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A Random Fixed Point Theorem for Set-Valued Mappings

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Calcolo delle probabilità. — *A Random Fixed Point Theorem for Set-Valued Mappings.* Nota di SIMEON REICH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si studia l'esistenza di punti fissi stocastici di funzioni multivoche in spazi di Fréchet.

In a recent paper [1, p. 653] A. T. Bharucha-Reid has expressed an interest in probabilistic fixed point theorems for set-valued mappings. It is the purpose of this note to present such a theorem for condensing upper semicontinuous set-valued mappings in Fréchet spaces.

Let E be a separable Fréchet space with metric d, K a closed convex subset of E, C (E) the family of nonempty compact convex subsets of E, and T a complete measure space. Recall that mappings $F : K \to C(E)$ and $G : T \to C(E)$ are said to be upper semicontinuous and measurable respectively if $\{x \in K : F(x) \cap B \neq \varnothing\}$ is closed and $\{t \in T : G(t) \cap B \neq \varnothing\}$ is measurable for each closed subset B of E. For x in K let $I(K, x)$ denote the set $\{z \in E : z = x + a(y - x)$ for some $y \in K$ and $a \geq 0\}$. $F : K \to C(E)$ will be called weakly inward if $F(x)$ is contained in the closure of $I(K, x)$ for each $x \in K$. It will be said to satisfy the Leray-Schauder condition if there is a point w in the interior of K such that for every y in the boundary of K and $z \in F(y)$, $z - w \neq m(y - w)$ for all $m > 1$. Let clco (D) denote the closed convex hull of $D \subset E$.

THEOREM. *Let* K *be a closed convex subset of a separable Fréchet space* E, T *a complete measure space, and* $F : T \times K \to C(E)$. *Suppose that for each* $t \in T$, $F(t, x)$ *is condensing and upper semicontinuous, has bounded range, and either is weakly inward or satisfies the Leray-Schauder condition. If for each* $x \in K$, $F(t, x)$ *is measurable, then there is a measurable* $f : T \to K$ *such that* $f(t)$ *is a fixed point of* $F(t, x)$ *for each* $t \in T$.

Proof. Let A be a dense countable subset of K, and define

$$G : T \times K \to C(E) \text{ by } G(t, x) = \bigcap_{n=1}^{\infty} \operatorname{clco}\left(\cup \{F(t, a) : a \in A, d(a, x) < 1/n\}\right)$$

(cf. [2, Theorem 16]). For each $(t, x) \in T \times K$, $G(t, x)$ is a nonempty subset of $F(t, x)$ because $F(t, x) : K \to C(E)$ is upper semicontinuous. Also, the graph of $G(t, x)$ is a closed subset of $K \times E$ for each $t \in T$. Therefore the proofs

(*) Nella seduta del 14 gennaio 1978.

of [4, Theorem 4.1] and [5, Theorem] show that for each $t \in \mathrm{T}$ the fixed point set $\mathrm{H}(t)$ of $\mathrm{G}(t, x)$ is nonempty. Now let K be assigned the σ–algebra of Borel subsets of K and let $\mathrm{T} \times \mathrm{K}$ be equipped with the corresponding product σ-algebra. Since for a fixed n and each closed $\mathrm{B} \subset \mathrm{E}$ the set

$$\left\{(t, x) : \left(\bigcup_{a \in \mathrm{A}} \{\mathrm{F}(t, a) : d(a, x) < 1/n\}\right) \cap \mathrm{B} \neq \varnothing\right\} =$$

$$= \bigcup_{a \in \mathrm{A}} \{t : \mathrm{F}(t, a) \cap \mathrm{B} \neq \varnothing\} \times \{x : d(a, x) < 1/n\}$$

is measurable, it follows [3] that $\mathrm{G} : \mathrm{T} \times \mathrm{K} \to \mathrm{C}(\mathrm{E})$ is measurable. Since

$$\{(t, x) : x \in \mathrm{H}(t)\} = \{(t, x) : d(x, \mathrm{G}(t, x)) = 0\} =$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{a \in \mathrm{A}} \{(t, x) : d(a, x) < 1/n\} \cap \{(t, x) : d(a, \mathrm{G}(t, x)) < 1/n\}$$

is measurable, $\mathrm{H} : \mathrm{T} \to \mathrm{C}(\mathrm{K})$ is measurable and has a measurable selector [3, Theorems 3.5 and 5.1]. This completes the proof.

Remark. Suppose that for each $t \in \mathrm{T}$ a continuous $\mathrm{F}(t, x)$ assigns to each x in K a nonempty closed convex subset of E. Then $d(x, \mathrm{F}(t, x)) : \mathrm{K} \to \mathrm{R}$ is a continuous function of x for each $t \in \mathrm{T}$. It follows that $d(x, \mathrm{F}(t, x) : \mathrm{T} \times \mathrm{K} \to \mathrm{R}$ is measurable (cf. [2, Theorem 6]) and that the theorem remains valid if we assume that for each $t \in \mathrm{T}$ and $x \in \mathrm{K}$, $\mathrm{F}(t, x)$ is not disjoint from the closure of $\mathrm{I}(\mathrm{K}, x)$.

References

[1] A. T. Bharucha–Reid (1976) – *Fixed point theorems in probabilistic analysis*, « Bull. Amer. Math. Soc. », *82*, 641–657.
[2] H. W. Engl – *Random fixed point theorems*, to appear.
[3] C. J. Himmelberg (1975) – *Measurable relations*, « Fund. Math. », *87*, 53–72.
[4] S. Reich (1972) – *Fixed points in locally convex spaces*, « Math. Z. », *125*, 17–31.
[5] S. Reich (1976) – *A remark on set-valued mappings that satisfy the Leray–Schauder condition*, « Atti Accad. Naz. Lincei », *61*, 193–194.