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The Set-Spectra of Decomposable Operators

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Analisi funzionale. — *The Set-Spectra of Decomposable Operators.* Nota di IVAN ERDELYI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si studiano le proprietà dei chiusi spettrali di un operatore decomponibile. Tale concetto, strettamente legato agli invarianti della decomposizione spettrale, ha varie applicazioni nella teoria degli operatori decomponibili. Si ottiene fra l'altro una proprietà di chiusura per questi insiemi rispetto ad unioni arbitrarie.

The spectral theory of a decomposable operator T gives a clear and efficient description of the restrictions $T \mid Y$ of T to the invariants Y of the spectral decomposition. The invariant subspaces Y, known as spectral maximal spaces, reveal most of the spectral properties of the given T. What special properties have the spectra $\sigma(T \mid Y)$ of those restrictions? This paper attempts to give some answers to this question. Implicitly, we obtain a few new interpretations and proving techniques for various spectral properties of the given decomposable T. For a full dress account of the theory of decomposable operators, the reader is directed to the monograph [4] by Colojoară and Foiaş. For completeness we include the definitions of the basic concepts and list some pertinent properties.

Given a bounded linear operator T on a Banach space X, a subspace (closed linear manifold) Y invariant under T is called *spectral maximal* for T if any invariant subspace Z with $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ is contained in Y. T is said to be decomposable on X if for every finite open cover $\{G_i\}_1^n$ of the spectrum $\sigma(T)$, there is a system $\{Y_i\}_1^n$ of spectral maximal spaces of T performing the spectral decomposition

$$\begin{cases} X = \sum_{i=1}^{n} Y_i , \\ \sigma(T \mid Y_i) \subset G_i , \qquad i = 1 , 2 , \cdots , n. \end{cases}$$

A decomposable operator T has the single valued extension property and for every closed subset F of the complex plane C,

$$X_T(F) = \{x \in X : \sigma_T(x) \subset F\}$$

is a spectral maximal space of T [5] ($\sigma_T(\cdot)$ denotes the local spectrum). Moreover,

$$\sigma[T \mid X_T(F)] \subset F \tag{1}$$

(*) Nella seduta del 14 gennaio 1978.

and every spectral maximal space Y of T has the representation [5]

$$Y = X_T [\sigma (T \mid Y)] . \tag{2}$$

1. PROPOSITION [5]. *Let* T *be decomposable on* X *and let* G *be any open subset of* C *with*

$$G \cap \sigma (T) \neq \varnothing .$$

Then there exists a nonzero spectral maximal space Y *of* T *such that*

$$\sigma (T \mid Y) \subset G .$$

2. PROPOSITION [1]. *Let* T *be decomposable on* X *and let* Y *be a spectral maximal space of* T. *Then the spectrum of the coinduced operator* T^Y *on the quotient space* X/Y *satisfies relation*

$$\sigma (T^Y) = \overline{\sigma (T) - \sigma (T \mid Y)} .$$

3. DEFINITION. *Given* T *decomposable on* X, *a closed subset* F *of* σ (T) *is called a set-spectrum of* T *if there exists a spectral maximal space* Y *of* T *such that*

$$F = \sigma (T \mid Y) . \tag{3}$$

It follows from the representation (2) that for a set-spectrum (3) we have

$$\sigma [T \mid X_T (F)] = F .$$

4. THEOREM [3]. *Given* T *decomposable on* X, *for every closed subset* F *of* σ (T) *there is a largest set-spectrum* F_0 *contained in* F. F_0 *is defined by*

$$F_0 = \sigma [T \mid X_T (F)] \tag{4}$$

and satisfies property

$$X_T (F_0) = X_T (F) .$$

Proof. F_0 as defined by (4) is a set-spectrum contained in F. Let $\sigma (T \mid Y)$ be any set-spectrum contained in F. The hypotheses on Y imply

$$Y = X_T [\sigma (T \mid Y)] \subset X_T (F) .$$

Thus, we have

$$\sigma (T \mid Y) \subset \sigma [T \mid X_T (F)] = F_0$$

and hence F_0 is the largest set-spectrum contained in F. Now, let x be arbitrary in $X_T (F)$. Then

$$\sigma_T (x) \subset \sigma [T \mid X_T (F)] = F_0$$

and hence $x \in X_T(F_0)$. Thus

$$X_T(F) \subset X_T(F_0) .$$

The opposite inclusion follows from the fact that F_0 is contained in F. □

5. THEOREM [3]. *Given* T *decomposable on* X, *let* $F \subset \sigma(T)$ *be closed and let* F^I *be the interior of* F *in the topology of* $\sigma(T)$. *Then for the largest set-spectrum* F_0 *contained in* F *we have*

$$\overline{F^I} \subset F_0 \subset F . \tag{5}$$

Proof. Let $\lambda \in \overline{F^I}$. For every neighborhood $V(\lambda)$ of λ open in the topology of $\sigma(T)$, the set $V(\lambda) \cap F^I$ is nonvoid and open in $\sigma(T)$. By Proposition 1, there exists a spectral maximal space Y of T and hence a set-spectrum

$$F_V = \sigma(T \mid Y)$$

such that

$$\emptyset \neq F_V \subset V(\lambda) \cap F^I .$$

We have

$$F_V \subset F^I \subset F .$$

Since F_0 is the largest set-spectrum contained in F, $F_V \subset F_0$ and hence

$$F_V \subset V(\lambda) \cap F_0 .$$

Thus $\lambda \in \overline{F}_0 = F_0$ and inclusions (5) follow. □

6. COROLLARY [3]. *Given* T *decomposable on* X, *let* F *be a closed subset of* $\sigma(T)$ *such that*

$$F = \overline{F^I} .$$

Then F *is a set-spectrum of* T. *In particular, every* $\overline{F^I}$ *is a set-spectrum of* T.

7. COROLLARY [3]. *Given* T *decomposable on* X, *for every closed* $F \subset \sigma(T)$ *let* F_0 *be the largest set-spectrum contained in* F. *Then* $F - F_0$ *has void interior in* $\sigma(T)$.

8. COROLLARY. *Given* T *decomposable on* X, *for every closed* $F \subset C$,

$$F_1 = \overline{\sigma(T) - F}$$

is a set-spectrum of T.

Proof. $\sigma(T) - F = \sigma(T) \cap F^c$ (^{c} for the complement in C) is open in $\sigma(T)$ and then Corollary 6 completes the proof. □

9. THEOREM. *Given* T *decomposable on* X, *let* Y *be a spectral maximal space of* T. *There exists a spectral maximal space* Z *of* T *such that*

$$\sigma(T \mid Z) = \sigma(T^{Y}) . \tag{6}$$

Proof. By Proposition 2,

$$\sigma(T^{Y}) = \overline{\sigma(T) - \sigma(T \mid Y)} \tag{7}$$

and then Corollary 8 implies that $\sigma(T^{Y})$ is a set-spectrum of T. By definition, there exists a spectral maximal space Z with the property expressed by (7). □

Next, we use the set-spectrum concept to prove a property of decomposable operators.

10. THEOREM. *Given* T *decomposable on* X, *for every open* $G \subset C$ *with*

$$G \cap \sigma(T) \neq \varnothing \qquad \textit{and} \quad \sigma(T) \not\subset \overline{G} ,$$

there is a proper spectral maximal space Y *of* T *satisfying the following properties*

$$\sigma(T \mid Y) \subset \overline{G} \quad \textit{and} \quad \sigma(T^{Y}) \cap G = \varnothing .$$

Proof. Put

$$F = \overline{G \cap \sigma(T)} .$$

Then $\overline{F^{i}} = F$ and by Corollary 6, F is a set-spectrum of T. Thus, there is a spectral maximal space

$$Y = X_T(F)$$

which by the hypotheses on G is a proper subspace of X. We have

$$\sigma(T \mid Y) = F \subset \overline{G} .$$

Now, with the help of (7), we obtain successively:

$$\sigma(T^{Y}) = \overline{\sigma(T) - F} = \overline{\sigma(T) - \overline{[G \cap \sigma(T)]}} \subset \overline{\sigma(T) - [G \cap \sigma(T)]} =$$
$$= \overline{\sigma(T) - G} \subset G^{c}. \qquad \square$$

11. COROLLARY. *Given* T *decomposable on* X, *for every open* $G \subset C$ *with*

$$G \cap \sigma(T) \neq \varnothing \qquad \textit{and} \quad \sigma(T) \not\subset \overline{G} , \tag{8}$$

there are proper spectral maximal spaces Y *and* Z *satisfying*

$$\sigma(T \mid Y) \subset \overline{G} \qquad \textit{and} \quad \sigma(T \mid Z) \subset G^{c} . \tag{9}$$

Proof. The existence of spectral maximal spaces Y and Z satisfying (9) follows from Theorem 9 and Theorem 10. $Y \neq \{0\}$ by Proposition 1.

$Z \neq \{0\}$ because otherwise by (6) and (7),

$$\sigma(T) - \sigma(T \mid Y) = \varnothing$$

which implying $Y = X$ contradicts the hypothesis $\sigma(T) \not\subset \overline{G}$. □

There is another way of expressing Corollary 11: If T is decomposable on X and an open G satisfies (8) then there are two spectral maximal spaces Y and Z such that either $Y + Z$ is a direct sum or

$$\sigma(T \mid Y \cap Z) \subset \partial G ,$$

where ∂ denotes the boundary.

A characterization of the local spectrum in terms of set-spectra is given by the following.

12. THEOREM. *For every* T *decomposable on* X *and* $x \in X$,

$$\sigma_T(x) = \cap \{F : F = \sigma[T \mid X_T(F)] \quad and \quad x \in X_T(F)\} . \tag{10}$$

Proof. Denote the right-hand side of the equality (10) by S_x. For $x \in X_T(F)$ we have $\sigma_T(x) \subset F$ and hence

$$\sigma_T(x) \subset S_x .$$

On the other hand, $Y = X_T[\sigma_T(x)]$ being a spectral maximal space of T which contains x, we have

$$S_x \subset \sigma(T \mid Y) = \sigma_T(x) . \quad \square$$

13. LEMMA [2]. *Given* T *decomposable on* X, *for every subset* W *of* x,

$$F = \overline{\cup \{\sigma_T(x) : x \in W\}}$$

is a set-spectrum of T.

Proof. First, we show that $W \subset X_T(F)$. Indeed, for every $x \in W$, we have

$$x \in X_T[\sigma_T(x)] \subset X_T(F) .$$

Next, we have (1)

$$\sigma[T \mid X_T(F)] \subset F .$$

On the other hand, in view of a property of the local spectrum [6] we have

$$\sigma[T \mid X_T(F)] = \cup \{\sigma_T(x) : x \in X_T(F)\} \supset \overline{\cup \{\sigma_T(x) : x \in W\}} = F. \quad \square$$

14. THEOREM. *Given* T *decomposable on* X, *let* $\{F_\alpha\}_{\alpha \in I}$ *be an arbitrary family of set-spectra for* T. *Then*

$$F = \overline{\cup \{F_\alpha : \alpha \in \overline{I}\}}$$

is a set-spectrum of T.

Proof. For every $\alpha \in \mathrm{I}$, there is a spectral maximal space Y_α of T with

$$\mathrm{F}_\alpha = \sigma(\mathrm{T} \mid \mathrm{Y}_\alpha) = \cup \{\sigma_{\mathrm{T}|\mathrm{Y}_\alpha}(x) : x \in \mathrm{Y}_\alpha) = \cup \{\sigma_{\mathrm{T}}(x) : x \in \mathrm{Y}_\alpha\} .$$

By denoting

$$\mathrm{W} = \cup \{\mathrm{Y}_\alpha : \alpha \in \mathrm{I}\}$$

we obtain

$$\mathrm{F} = \overline{\cup\{\mathrm{F}_\alpha : \alpha \in \mathrm{I}\}} = \overline{\cup\{\sigma(\mathrm{T} \mid \mathrm{Y}_\alpha) : \alpha \in \mathrm{I}\}} = \overline{\cup\{\sigma_{\mathrm{T}}(x) : x \in \mathrm{W}\}} .$$

By Lemma 13, F is a set-spectrum of T. □

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