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On the eventual asymptotic behaviour of perturbed functional differential equations

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Equazioni differenziali ordinarie. — *On the eventual asymptotic behaviour of perturbed functional differential equations.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà condizioni necessarie e sufficienti per la stabilità eventuale uniformemente asintotica di un sistema funzionale di equazioni differenziali. Si studiano effetti delle perturbazioni generali sull'eventuale comportamento asintotico dell'insieme $\varphi = 0$.

§ 1. INTRODUCTION

Consider a system of ordinary differential equations of the form

$$\frac{dx}{dt} = f(t, x) \quad , \quad x(t_0) = x_0 , \tag{1}$$

where $f \in C(R^+ \times R^n, R^n)$ and $C(R^+ \times R^n, R^n)$ is the space of continuous functions with domain $R^+ \times R^n$ and range R^n; and the perturbed system

$$\frac{dx}{dt} = f(t, x) + R(t, x) \quad , \quad x(t_0) = x_0 , \tag{2}$$

where $f, R \in C(R^+ \times R^n, R^n)$.

In our earlier work [1], we discussed a more general type of necessary conditions for the set $x = 0$ of (1) to be eventually uniformly asymptotically stable and obtained some more general conditions on the term $R(t, x)$ of (2) to ensure that the eventual uniform asymptotic stability property of the set $x = 0$ of (1) will also be a property of the set $x = 0$ of (2).

In this paper, we wish to extend to functional differential equations our results in [1] obtained for ordinary differential equations. We shall consider, instead of the ordinary differential equations (1) and (2) the functional differential system

$$\frac{dx}{dt} = f(t, x_t) \tag{3}$$

and the perturbed functional differential system

$$\frac{dx}{dt} = f(t, x_t) + R(t, x_t) . \tag{4}$$

(*) Nella seduta del 14 gennaio 1978.

Here, for any $\tau > 0$, $\mathscr{C}^n = C([-\tau, 0], R^n)$ denotes the space of continuous functions with domain $[-\tau, 0]$ and range in R^n, with the norm of an element $\phi \in \mathscr{C}^n$ defined by $\|\phi\|_0 = \max_{-\tau \leq s \leq 0} \|\phi(s)\|$.

x_t is an element of $\mathscr{C}^n$ defined by $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ and for $\rho > 0$, $C_\rho = [\phi \in \mathscr{C}^n : \|\phi\|_0 < \rho]$. Also we assume that $f, R \in C(R^+ \times C_\rho, R^n)$ where $C(R^+ \times C_\rho, R^n)$ denotes the space of continuous functions with domain $R^+ \times C_\rho$ and range in R^n. In section 2, we shall be concerned with the problem of obtaining necessary and sufficient conditions for the set $\phi = 0$ to be eventually uniformly asymptotically stable with respect to the system (3). In section 3, we apply our results of section 2 to study the perturbed system (4) under a more general type of conditions on the perturbation term $R(t, x_t)$ which ensures that the eventual uniform asymptotic stability property of the set $\phi = 0$ of (3) is inherited by the set $\phi = 0$ of (4).

§ 2. MAIN RESULTS

The following are the definitions which we need in this paper.

DEFINITION 2.1. A function $x(t_0, \phi_0)$ is said to be a solution of (3) with the given initial function $\phi_0 \in C_\rho$ at $t = t_0 \geq 0$ if there exists a number $A > 0$ such that (i) $x(t_0, \phi_0)$ is defined and continuous on $[t_0 - \tau, t_0 + A]$ and $x_t(t_0, \phi_0) \in C_\rho$ for $t_0 \leq t \leq t_0 + A$; (ii) $x_{t_0}(t_0, \phi_0) = \phi_0$; (iii) the derivative $x'(t_0, \phi_0)$ at t, $x'(t_0, \phi_0)(t)$ exists for $t \in [t_0, t_0 + A)$ and satisfies the system (3) for $t \in [t_0, t_0 + A)$.

DEFINITION 2.2. The set $\phi = 0$ is said to be eventually uniformly stable with respect to the system (3) if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ and $\tau_0 = \tau_0(\varepsilon) > 0$ such that

$$\|x_t(t_0, \phi_0)\| < \varepsilon, \qquad t \geq t_0 \geq \tau_0,$$

provided $\|\phi_0\|_0 \leq \delta$, where $x_t(t_0, \phi_0)$ is the solution of the system (3).

DEFINITION 2.3. The set $\phi = 0$ is eventually uniformly asymptotically stable if Definition 2.2 holds and there exists positive number $T = T(\varepsilon)$ such that $\|\phi_0\|_0 \leq \delta$ implies

$$\|x_t(t_0, \phi_0)\| < \varepsilon, \qquad t \geq t_0 + T$$

for $t_0 \geq \tau_0$.

In what follows the classes $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{K} \times \mathscr{K}$ are the classes of functions defined in § 2 of [1]. We now state a number of results that give us the set of conditions necessary for the eventual uniform asymptotic stability of the system (3).

THEOREM 2.4. *Assume that there exist a functional* $V(t, \phi)$ *and a function* $g(t, u)$ *satisfying the following properties*

(i) $V \in C(R^+ \times C_\rho, R^+)$ *and for* $t \geq t_0$,

$$D^+ V(t, x_t(t_0, \phi_0)) = \limsup_{h \to 0^+} h^{-1} [V(t+h, x_{t+h}(t_0, \phi_0)) - V(t, x_t(t_0, \phi_0))] \leq g(t, V(t, x_t(t_0, \phi_0)));$$

for every $\phi \in C_\rho$ *such that* $0 < \alpha < \| \phi_0 \|_0 < \rho$ *and* $t \geq \theta(\alpha)$;

(ii) *there exists* $a, b \in \mathcal{K}$ *and* $\theta(u)$ *continuous and monotinic decreasing in* u *for* $0 < u < \rho$, $V(t, \phi)$ *is locally Lipschitzian in* ϕ *and*

$$b(\| \phi \|_0) \leq V(t, x_t(t_0, \phi_0)) \leq a(\| \phi \|_0),$$

for

$$0 < \alpha < \| \phi \|_0 < \rho \quad \text{and} \quad t \geq \theta(\alpha).$$

(iii) $g \in C(R^+ \times R^+, R^+)$, $g(t, 0) = 0$ *and the trivial solution* $u = 0$ *is eventually uniformly stable with respect to the scalar differential equation*

$$\frac{du}{dt} = g(t, u) \qquad u(t_0) = u_0. \tag{5}$$

Then the set $\phi = 0$ *is eventually uniformly stable with respect to the system* (3).

Proof. On the basis of the arguments used in Theorem 8.5.1 of [2], the proof is straightforward.

THEOREM 2.5. *Assume that* (ii) *of Theorem 2.4 holds. Suppose* $f \in C(R^+ \times C_\rho, R^n)$, $V \in C(R^+ \times C_\rho, R^+)$ *and*

$$D^+ V(t, \phi) \leq -C(\| \phi \|_0) \left\{ 1 + \sum_{k=1}^{n} \eta (t - \tau_0)^k \right\} \tag{6}$$

where $\eta > 0$ *for every* $\phi \in C_\rho$ *such that* $0 < \alpha < \| \phi_0 \|_0 < \rho$, $t \geq \tau_0 \geq t_0 \geq \theta(\alpha)$, *and* $C \in \mathcal{K}$.

Then the set $\phi = 0$ *is eventually uniformly asymptotically stable with respect to* (3). *In particular if* $\| \phi \|_0$ *is replaced by* $V(t, \phi)$ *in* (6) *then the set* $\phi = 0$ *is eventually uniformly asymptotically stable.*

Proof. Setting $g(t, u) \equiv 0$ in the last theorem and using (6), we have the eventual uniform stability of the set $\phi = 0$. Hence given $\varepsilon > 0$, $t_0 \in R^+$, there exist $\delta(\varepsilon)$, and $\tau = \tau(\varepsilon)$ such that

$$\| x_t(t_0, \phi_0) \| < \varepsilon \qquad \text{for} \quad t \geq t_0 \geq \tau(\varepsilon).$$

We now show that it is asymptotically stable. Let $\varepsilon > 0$, $t_0 \in R^+$ be given and $0 < \varepsilon < \rho$. Set $\delta_0 = \delta(\rho)$, $\tau = \tau(\rho)$, let $\tau^* = \max[\tau(\rho), \theta(\alpha)]$ and choose $T(\varepsilon) = \tau(\varepsilon) + \dfrac{a(\rho)}{C(\delta(\varepsilon))}$. Let $t_0 \geq \tau^*$ and $\| \phi_0 \| \leq \delta_0$. We claim that there exists $t^* \in [t_0 + \tau(\varepsilon), t_0 + T(\varepsilon)]$ such that $\| x_{t^*}(t_0, \phi_0) \| < \delta(\varepsilon)$. Suppose not, then $t \in [t_0 + \tau(\varepsilon), t_0 + T(\varepsilon)]$ such that $\delta(\varepsilon) \leq \| x_t(t_0, \phi_0) \| < \rho$.

Integrating (6) from $t_0 + \tau(\varepsilon)$ to $t_0 + \mathrm{T}(\varepsilon)$,

$$\mathrm{V}(t_0 + \mathrm{T}(\varepsilon), x_{t_0+\mathrm{T}(\varepsilon)}(t_0, \phi_0)) \leq \mathrm{V}(t_0 + \tau(\varepsilon), x_{t_0+\tau(\varepsilon)}(t_0, \phi_0)) - \\ - \mathrm{C}(\delta(\varepsilon)) \int_{t_0+\tau(\varepsilon)}^{t_0+\mathrm{T}(\varepsilon)} \left\{ 1 + \sum_{k=1}^{n} \eta (s - \tau_0)^k \right\} ds$$

$$0 < b(\varepsilon) \leq \alpha(\| x_{t_0+\mathrm{T}(\varepsilon)}(t_0, \phi_0)\|) - \mathrm{C}(\delta(\varepsilon)) \int_{t_0+\tau(\varepsilon)}^{t_0+\mathrm{T}(\varepsilon)} ds - \mathrm{C}(\delta(\varepsilon)) \sum_{k=1}^{n} \eta \int_{t_0+\tau(\varepsilon)}^{t_0+\mathrm{T}(\varepsilon)} (s - \tau_0)^k \, ds \leq$$

$$\leq a(\rho) - \mathrm{C}(\delta(\varepsilon)(\mathrm{T}(\varepsilon) - \tau(\varepsilon)) - \mathrm{C}(\delta(\varepsilon)) \sum_{k=1}^{n} \int_{t_0+\tau(\varepsilon)}^{t_0+\mathrm{T}(\varepsilon)} \eta (s - \tau_0)^k \, ds =$$

$$= a(\rho) - a(\rho) - \mathrm{C}(\delta(\varepsilon)) \sum_{k=1}^{n} \int_{t_0+\tau(\varepsilon)}^{t_0+\mathrm{T}(\varepsilon)} \eta (s - \tau_0)^k \, ds \leq 0 ,$$

which is a contradiction hence $\phi = 0$ is asymptotically stable and the proof is complete.

In the following theorem, we wish to treat the solutions of the functional differential equation (3) as elements of a Euclidean space for $t > t_0$ except at the initial time. We then use a Lyapunov function instead of a functional and the following theorem gives a set of conditions in terms of such functions for the eventual asymptotic stability of the set $\phi = 0$ with respect to (3).

THEOREM 2.6. *Assume that*

(i) $\mathrm{V} \in \mathrm{C}\,[-\tau, \infty) \times \mathrm{S}_\rho, \mathrm{R}^+)$, $\mathrm{V}(t, x)$ *is locally Lipschitzian in* x *and*

$$b(\| x \|) \leq \mathrm{V}(t, x) \leq a(\| x \|)$$

for $0 < \alpha < \| x \| < \rho$ *and* $t \geq \theta(\alpha)$ *where* $a, b \in \mathscr{K}$, $\theta(u)$ *is continuous and monotonic decreasing in* u *for* $0 < u < \rho$ *and* $\mathrm{S}_\rho = \{x \in \mathrm{R}^n : \| x \| < \rho\}$.

(ii) $f \in \mathrm{C}(\mathrm{R}^+ \times \mathrm{C}_\rho, \mathrm{R}^n)$ *and*

$$\mathrm{D}^+ \mathrm{V}(t, \phi(0), \phi) = \limsup_{h \to 0^+} h^{-1} [\mathrm{V}(t + h, \phi(0) + hf(t, \phi)) - \mathrm{V}(t, \phi(0))] \leq \\ \leq - \mathrm{C}(\| \phi(0) \|) \left\{ 1 + \sum_{k=1}^{n} \eta (t - \tau_0)^k \right\}$$

for every $\phi \in \mathrm{C}_\rho$ *such that* $0 < \alpha < \| \phi(0) \| < \rho$, $t \geq \tau_0 \geq t_0 \geq \theta(\alpha)$ *and* $\mathrm{C} \in \mathscr{K}$. *Then the set* $\phi = 0$ *is eventually uniformly asymptotically stable with respect to* (3).

Proof. The eventual uniform stability of the set $\phi = 0$ follows from Corollary 8.5.1 of [2]. With appropriate changes the rest of the proof can be constructed as in the last theorem.

Remark. If $t \equiv \tau_0$, our result reduces to Theorem 8.5.3 of [2]. We now state a result which does not demand that $V(t, \phi)$ be positive definite.

THEOREM 2.7. *Assume that*

(i) $f \in C(R^+ \times C_\rho, R^n)$ *and* $\|f(t, \phi)\| \leq M$ $t \in R^+, \|\phi\|_0 \leq \rho^* < \rho$,

(ii) $V \in C([-\tau, \infty) \times S_\rho, R^+)$, $V(t, x)$ *is locally Lipschitzian in* x *and*

$$D^+ V(t, \phi(0), \phi) \leq -C[\phi(0)] \left\{1 + \sum_{k=1}^{n} \eta (t - \tau_0)^k\right\}$$

for every $\phi \in \Omega$, $t \in R^+$ *where* $C(r)$ *is positive definite with respect to a closed set* $\Omega \subset C_\rho$, *and* $t \geq \tau_0$;

(iii) *all the solutions* $x(t_0, \phi_0)$ *of* (3) *are bounded for* $t \geq t_0$. *Then every solution of the functional differential equation* (3) *approaches the set* Ω *as* $t \to \infty$.

Proof. Set $\phi = x_t(t_0, \phi_0)$ so that $\phi(0) = x(t_0, \phi_0)(t)$. Let $x_t(t_0, \phi_0)$ be any solution of (3), then by (iii) there exists a compact set $B \subset S_\rho$ such that $x(t_0, \phi_0)(t) \in B$ for $t \geq t_0$, and also $\|x_t(t_0, \phi_0)\|_0 \leq \rho^* < \rho$ for $t \geq t_0$. Hence by hypothesis (i), $\|f(t, x_t(t_0, \phi_0))\| \leq M$. Assume that the conclusion of the theorem is not true, then for $\varepsilon > 0$ there exists $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that $x(t_0, \phi_0)(t_k) \in S(\Omega, \varepsilon)^c \cap B$ where $S(\Omega, \varepsilon)^c$ is the complement of the set $S(\Omega, \varepsilon) = \{x : d(x, \Omega) < \varepsilon)$ [2, Def. 3.15.2]. Assume that t_1 is large enough so that for $t_k \leq t \leq t_k + \dfrac{\varepsilon}{2M}$, $x(t_0, \phi_0)(t) \in S(\Omega, \varepsilon)^c \cap B$. We further assume that the intervals $\left[t_k, t_k + \dfrac{\varepsilon}{2M}\right]$, $k = 1, 2, \cdots$ are disjoint. If not we take a subsequence of $\{t_k\}$ to ensure this.

For $t \geq t_0$ define

$$m(t) = V(t, x(t_0, \phi_0)(t)) + \int_{t_0}^{t} C[x(t_0, \phi_0)(s)] \, d(s)$$
$$+ \int_{t_0}^{t} C([x(t_0, \phi_0)(s)]) \sum_{k=1}^{n} \eta (s - \tau_0)^k \, ds.$$

Then,

$$\frac{m(t+h) - m(t)}{h} = \frac{V(t+h, x(t_0, \phi_0)(t+h)) - V(t, x(t_0, \phi_0)(t))}{h} +$$
$$+ \frac{1}{h} \int_{t}^{t+h} C([x(t_0, \phi_0)(s)]) \, ds + \frac{1}{h} \int_{t}^{t+h} C([x(t, \phi_0)(s)]) \sum_{k=1}^{n} \eta (s - \tau_0)^k \, ds$$

$$D^+ m(t) = \limsup_{h\to 0^+} h^{-1}[V(t+h, x(t_0, \phi_0)(t) + hf(t,\phi)) - V(t, x(t_0,\phi_0)(t))] +$$

$$+ C[x(t_0, \phi_0)(t)] + C[x(t_0, \phi_0)(t)] \sum_{k=1}^{n} \eta (t-\tau_0)^k \leq$$

$$\leq D^+ V(t, \phi(0), \phi) + C[\phi(0)] \left\{1 + \sum_{k=1}^{n} \eta(t-\tau_0)^k\right\} \leq 0 .$$

Thus

$$m(t) \leq m(t_0)$$

and

$$m(t_0) = \| V_t \|_0 = \sup_{-\tau\leq s\leq 0} V(t_0, \phi_0(s)) .$$

Thus for $t \geq t_0$,

$$V(t, x(t_0, \phi_0)(t)) \leq \sup_{-\tau\leq s\leq 0} V(t_0, \phi_0(s)) - \int_{t_0}^{t} C(x(t_0, \phi_0)(s))\, ds -$$

$$- \eta \int_{t_0}^{t} C(x(t_0, \phi_0))(s) \sum_{j=1}^{n} (s-\tau_0)^j\, ds .$$

Now since $C(r)$ is positive definite with respect to Ω, [2, Def. 3.15.1], then $x(t_0, \phi_0)(t) \in S\left(\Omega, \frac{\varepsilon}{2}\right)^c \cap B$ implies there exists

$$\delta = \delta\left(\frac{\varepsilon}{2}\right) \text{ such that } C((x(t_0, \phi_0)(t)) \geq \delta , \quad \text{for } t_k \leq t \leq t_k + \frac{\varepsilon}{2M} .$$

Therefore,

$$V\left(t_k + \frac{\varepsilon}{2M}, x(t_0, \phi_0)\left(t_k + \frac{\varepsilon}{2M}\right)\right) \leq \sup_{-\leq s\leq 0} V(t_0, \phi_0(s)) - \sum_{i=1}^{k} \int_{t_i}^{t_i+\frac{\varepsilon}{2M}} \delta\, ds -$$

$$- \eta\delta \sum_{i=1}^{k} \int_{t_i}^{t_i+\frac{\varepsilon}{2M}} \sum_{j=1}^{n} (s-\tau_0)^j\, ds .$$

Let $N = \sup_{t_i\leq s\leq t_i+\frac{\varepsilon}{2M}} \sum_{j=1}^{n} (s-\tau_0)^j$, then, we have

$$V\left(t_k + \frac{\varepsilon}{2M}, x(t_0, \phi_0)\left(t_k + \frac{\varepsilon}{2M}\right)\right) \leq \sup_{-\tau\leq s\leq 0} V(t_0, \phi_0(s)) - \delta k \frac{\varepsilon}{2M} -$$

$$- \eta\delta k \frac{N\varepsilon}{2M} \leq \sup_{-\tau\leq s\leq 0} V(t_0, \phi_0(s)) - \frac{\delta\varepsilon}{2M}(1 + \eta N) k .$$

As $k \to \infty$, we have a contradiction since $V(t, \phi) \geq 0$. Hence any solution $x(t_0, \phi_0)(t)$ tends to the set Ω as $t \to \infty$.

THEOREM 2.8. *Assume that the set* $\phi = 0$ *is eventually uniformly asymptotically stable with respect to the system* (3). *Suppose that*

$$\| f(t, \phi) - f(t, \psi) \| \leq L(t) \| \phi - \psi \|_0$$

for $(t, \phi), (t, \psi) \in R^+ \times C_\rho$, *where* $L(t) \geq 0$ *is continuous on* R^+ *and*

$$\int_t^{t+u} L(s) \, ds \leq Ku, \qquad u \geq 0.$$

Then there exists a functional $V(t, \phi)$ *with the following properties*:

(i) $V \in C(R^+ \times C_\rho, R^+)$ *and* $V(t, \phi)$ *satisfies*,

$$\| V(t, \phi) - V(t, \psi) \| \leq M \| \phi - \psi \|_0$$

for $t \in R^+, \phi, \psi \in C_{\delta(\delta_0)}$ $0 < \alpha < \| \phi \|_0 < \delta_0$ $0 < \| \psi \|_0 < \delta_0$ *and* $t \geq \tau_0 \geq \theta(\alpha)$;

(ii) $b(\| \phi \|_0) \leq V(t, \phi) \leq a(\| \phi \|_0)$, for $a, b \in \mathscr{K}$,

$$\| \phi \|_0 < \delta_0, \ t \geq \tau_0 \geq \theta(\alpha);$$

(iii) $D^+ V(t, \phi) \leq -C[V(t, \phi)] \left(1 + \sum_{k=1}^{n} \eta (t - \tau_0)^k \right)$ *for* $C \in \mathscr{K}$,

$$\eta > 0 \quad , \quad 0 < \alpha \| \phi \|_0 < \delta_0 \quad \textit{and} \quad t \geq \tau_0 \geq \theta(\alpha).$$

Proof. As in Theorem 2.5 of [1], $\exists \delta(\varepsilon), \tau(\varepsilon) > 0$ and $T(\varepsilon)$ such that $\delta \in \mathscr{K}$, and $\tau, T \in \mathscr{L}$. Let $G(r)$ be the function of Theorem 3.6.9 of [2] and set $\delta_0 = \delta(\varepsilon)$, $\tau_0 = \tau(\varepsilon)$. Define a functional

$$V(t, x(t_0, \phi_0)(t)) = \sup_{\sigma \geq 0} G(\| x(t_0, \phi_0)(t + \sigma) \|_0) \left\{ \frac{1 + \left[\alpha - \sum_{k=1}^{n} (t - \tau_0)^k \right] \sigma}{1 + \sigma} \right\};$$

where $\alpha > 1$, $\| \phi \|_0 \leq \delta_0$ and for $t \geq \tau_0$.

The rest of the proof can then be constructed as in Theorem 2.5 of [1] and so we leave details.

Remarks. Theorem 2.8 is the converse theorem for eventual uniform asymptotic stability of the set $\phi = 0$ which extends our result in [1] obtained for ordinary differential equations. It is easy to state and prove other variations of Theorem 2.8 analogous to the corresponding results in differential equation in Euclidean spaces as in [1].

§ 3. ASYMPTOTIC BEHAVIOUR OF PERTURBED SYSTEMS

In this section, we shall consider the perturbed functional differential system (4) and utilize our results of the preceding section to study the eventual asymptotic stability of the set $\phi = 0$ with respect to (4) under a more general type of perturbation. We shall assume that the set $\phi = 0$ is eventually uniformly asymptotically stable with respect to the system (3) and use the Lyapunov functionals constructed to show that the set $\phi = 0$ is eventually uniformly asymptotically stable.

THEOREM 3.1. *Assume that the set* $\phi = 0$ *is eventually uniformly asymptotically stable with respect to* (3). *Suppose that*

$$\|f(t, \phi) - f(t, \psi)\| \leq L(t) \|\phi - \psi\|_0$$

for, $\phi, \psi \in C_\rho$ *and* $\int_u^{t+u} L(s)\, ds \leq Lu$, $u \geq 0$. *Assume that the perturbation term* $R(t, x_t)$ *of the system* (4) *is such that* $\alpha, \beta > 0$

$$\| R(t, x_t) \| \leq \lambda_{\alpha,\beta}(t)$$

whenever $0 < \alpha \leq \|\phi\|_0 \leq \beta$, $t \in R^+$, $\lambda_{\alpha,\beta} \in C(R^+, R^+)$ *and*

$$(7) \qquad \int_{t_0}^{t} \lambda_{\alpha,\beta}(s)\, ds \leq q_0(t_0) + q_1(t_0)(t - t_0) + \cdots + q_{n+1}(t_0)(t - t_0)^{n+1},$$

where $q_k \in \mathscr{L}$, $k = 0, 1, 2, \cdots, n + 1$, *and* $t \geq t_0$.

Then the set $\phi = 0$ *is eventually uniformly asymptotically stable with respect to the system* (4).

Proof. By the assumptions the conclusions of Theorem 2.8 hold. Let $t \geq t_0 \geq \tau_0$ and $\phi \in C_\rho$, then

$$D^+ V(t, \phi)_{(4)} = \limsup_{h \to 0^+} h^{-1} [V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)] =$$
$$= \limsup_{h \to 0^+} h^{-1} [V(t + h, x_{t+h}(t, \phi)) - V(t + h, y_{t+h}(t, \phi) +$$
$$+ V(t + h, y_{t+h}(t, \phi) - V(t, \phi)]$$

where $y(t, \phi)$ is any solution of (3) with initial function ϕ at time t. Therefore,

$$(8) \qquad D^+ V(t, \phi)_{(4)} \leq D^+ V(t, \phi)_{(3)} + \limsup_{h \to 0^+} h^{-1} M(t + h) \| x_{t+h}(t, \phi) -$$
$$- y_{t+h}(t, \phi) \|_0 = D^+ V(t, \phi)_{(3)} + M(t) \| R(t, \phi) \| .$$

Let $\varepsilon > 0$, $t_0 \in R^+$ and choose δ such that $\alpha(\delta) < \frac{b(\varepsilon)}{2}$ and assume that $\|\phi\|_0 < \delta$. Then there exists $\tau = \tau(\varepsilon)$ such that

$$\delta \leq \| x_t(t_0, \phi_0) \| < \varepsilon \qquad \text{for} \quad t \geq t_0 \geq \tau(\varepsilon)$$

where $x_t(t_0, \phi_0)$ is any solution of (4). Suppose not then there exists $t_1, t_2 > t_0$ such that

$$\| x_{t_1}(t_0, \phi_0) \| = \delta \quad , \quad \| x_{t_2}(t_0, \phi_0) \|_0 = \varepsilon$$

and

$$\delta \leq \| x_t(t_0, \phi_0) \| \leq \varepsilon \qquad \text{for} \quad t \in [t_1, t_2] .$$

Integrating (8), hypothesis (iii) of Theorem 2.8 implies

$$b(\varepsilon) = V(t_2, x_{t_2}(t_0, \phi_0)) \leq V(t_1, x_{t_1}(t_0, \phi_0) - \int_{t_1}^{t_2} C(\|\phi(s)\|_0) \cdot$$

$$\cdot \left\{ 1 + \sum_{k=1}^{n} \eta (s - \tau_0)^k \right\} ds + M(r) \int_{t}^{t_2} \lambda_{\alpha,\beta}(s) \, ds .$$

So,

$$b(\varepsilon) \leq a(\| x_{t_1} \|_0) - C(\delta) \int_{t_1}^{t_2} ds - \sum_{k=1}^{n} C(\delta) \eta \int_{t_1}^{t_2} (s - \tau_0)^k \, ds +$$

$$+ M(r) q_0(t_0) + M(r) \sum_{k=1}^{n+1} q_k(t_0)(t_2 - t_1)^k .$$

Choosing t_0 large enough so that for $t_0 \geq \tau_1(\varepsilon)$, $\sum_{k=1}^{n+1} q_k(t_0)(t_2 - t_1)^{k-1} = \frac{C(\delta)}{M(r)}$, $t_0 \geq \tau_2(\varepsilon)$ such that $q_0(t_0) < \frac{b(\varepsilon)}{M(r)}$ and set $\tau(\varepsilon) = \max\{\tau_0, \tau_1(\varepsilon), \tau_2(\varepsilon)\}$. Then for $t_0 \geq \tau(\varepsilon)$,

$$b(\varepsilon) < \frac{b(\varepsilon)}{2} + \frac{b(\varepsilon)}{2} - C(\delta) \sum_{k=1}^{n} \int_{t_1}^{t_2} \eta (s - \tau_0)^k \, ds < b(\varepsilon) ,$$

which is a contradiction. Hence $\| x_t(t_0, \phi_0) \| < \varepsilon$ for $t \geq t_0 \geq \tau(\varepsilon)$. Finally we show eventual asymptotic stability. Let $\varepsilon > 0$, $t_0 \in R^+$ be given and $0 < \varepsilon < \rho$. Set $\delta_0 = \delta(\rho)$, $\tau^* = \tau(\rho)$ and choose $T(\varepsilon) = \tau(\varepsilon) + \frac{2\alpha(\rho) + (M+1) M(r)}{C(\delta)}$. Proceeding from this point on as in Theorem 2.5 of [1] we can show that there exists $\tau^{**}(\varepsilon)$ such that if $\|\phi_0\|_0 \leq \delta$, then $t^* \in [t_0 + \tau(\varepsilon), t_0 + T(\varepsilon)]$ such that $\| x_{t^*}(t_0, \phi_0) \|_0 < \delta(\varepsilon)$, for $t_0 \geq \tau^{**}(\varepsilon)$. Hence the set $\phi = 0$ is eventually uniformly asymptocially stable.

Remarks. This result extends the perturbation result discussed in [1] for ordinary differential equations to functional differential system (4). The results of Theorem 8.5.2 of [2] is a special case of Theorem 3.1 if we note that instead of Lyapunov functionals one could use Lyapunov functions to obtain the conclusions of Theorem 2.8.

COROLLARY 3.2. *Let the asymptotically self-invariant set* $\phi = 0$ *be uniformly asymptotically stable with respect to the system* (3). *Suppose*

$$\|f(t, \phi) - f(t, \psi)\| \leq L(t) \|\phi - \psi\|_0 \qquad \text{for} \quad t \geq 0$$

$\phi, \psi \in C_\rho$ *and* $\int_t^{t+u} L(s)\, ds \leq ku$, $u \geq 0$. *Assume that the perturbations* $R(t, \phi)$ *of the system* (4) *is such that for each* $\alpha, \beta > 0$ *there exist* $\lambda_{\alpha,\beta} \in C(R^+, R^+)$ *such that*

$$\|R(t, \phi)\| \leq \lambda_{\alpha,\beta}(t) \qquad \text{where} \quad \alpha \leq \|\phi\| \leq \beta, \ t \in R^+,$$

and

$$\int_{t_0}^{t} \lambda_{\alpha,\beta}(s)\, ds \leq q_0(t_0) + q_1(t_0)(t - t_0), \qquad t \geq t_0, q_0, q_1 \in .$$

Then the set $\phi = 0$ *is asymptotically self-invariant and it is eventually uniformly asymptotically stable with respect to the system* (4).

REFERENCES

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