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On the ordinal and absolute stability properties

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Fisica matematica. — *On the ordinal and absolute stability properties* (*). Nota di FRANCESCA VISENTIN (**), presentata (***) dal Socio C. CATTANEO.

RIASSUNTO. — Si discutono e si ampliano alcuni concetti, introdotti da T. Ura, concernenti la stabilità di un insieme compatto rispetto ad un sistema dinamico ordinario. In particolare la nozione di stabilità assoluta viene estesa a proprietà che, come ad esempio la stabilità asintotica, non sono definibili mediante i « prolungamenti » di Ura.

1. Let E be a locally compact metric space and π a dynamical system defined on E. It is well-known that T. Ura [5] has given some concepts concerning the " degree " of a certain stability property up to a compact set $M \subset E$. In the work of T. Ura these properties are defined by means of so called prolongations which are suitable compositions of maps from E on 2^E. T. Ura, and successively J. Auslander and P. Seibert [1], individuate some properties of a compact set M with respect to a dynamical system, which can occur to several degrees. If M possesses one of these properties at any degree, then M is said to have such property absolutely. Characterizations of these behaviors are given by T. Ura [6] and J. Auslander and P. Seibert [1], always by means of prolongations.

In this paper we give a generalization of the concept of a property possessed absolutely by a compact set M in such a way to include properties which are not necessarily defined by prolongations. One of these properties is the asymptotic stability. Properties possessed at a given order can be analogously defined; but they cannot be in general reduced to the well-known ones defined by prolongations. At last we individuate some properties which have intrinsic absolute character, in the sense that if any compact set has such a property, then it has the same property for each degree. For example total stability, as defined in [1], and again asymptotic stability are intrinsic absolute properties.

2. Let E be a locally compact metric space and ρ the distance in E. Let $\mathbf{R}$ be the set of real numbers, and $\{\mathbf{R}, E, \pi\}$ a dynamical system from $\mathbf{R} \times E$ on E (see [2]), which we denote by π. For each $x \in E$ let $\gamma^+(x)$ be the positive π-semitrajectory through x. We consider the set $\mathcal{A}(E)$ of all mappings from E on 2^E; for a map $\Gamma \in \mathcal{A}(E)$ and for a non-empty subset U of E, we set $\Gamma(U) = \cup \{\Gamma(x) : x \in U\}$. Further, for $\varepsilon > 0$ we put $S(U, \varepsilon) = \{x \in E : \rho(x, U) < \varepsilon\}$. We shall denote by Σ the set of all

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dynamical systems defined in $\mathbf{R}\times E$, and if $E = \mathbf{R}^n$, we denote by $\mathcal{G}$ the set of the functions from $\mathbf{R}\times\mathbf{R}^n$ on $\mathbf{R}^n$ satisfying Caratheodory conditions. Let also $\mathcal{G}^*$ the set consisting of all the continuous functions from $\mathbf{R}^n$ on $\mathbf{R}^n$ such that for every $f \in \mathcal{G}^*$ uniqueness and global existence in $\mathbf{R}^n$ of solutions of equation

$$\dot{x} = f(x) \tag{2.1}$$

hold. For $x \in \mathbf{R}^n$ we denote by $\pi_f(\cdot, x)$ the solution of (2.1) such that $\pi_f(0, x) = x$. Then the triplet $\{\mathbf{R}, \mathbf{R}^n, \pi_f\}$ defines a dynamical system and we obtain a subset Σ^* of Σ. We need the following definition.

DEFINITION 2.1. Let $M \subset E$ be a compact set and π a dynamical system defined in $\mathbf{R}\times E$. M is said to be

2.1.1. π-stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in S(M, \delta)$ implies $\pi(t, x) \in S(M, \varepsilon)$ for any $t \in \mathbf{R}^+$.

2.1.2. π-asymptotically stable if M is π-stable and there exists $\sigma > 0$ such that, for $x \in S(M, \sigma)$, $\rho(\pi(t, x), M) \to 0$ as $t \to +\infty$.

2.1.3. (Σ, π)-totally stable if for any $\varepsilon, \bar{t} > 0$, there exists two numbers $\delta_1, \delta_2 > 0$ such that if $x \in S(M, \delta_1)$ and $p \in \Sigma$, with $\rho(\pi(t, z), p(t, z)) < \delta_2$ in $[0, \bar{t}]\times E$, then $p(t, x) \in S(M, \varepsilon)$ for any $t \in R^+$.

2.1.4. For $f \in \mathcal{G}^*$, $(\mathcal{G}, f)$-totally stable when for every $\varepsilon > 0$ there exist two numbers $\delta_1, \delta_2 > 0$ such that for any $t_0 \geq 0$, $x_0 \in S(M, \delta_1)$, $g \in \mathcal{G}$ with $\rho(f(z), g(t, z)) < \delta_2$ in $\mathbf{R}^+\times S(M, \varepsilon)$ then $x_g(t, t_0, x_0) \in S(M, \varepsilon)$ $\forall t \geq t_0$, where $x_g(t, t_0, x_0)$ is any solution of $\dot{x} = g(t, x)$ through (t_0, x_0).

It is possible to prove that Definition 2.1.4. is equivalent to that used by J. Auslander and P. Seibert in [1]. Further, when $f \in \mathcal{G}^*$ is locally Lipschitzian in x, one proves that if M is (Σ^*, π_f)-totally stable then M is $(\mathcal{G}^*, f)$-totally stable (see [3]).

DEFINITION 2.2. Let $Q \in \mathcal{A}(E)$ be a prolongation and, for any ordinal number α, let Q_α be the prolongation of order α constructed by means of Q (see [1]). Let $M \subset E$ be a compact set. M is said to be

2.2.1. Q-invariant if $Q(M) = M$.

2.2.2. Q-stable of order α, or Q_α-stable, if $Q_\alpha(M) = M$.

2.2.3. Q-absolutely stable if M is Q_α-stable for any ordinal number α.

We notice that there exists a prolongation D such that Liapunov stability coincides with D-stability of order 1 (see [1, 2, 5]). For Q_α-stability and Q-absolute stability of a compact set $M \subset E$ the following statements hold.

THEOREM 2.3 ([6]). *Let $Q \in \mathcal{A}(E)$ be a prolongation and let M be a compact subset of E. M is Q_α-stable if and only if there exists a fundamental system of neighborhoods $\{U_n\}_{n\in N}$ of M such that $Q_\beta(U_n) = U_n$ for any $\beta < \alpha$.*

THEOREM 2.4 ([1], Theorem 6). *Let* $Q \in \mathcal{A}(E)$ *be a prolongation and let* M *be a compact* Q*-invariant subset of* E. *Then the following conditions are equivalent*:

2.4.1. *There exist a* Q*-invariant neighborhood* W *of* M *and a continuous function* $V : W \to \mathbf{R}^+$ *satisfying*

(i) $V(x) = 0$ *for any* $x \in M$, $V(x) > 0$ *for any* $x \in W \setminus M$;

(ii) $V(y) \leq V(x)$ *for any* $x \in W$ *and* $y \in Q(x)$.

2.4.2. M *is* Q*-absolutely stable.*

2.4.3. M *possesses a fundamental system of* Q*-absolutely compact neighborhoods.*

3. Some Authors [1, 5, 6] have put the problem of establish the conditions under which a compact set possesses absolutely a property. This has been done by the use of ordinal prolongations. We remark that this approach to the question concernes necessarily properties relative to dynamical systems, because the existence of trajectories is necessary to construct a prolongation. In this setup J. Auslander and P. Seibert [1] prove that every compact set $(\mathcal{G}, f)$-totally stable possesses this property absolutely. On the other hand this statement seems not true for the (Σ, π)-total stability; however sufficient conditions in order this result are in [4].

Here we shall give a generalization of this concept so that one can include also properties which cannot be defined by means of prolongations (for instance asymptotic stability, see [1]).

Let E be a locally compact metric space and $\mathcal{C}$ the class of all compact subsets of E, and let $\mathcal{P}$ be a property defined in $\mathcal{C}$. If $M \in \mathcal{C}$ has the property $\mathcal{P}$ we write $M \wedge \mathcal{P}$. Let $\mathcal{H} = \mathcal{H}(M)$ be the class of subsets of 2^E defined in the following way. $\mathcal{F}$ is an element of $\mathcal{H}$ if $\mathcal{F}$ is a system of sets $\in \mathcal{C}$ consisting of: the set M; a fundamental system of compact neighborhoods $\{U_n\}_{n \in N}$ of M; a fundamental system of compact neighborhoods $\{U_{nm}\}_{m \in N}$ of every set U_n; a fundamental system of compact neighborhoods $\{U_{nms}\}_{s \in N}$ of every set U_{nm}; and so on (inductively).

DEFINITION 3.1. Let $\mathcal{P}$ be a property defined in $\mathcal{C}$. We say that $M \in \mathcal{C}$ has absolutely the property $\mathcal{P}$, or M is absolutely $\mathcal{P}$, if there exists $\mathcal{F} \in \mathcal{H}$ such that $\mathcal{P}$ holds for any $U \in \mathcal{F}$.

Remark 3.2. If a set M is absolutely $\mathcal{P}$, then the same happens for every set $U \in \mathcal{F}$.

DEFINITION 3.3. A property $\mathcal{P}$ defined in $\mathcal{C}$ derives from a prolongation Q if for $M \in \mathcal{C}$

$$M \wedge \mathcal{P} \Longleftrightarrow Q(M) = M .$$

For example stability, $(\mathcal{G}, f)$-total stability, (Σ, π)-total stability derive from a prolongation, but there exists no prolongation defining the asymptotic stability as it is proved in [1], and a similar proof shows that there exists no prolongation for instability. When a property $\mathscr{P}$ derives from a prolongation Q, we can establish the relation between Q-absolute stability of a compact set M and the condition that M is absolutely $\mathscr{P}$. Precisely the following theorem holds.

THEOREM 3.4. *Let $\mathscr{P}$ be a property deriving from a prolongation* Q. *A compact set* $M \subset E$ *is absolutely $\mathscr{P}$ if and only if* M *is Q-absolutely stable.*

Proof. (a) Suppose that M is absolutely $\mathscr{P}$; it suffices to prove that there exists a function V satisfying the conditions of Theorem 3.4. We can obtain this result using Remark 3.2 and similar arguments as in Theorem 3.4 (see [1]).

(b) Conversely, if M is a compact Q-absolutely stable set, then $Q(M) = M$; further M possesses a fundamental system of compact neighborhoods $\{U_n\}_{n \in N}$ such that $Q(U_n) = U_n$, and every set U_n possesses a fundamental system of compact neighborhoods $\{U_{nm}\}_{m \in N}$ with $Q(U_{nm}) = U_{nm}$, and so on; therefore M is absolutely $\mathscr{P}$.

Remark 3.5. If a property $\mathscr{P}$ derives from two prolongations Q and Q′, by Theorem 3.4 we have that M is Q-absolutely stable if and only if M is Q′-absolutely stable.

4. Relative to the question of the order of a property $\mathscr{P}$ for a compact set $M \subset E$, we give the following definition.

DEFINITION 4.1. Let $\mathscr{P}$ be a property defined in $\mathscr{C}$ and $M \in \mathscr{C}$. $M \wedge \mathscr{P}$ at order 2 (M is $\mathscr{P}$ at order 2), if M satisfies the property $\mathscr{P}$ and there exists a fundamental system of compact neighborhoods $\{U_n\}_{n \in N}$ with $U_n \wedge \mathscr{P}$, $n \in N$.

By using transfinite induction, it is possible to define the order $\alpha \geq 2$ (α ordinal number) of the property $\mathscr{P}$ for the set M.

DEFINITION 4.2. $M \wedge \mathscr{P}$ at order α ($\alpha \geq 2$) if $M \wedge \mathscr{P}$ and there exists a fundamental system of compact neighborhoods of M satisfying the property $\mathscr{P}$ at order β, for each ordinal number $\beta < \alpha$.

We observe that if M is absolutely $\mathscr{P}$, then $M \wedge \mathscr{P}$ at order α, for any ordinal number α. If the property $\mathscr{P}$ derives from a prolongation Q, we will establish the relation between Q_α-stability of a compact set M and the condition that $M \wedge \mathscr{P}$ at order α. The following result holds.

THEOREM 4.3. *Let $\mathscr{P}$ be a property deriving from a prolongation* Q *and* $M \in \mathscr{C}$. *If* $M \wedge \mathscr{P}$ *at order* α, *then* M *is Q-stable at the same order.*

Proof. We prove the theorem for $\alpha = 2$. If $M \wedge \mathscr{P}$ at order 2, there exists a fundamental system of compact neighborhoods $\{U_n\}_{n \in N}$ of M with the same property $\mathscr{P}$. On the other hand there exists a prolongation Q defining the property $\mathscr{P}$, then $Q(M) = M$ and $Q(U_n) = U_n$, $n \in N$; but in such a case $Q_2(M) = M$ (see Theorem 2.4), i.e. M is Q-stable at order 2. For $\alpha > 2$ the statement can be proved by transfinite induction.

Generally the converse statement is not true, because if M is Q-stable at order α, then for every $\beta < \alpha$ there exists a fundamental system of neighborhoods $\{U_n\}_{n \in N}$ of M with $Q_\beta(U_n) = U_n$, but these neighborhoods are not necessarily compact. The equivalence between the two conditions occurs under further requirements. An example is given below by a slight modification of a dynamical system already considered in [2], p. 125.

Example 4.4 ([2]). We consider a sequence of points of the real line $P_n \to 0$, where $P_1 > P_2 > \cdots > P_n > \cdots > 0$. For all points P_k of the sequence we introduce a sequence $P_k^n \to P_k$, where $P_k^1 > P_k^2 > \cdots > P_k^n > \cdots > P_k$. We consider a dynamical system on the real line such that 0, P_n, P_k^n $(n, k = 1, 2, \cdots)$ are equilibrium points, and the motions between any two successive equilibrium points and those on $\mathbf{R}^- \setminus \{0\}$ evolve from left to right. The set $\{0\}$ is stable. We prove that $\{0\}$ possesses a fundamental system of stable compact neighborhoods. Let us consider the neighborhoods $\{U_n\}_{n \in N}$, where $U_n = [-P_n, P_n]$. These are compact sets and constitue a fundamental system of neighborhoods of $\{0\}$. We prove that they are stable. Let U be a neighborhood of U_n, also U is a neighborhood of the points $-P_n$ and P_n, then there exists $m \in N$ such that $-P_n^m, P_n^m \in U$ and therefore $U_n \subset]-P_n^m, P_n^m[\subset [-P_n^m, P_n^m] \subset U$. On the other hand the motions are from left to right and P_n^m is an equilibrium point, then $\gamma^+([-P_n^m, P_n^m]) = [-P_n^m, P_n^m] \subset U$. This proves that every set U_n is stable. Then the set $\{0\}$ is Liapunov stable at order 2.

By introducing other sequences of equilibrium points, it is possible to consider a dynamical system such that the set $\{0\}$ is stable at order n, for any $n \in N$.

5. Among all the properties included in Definition 3.1. we individuate some properties $\mathscr{P}$ satisfying the following stronger condition: $M \wedge \mathscr{P}$ implies M is absolutely $\mathscr{P}$. These properties are said to be intrinsic absolute properties. That is we give the following definition.

DEFINITION 5.1. $\mathscr{P}$ is said to be an intrinsic absolute property when for every compact set M such that $M \wedge \mathscr{P}$, there exists a fundamental system of compact neighborhoods $\{U_n\}_{n \in N}$ of M for which $U_n \wedge \mathscr{P}$, $n \in N$.

Examples of intrinsic absolute properties are asymptotic stability, instability, $(\mathscr{G}, f)$-total stability. Other examples are the compactedness and, if E is a locally connected metric space, the connectedness. On the contrary Liapunov stability is not an instrinsic absolute property (see [1]) and it seems that also (Σ, π)-total stability is not an intrinsic absolute one (see [4]); but these properties can be possessed absolutely by a compact set M. In order to (Σ, π)-total stability we recall that the following statement holds.

THEOREM 5.1. ([4]). *Let* M *be a compact set. If* M *possesses a fundamental system of asymptotically stable compact neighborhoods* $\{U_n\}_{n \in N}$, *then* M *is* (Σ, π)-*absolutely totally stable.*

By virtue of Definition 3.1, Theorem 5.1 can be proved more easy than in [4]. In fact, since asymptotic stability is an absolute property, every set U_n possesses a fundamental system of asymptotically stable compact neighborhoods $\{U_{nm}\}_{m \in N}$, and so on. On the other hand, asymptotic stability implies (Σ, π)-total stability (see [3]), thus M is (Σ, π)-absolutely totally stable.

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