If T has a strong spectral capacity E then (iv′) and (v) imply that the domain D_T is dense in X. It was proved in [4] that T at most has one strong spectral capacity. As a global type of characterization, an operator T with a strong spectral capacity has the single valued extension property [4]. Among the "locally compact" properties of such an operator, we recall from [4] that for every $K \in \mathbf{K}$, $E(K)$ is a spectral maximal space and $T \mid E(K)$ is a bounded decomposable operator on $E(K)$.

In what will follow we shall need the following

5. LEMMA [6, IV]. *Let* Y_1, Y_2 *be subspaces of* X *such that*

$$X = Y_1 + Y_2$$

and let $f : D_f \to X$ *be analytic on an open* $D_f \subset \pi$. *Then for every* $\lambda \in D_f$ *there is a neighborhood* $V (\subset D_f)$ *of* λ *and there are analytic functions*

$$f_i : V \to Y_i, \qquad i = 1, 2$$

such that

$$f(\mu) = f_1(\mu) + f_2(\mu), \qquad \mu \in V.$$

Now we give a characterization of some invariant subspaces $E(F)$ for F not necessarily compact.

6. THEOREM. *Given* T *with the strong spectral capacity* E, *let* $G \subset \pi$ *be open such that* $E(\overline{G}) \subset D_T$. *Then* $E(\overline{G})$ *is an analytically invariant subspace under* T.

Proof. Denote $G_1 = G$ and let $f : D_f (\subset G_1) \to D_T$ be analytic and satisfy condition

$$(\lambda - T) f(\lambda) \in E(\overline{G}_1) \quad \text{on } D_f. \tag{3}$$

Choose an open G_2 such that

$$\pi = G_1 \cup G_2 \quad \text{and} \quad D_f \cap \overline{G}_2 = \varnothing.$$

By (iii′) of Definition 1,

$$X = E(\overline{G}_1) + E(\overline{G}_2).$$

In view of Lemma 5, for every $\lambda \in D_f$ there is a neighborhood $V (\subset D_f)$ of λ and there are analytic functions

$$f_i : V \to E(\overline{G}_i), \qquad i = 1, 2$$

such that

$$f(\mu) = f_1(\mu) + f_2(\mu), \quad \text{for all } \mu \in V. \tag{4}$$

Since the ranges of both f and f_1 are contained in D_T, we have

$$f_2(V) \subset E(\overline{G}_2) \cap D_T.$$

By condition (3),

$$(\mu - T) f(\mu) = x_\mu \tag{5}$$

where for every $\mu \in V$, $x_\mu \in E(\overline{G}_1)$. With the help of (4), relation (5) can be written as

$$x_\mu - (\mu - T) f_1(\mu) = (\mu - T) f_2(\mu) \in E(\overline{G}_1) \cap E(\overline{G}_2) = E(\overline{G}_1 \cap \overline{G}_2).$$

Therefore

$$g(\mu) = (\mu - T) f_2(\mu) \in E(\overline{G}_1 \cap \overline{G}_2) \qquad \text{on } V.$$

Since

$$\sigma[T \mid E(\overline{G}_1 \cap \overline{G}_2)] \subset \overline{G}_1 \cap \overline{G}_2 \subset \overline{G}_2,$$

$\mu \in \rho[T \mid E(\overline{G}_1 \cap \overline{G}_2)]$ Hence the function

$$h(\mu) = R[\mu; T \mid E(\overline{G}_1 \cap \overline{G}_2)] g(\mu) \in E(\overline{G}_1 \cap \overline{G}_2)$$

is analytic on V and

$$(\mu - T)[h(\mu) - f_2(\mu)] = 0 \qquad \text{on } V.$$

Since T has the single valued extension property it follows that

$$h(\mu) = f_2(\mu) \in E(\overline{G}_1 \cap \overline{G}_2) \subset E(\overline{G}_1) \qquad \text{on } V$$

and hence

$$f(\mu) = f_1(\mu) + f_2(\mu) \in E(\overline{G}_1) \qquad \text{on } V$$

and on all of D_f by analytic continuation. Thus $E(\overline{G}) = E(\overline{G}_1)$ is analytically invariant. □

It was shown [4] that every operator T with a strong spectral capacity E has decomposable restrictions $T \mid E(K)$, for all $K \in \mathbf{K}$. When T has a compact spectrum we can obtain a spectral decomposition for T itself. This, however, needs some preparation. First we extend a concept introduced by Apostol [1]. With the original concept of spectral capacity E, Apostol defined the support of E by

$$\operatorname{supp} E = \cap \{F \in \mathbf{F} : E(F) = X\}.$$

7. LEMMA. *Let* T *possess the strong spectral capacity* E. *If* G *is any open set such that* $G \cap \operatorname{supp} E \neq \varnothing$ *then* $E(\overline{G}) \neq \{0\}$.

Proof. There exists a second open set H such that G and H cover π and

$$\overline{H} - \operatorname{supp} E \neq \varnothing.$$

Then, by (iii′) of Definition 1,

$$X = E(\overline{G}) + E(\overline{H}).$$

If $E(\overline{G}) = \{0\}$, then $E(\overline{H}) = X$ and consequently

$$X = E(\operatorname{supp} E \cap \overline{H}). \tag{6}$$

But (6) contradicts the definition of supp E. Thus $E(\overline{G}) \neq \{0\}$. □

8. LEMMA. *If* T *has a strong spectral capacity* E *then*

$$X = E[\sigma(T)].$$

Proof. We have to show that supp $E \subset \sigma(T)$. Assume to the contrary that there is a $\lambda \in \operatorname{supp} E - \sigma(T)$. Then there is a closed disk F with center at λ and disjoint from $\sigma(T)$. By Lemma 7, $E(F) \neq \{0\}$ and then by (vii) Definition 2,

$$\sigma[T \mid E(F) \cap D_T] \cap \sigma(T) \subset F \cap \sigma(T) = \varnothing$$

but this is impossible. Hence supp $E \subset \sigma(T)$ and it follows that

$$X = E(\operatorname{supp} E) \subset E[\sigma(T)] \subset X. \quad \square$$

9. PROPOSITION. *If* T *has a strong spectral capacity* E *then*

$$\sigma[T \mid E(F) \cap D_T] \subset \sigma(T), \qquad \text{for all} \quad F \in \mathbf{F}. \tag{7}$$

Proof. Lemma 8, (ii) Definition 1 and (vii) Definition 2 imply

$$\sigma[T \mid E(F) \cap D_T] = \sigma[T \mid E(F) \cap E(\sigma(T)) \cap D_T] =$$
$$= \sigma[T \mid E(F \cap \sigma(T)) \cap D_T] \subset F \cap \sigma(T),$$

and hence property (7) follows. □

Now all pieces are assembled for the spectral decomposition of T.

10. THEOREM. *If* T *has a strong spectral capacity and* $\infty \in \rho(T)$, *then* T *is weakly decomposable.*

Proof. Let $\{G_i\}_1^n$ be a finite open cover of $\sigma(T)$. Since by hypothesis, $\sigma(T)$ is compact, there is a relatively compact open neighborhood H of $\sigma(T)$. The sets

$$H_i = H \cap G_i, \qquad i = 1, 2, \cdots, n$$

form a relatively compact open cover of $\sigma(T)$. Let H_0 be an open set such that $\{H_i\}_0^n$ covers π and $\overline{H}_0 \cap \sigma(T) = \varnothing$. Let E be the strong spectral capacity possessed by T. By (vii) and Proposition 9, we have

$$\sigma[T \mid E(\overline{H}_0) \cap D_T] \subset \overline{H}_0 \cap \sigma(T) = \varnothing$$

and hence

$$E(\overline{H}_0) \cap D_T = \{0\} . \tag{8}$$

By (iii′) Definition 1, we have

$$X = \sum_{i=0}^{n} E(\overline{H}_i) . \tag{9}$$

Relations (8), (9) imply

$$D_T \subset \sum_{i=1}^{n} E(\overline{H}_i) ,$$

and since D_T is dense in X, we have

$$X = \overline{\sum_{i=1}^{n} E(\overline{H}_i)} .$$

Furthermore, (vii) Definition 2 implies

$$\sigma [T \mid E(\overline{H}_i)] \subset \overline{H}_i \subset \overline{G}_i , \qquad i = 1, 2, \cdots, n .$$

Since, for every i, $\overline{H}_i$ is compact, $E(\overline{H}_i)$ is spectral maximal for T. □

REFERENCES

[1] C. APOSTOL (1968) – *Spectral decompositions and functional calculus*, « Rev. Roum. Math. Pures Appl. », *13*, 1481–1528.

[2] I. COLOJOARA and C. FOIAS (1968) – *Theory of generalized spectral operators*, Gordon and Breach, New York.

[3] N. DUNFORD and J. T. SCHWARTZ (1971) – *Linear operators*, Part. III, Wiley, New York.

[4] I. ERDELYI (1975) – *Unbounded operators with spectral capacities*, « J. Math. Anal. Appl. », *52*, 404–414.

[5] C. FOIAS (1963) – *Spectral maximal spaces and decomposable operators*, « Arch. Math. (Basel) », *14*, 341–349.

[6] C. FOIAS (1968) – *Spectral capacities and decomposable operators*, « Rev. Roum. Math. Pures Appl. », *13*, 1539–1545.

[7] S. FRUNZA (1973) – *The single-valued extension property for coinduced operators*, « Rev. Roum. Math. Pures Appl. », *18*, 1061–1065.