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Unsteady magnetoaerodynamic forces on an oscillating circular cylindrical shell of finite length. I. Simple harmonic motion

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Magnetofluidodinamica. — *Unsteady magnetoaerodynamic forces on an oscillating circular cylindrical shell of finite length.* I. *Simple harmonic motion.* Nota di LIVIU LIBRESCU, presentata (*) dal Socio C. FERRARI.

RIASSUNTO. — Il presente lavoro è dedicato alla determinazione analitica delle forze magnetoaerodinamiche, agenti su un pannello cilindrico circolare che oscilla armonicamente in una corrente di gas supersonico, perfetto conduttore dell'elettricità, in presenza di un campo magnetico.

1. The study of the structural instability phenomena resulting from the reciprocal interaction between the electrodynamic field, a conducting gas flow and an elastic thin body (interaction occuring e.g. when an electrically conducting gas flows past an elastic thin body, a magnetic field being present), plays an important role in the design of some modern technological devices (space vehicles, ionic engines, etc.).

In this context, the importance of investigating the magneto-aeroelastic stability of *finitely long circular cylindrical thin shells* becomes obvious.

In order to undertake such a stability analysis, an indispensable prerequisite is the determination of the appropriate magneto-aerodynamic unsteady forces; it is just this determination that constitutes the subject of the present Note.

2. Let us consider a circular cylindrical thin shell of finite length l placed in an external and separately in an internal supersonic ideally conducting gas flow, a magnetic field $\mathbf{H}_0$ (with $\mathbf{U} \parallel \mathbf{H}_0$) being also present, where $\mathbf{U}$, i.e. the velocit vector of the gas is considered to be parallel to the cylinder axis.

Let us refer the points of the space to the cylindrical coordinate system (x_1, x_2, x_3), where x_1 denotes the streamwise, $x_2 = \mathrm{R}\theta$—the circumferential and $x_3 \equiv r$—the radial coordinates, respectively, (R being the radius of the cylinder mid-surface) and let the initial and terminal sections of the cylinder bee defined by $x_1 = 0 ; l$, respectively.

In approaching the proposed problem we start from the linearized magnetoaerodynamic field equations established in [1], independently of any particular coordinate system, for the case of *simple media* (see in this sense [2]), by also disregarding the thermoelectric and viscosity effects. By further considering the case of an ideally conducting medium,

(*) Nella seduta del 14 maggio 1977.

in the absence of the Hall effect, the pertinent field equations write:

$$
\begin{aligned}
&\frac{\mathrm{D}\boldsymbol{h}}{\mathrm{D}t} = \mathrm{H}_1 \frac{\partial \boldsymbol{v}}{\partial x_1} + \frac{\mathrm{H}_1}{\rho_0}\frac{\mathrm{D}\hat{\rho}}{\mathrm{D}t}\mathbf{I}_1\,,\\
(1)\qquad &\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho_0}\,\mathrm{grad}\left(a_0^2\,\hat{\rho} + \frac{1}{4\pi}\,\mathrm{H}_1 h_1\right) + \frac{\mathrm{H}_1}{4\pi\rho_0}\frac{\partial \boldsymbol{h}}{\partial x_1}\,,\\
&\frac{\mathrm{D}\hat{\rho}}{\mathrm{D}t} + \rho_0\,\mathrm{div}\,\boldsymbol{v} = 0\,,\\
&\hat{p} = a_0^2\,\hat{\rho}\,,\qquad (a_0^2 = \varkappa\, p_0/\rho_0)\,;\qquad p = p_0 + \hat{p}
\end{aligned}
$$

where $\boldsymbol{h}\,(\equiv h^i\,\mathbf{I}_i)$; $\boldsymbol{v}\,(\equiv v^i\,\mathbf{I}_i)$; $\hat{\rho}$ and $\hat{p}$ denote the perturbations of the primary fields; i.e. of the applied magnetic field $\mathbf{H}_0$, of the velocity $\mathbf{U}$, of the density ρ_0 and of the pressure p_0, respectively, where the perturbations are considered to be functions of x_i and t, while the primary fields are considered to be constant quantities in both time and space; $\mathbf{I}_i$ $(i = 1, 2, 3)$ are the unit base vectors; $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \mathrm{U}\partial/\partial x_1$, denotes the substantial derivative, while a_0 denotes the speed of sound.

The solution to the above exhibited field equations is to be subjected to the appropriate conditions at infinity (in the case of external flow), to the finiteness condition at $r = 0$ (in the case of internal flow) and to the impenetrability condition at $r = \mathrm{R}$, i.e.

$$
(2)\qquad v_3\,|_{x_3=\mathrm{R}} = \begin{cases} -\dfrac{\mathrm{D}w}{\mathrm{D}t} & \text{for}\quad (x_1, x_2) \in \Omega \\ 0 & \text{for}\quad (x_1, x_2) \notin \Omega \end{cases}
$$

where Ω denotes the domain of the cylinder mid-surface, while $w\,(x_1, x_2, t)$, its radial deflection.

3. The pertinent field equations being already exhibited, we may pass to the determination of the pressure $\mathrm{P}\,|_{x_3=\mathrm{R}} = (p_+ - p_- - t_{33})_{x_3=\mathrm{R}}$ acting on the cylinder mid-surface, where the subscripts $\pm$ mean that the quantities thus affected are obtained for $x_3 = \mathrm{R} \pm 0$, respectively, while the transversal component t_{33} of the linearized Maxwell tensor t_{ij} $(i, j = 1, 2, 3)$, is given in the present instance by

$$
(3)\qquad t_{33} = -\mathrm{H}_1 h_1/(4\pi)\,.
$$

Consistent with the cylinder deflection-function defined by

$$
(4)\qquad w\,(x_1, x_2, t) = \mathrm{W}\,(x_1)\, e^{j\omega t}\cos n\theta\,, \qquad (j = (-1)^{\frac{1}{2}})
$$

the remaining unknown functions are adequately represented as:

$$
(5)\qquad \begin{Bmatrix} f\,(x_1, x_2, x_3, t) \\ g\,(x_1, x_2, x_3, t) \end{Bmatrix} = \begin{Bmatrix} \bar{f}\,(x_1, x_3)\cos n\theta \\ \bar{g}\,(x_1, x_3)\sin n\theta \end{Bmatrix} e^{j\omega t}\,,
$$

$f(x_i, t)$ and $g(x_i, t)$ denoting generically one of the functions $v_1(x_i, t)$, $v_3(x_i, t)$, $h_1(x_i, t)$, $h_3(x_i, t)$ and $v_2(x_i, t)$, $h_2(x_i, t)$, respectively, while n denotes the circumferential wave number.

Making use in the field equations of the dimensionless coordinates $\bar{x}_i = x_i/l$, $(i = 1, 2)$, $\bar{x}_3 = r/R$ and of (5) and taking in the so obtained equations the Laplace transform (henceforth denoted by L. T.) with respect to the $\bar{x}_1$-coordinate, i.e.

$$\mathring{f}(s, \bar{x}_3) = \int_0^\infty f(\bar{x}_1, \bar{x}_3)\, e^{-s\bar{x}_1}\, d\bar{x}_1 \equiv \mathscr{L}\{f\} \tag{6}$$

(s denotes L. T. variable) and using the continuity condition of disturbances at $\bar{x}_1 = 0$ (expressed generically as $f(0^+, \bar{x}_3) = f(0^-, \bar{x}_3) \equiv 0$), it is easy to see that $\mathring{v}_i(s, \bar{x}_3)$ and $\mathring{h}_i(s, \bar{x}_3)$, expressed in terms of $\mathring{\rho}(s, \bar{x}_3)$ as:

$$\begin{gathered}
\mathring{h}_1 = \frac{H_1(\check{\mu}^2 - s^2)}{\rho_0 \check{\mu}^2}\,\mathring{\rho} \quad , \quad \mathring{h}_2 = \frac{H_1 s}{\rho_0 \check{\mu}^2}\,\frac{nl}{r}\,\frac{\check{\mu}^2 + \lambda^2(\check{\mu}^2 - s^2)}{\check{\mu}^2 \lambda^2 s^2}\,\mathring{\rho}\,, \\
\mathring{h}_3 = -\frac{H_1 s}{\rho_0}\,\frac{l}{R}\,\frac{\check{\mu}^2 + \lambda^2(\check{\mu}^2 - s^2)}{\check{\mu}^2 - \lambda^2 s^2}\,\mathring{\rho}_{,3} \quad , \quad \mathring{v}_1 = -\frac{a_0 s}{\rho_0 \check{\mu}}\,\mathring{\rho}\,, \\
\mathring{v}_2 = \frac{a_0}{\rho_0}\,\frac{nl}{r}\,\frac{\check{\mu}^2 + \lambda^2(\check{\mu}^2 - s^2)}{\check{\mu}(\check{\mu}^2 - \lambda^2 s^2)}\,\mathring{\rho} \quad , \quad \mathring{v}_3 = -\frac{a_0 l}{\rho_0 R}\,\frac{\check{\mu}^2 + \lambda^2(\check{\mu}^2 - s^2)}{\check{\mu}(\check{\mu}^2 - \lambda^2 s^2)}\,\mathring{\rho}_{,3} \\
((\ )_{,3} \equiv \partial/\partial x_3)\,,
\end{gathered} \tag{7}$$

satisfy identically Eqs. $(1)_{1,2,3}$ (transformed through (6)) while Eq. $(1)_4$ in conjunction with (7) (also transformed through (6)) leads to the governing equation for $\mathring{\rho}(s, \bar{x}_3)$:

$$\mathring{\rho}_{,\zeta\zeta} + \frac{1}{\zeta}\,\mathring{\rho}_{,\zeta} - \left(1 + \frac{n^2}{\zeta^2}\right)\mathring{\rho} = 0\,, \qquad ((\ )_{,\zeta} \equiv \partial/\partial\zeta) \tag{8}$$

which is a modified Bessel equation, where

$$\zeta^2 = \frac{r}{R}\,\frac{R^2}{l^2}\,\frac{(\check{\mu}^2 - s^2)(\check{\mu}^2 - \lambda^2 s^2)}{\check{\mu}^2 + \lambda^2(\check{\mu}^2 - s^2)} \tag{9}$$

while

$$\check{\mu} \equiv M(j\check{\omega} + s) \quad , \quad \check{\omega} \equiv \frac{\omega l}{U}\ ;\ M \equiv \frac{U}{a_0} \text{ — (Mach number)}, \tag{10}$$

$$\lambda^2 = \frac{H_1^2}{4\pi\rho_0 a_0^2} \text{ — (Alfvén number).}$$

Satisfying the condition of finitness at infinity (in the case of the external flow) and the "impenetrability condition"

$$\mathring{v}_3|_{\bar{x}_3=1} = \begin{cases} -\dfrac{a_0 \check{\mu}}{l}\,\mathring{W} & (0 < \bar{x}_1 < 1) \\ 0 & (1 < \bar{x}_1 < 0) \end{cases} \tag{11}$$

where $\overset{\circ}{W}(s) \equiv \mathscr{L}\{W(\bar{x}_1)\}$, the solution to Eq. (8) turns out to be

$$\overset{\circ}{\rho}(s, \bar{x}_3) = \rho_0 \frac{R}{l^2} \frac{\breve{\mu}^2 (\breve{\mu}^2 - \lambda^2 s^2)}{\breve{\mu}^2 + \lambda^2 (\breve{\mu}^2 - s^2)} \frac{K_n(\zeta)}{[\zeta K_n'(\zeta)]_{\bar{x}_3=1}} \overset{\circ}{W}(s) \tag{12}$$

where $K_n(\zeta)$ denotes the modified Bessel functions.

Now, by using adequatelly Eqs. (12), (3), $(7)_1$, $(1)_{4,5}$ and assimilating the pressure into vacuum with p_0, one gets:

$$\begin{aligned} \overset{\circ}{P}|_{\bar{x}_3=1} &\equiv (p_+ - p_0 - t_{33})^0|_{\bar{x}_3=1} = \\ &= a_0^2 \rho_0 \frac{R}{l^2} (\breve{\mu}^2 - \lambda^2 s^2) \frac{K_n(\breve{\zeta})}{\breve{\zeta} K_n'(\breve{\zeta})} \overset{\circ}{W}(s) \qquad (\breve{\zeta} \equiv \zeta|_{r=R}) . \end{aligned} \tag{13}$$

At this stage, the difficulty consists in performing the inverse L. T. of (13), as to obtain $P|_{\bar{x}_3=1}$ explicitely. For this purpose, two asymptotic approximations of $\Psi(\breve{\zeta}) \equiv K_n(\breve{\zeta})/(\breve{\zeta} K_n'(\breve{\zeta}))$ will be used:

i) Within the first one (see in this sense [3, 4])

$$\Psi(\breve{\zeta}) \cong -(n^2 + \breve{\zeta}^2)^{-\frac{1}{2}}, \tag{14}$$

evaluation valid for large n and arbitrary $\mathrm{Re}(\breve{\zeta}) > 0$.

Inserting (14) into Eq. (13), having in view (10) and making use of the convolution and the shifting theorems to the resulting equation, we get the original as:

$$\begin{aligned} P|_{\bar{x}_3=1} &= \mathscr{C} \left[c_1 \left.\frac{dW(\bar{x}_1)}{d\bar{x}_1}\right|_{\bar{x}_1=0} K(\bar{x}_1) + \int_0^{\bar{x}_1} K(\bar{x}_1 - \bar{\xi}_1) \times \right. \\ &\left. \times \left(c_1 \frac{d^2}{d\bar{\xi}_1^2} + 2jM^2 \breve{\omega} \frac{d}{d\bar{\xi}_1} - M^2 \breve{\omega}^2 \right) W(\bar{\xi}_1) d\bar{\xi}_1 \right] e^{j\omega t} , \\ &\left(\mathscr{C} \equiv -\frac{a_0^2 \rho_0}{l} \mathscr{C}_1 \cos n\theta \right), \end{aligned} \tag{15}$$

in which the Kernel function $K(\bar{x}_1)$ expresses as:

$$K(x) = J_0(\Gamma x) \exp(-j\bar{\omega} x) \tag{16}$$

where

$$\bar{\omega} = \frac{M^2 \breve{\omega}}{M^2 - \lambda^2} \quad ; \quad \mathscr{C}_1 = \frac{\lambda}{(M^2 - \lambda^2)^{\frac{1}{2}}} \quad ; \quad c_1 = M^2 - \lambda^2 \quad \text{for } \lambda^2 \gg 1$$

and

$$\bar{\omega} = \frac{M^2 \breve{\omega}}{M^2 - 1} \quad ; \quad \mathscr{C}_1 = (M^2 - 1)^{-\frac{1}{2}} \quad ; \quad c_1 = M^2 - \lambda^2 \quad \text{for } \lambda^2 \ll 1$$

while

$$\Gamma^2 = \begin{cases} \dfrac{M^2 \breve{\omega}^2 \lambda^2}{(M^2 - \lambda^2)^2} + \left(\dfrac{nl}{R}\right)^2 \dfrac{\lambda^2}{M^2 - \lambda^2} & \text{for } \lambda^2 \gg 1 \\[2ex] \dfrac{M^2 \breve{\omega}^2}{(M^2 - 1)^2} + \left(\dfrac{nl}{R}\right)^2 (M^2 - 1)^{-1} & \text{for } \lambda^2 \ll 1 . \end{cases}$$

Another form ot the magnetoaerodynamic pressure given by:

$$(17)\qquad \mathrm{P}\,|_{\bar{x}_3=1} = \mathscr{C} c_1 \left[\frac{\partial w}{\partial \bar{x}_1} + \frac{l}{\mathrm{U}}\left(\frac{2\mathrm{M}^2}{c_1} - \tilde{\Psi}\right)\frac{\partial w}{\partial t} + \int_0^{\bar{x}_1} \mathrm{Q}(\bar{\xi}_1)\, w(\bar{x}_1 - \bar{\xi}_1)\, d\bar{\xi}_1\right]$$

may easily be obtained from (15) through successive integrations by parts, where

$$\tilde{\Psi} = \begin{cases} \mathrm{M}^2(\mathrm{M}^2 - 1)^{-1} & \text{for } \lambda^2 \leqslant 1 \\ \mathrm{M}^2(\mathrm{M}^2 - \lambda^2)^{-1} & \text{for } \lambda^2 \geqslant 1 \end{cases}$$

while

$$\mathrm{Q}(\xi) = e^{-j\bar{\omega}\xi}\left[-\left(\bar{\omega}^2 + \frac{\Gamma^2}{2} - \frac{2\mathrm{M}^2\check{\omega}\bar{\omega}}{c_1} + \frac{\mathrm{M}^2\check{\omega}^2}{c_1}\right)\mathrm{J}_0(\Gamma\xi) + \right.$$
$$\left. + 2j\Gamma\left(\bar{\omega} - \frac{\mathrm{M}^2\check{\omega}}{c_1}\right)\mathrm{J}_1(\Gamma\xi) + \frac{\Gamma^2}{2}\mathrm{J}_2(\Gamma\xi)\right]$$

J_n being Bessel functions of the first kind.

It is worth remarking that Eqs. (15) and (17) specialized for the non-conducting gas flow, coincide with those derived in [5] and in [6], respectively. In the instance $\check{\omega}^2 \ll 1$, the pressure form (17) reduces to the magnetoaerodynamic quasi-steady approximation, whose classical counterpart has been widely used in the supersonic flutter of cylindrical shells.

ii) Another approximation of $\Psi(\breve{\zeta})$ (see in this sense [7, 8]) valid for large n and small $\breve{\zeta}$ $(l/\mathrm{R} \gg 1)$ results from (14) as

$$(18)\qquad \Psi(\breve{\zeta}) \cong -n^{-1}.$$

Making use of (18) and (10) in Eq. (13), we may compute its original to obtain the following pressure-expression

$$(19)\qquad \mathrm{P}\,|_{\bar{x}_3=1} = -\frac{\mathrm{U}^2\rho_0\mathrm{R}}{nl^2}\left[(1-\lambda^2)w_{,11} + 2j\check{\omega}\, w_{,1} - \check{\omega}^2 w\right],\quad \forall\lambda^2 > 0$$
$$((\ )_{,1} \equiv \partial/\partial\bar{x}_1),$$

which constitutes the magnetoaerodynamic generalization of the so called "slender-body theory aerodynamics" derived first in [7, 8] and subsequently in [9]; it is also discussed in [10, 11].

4. In the case of finitely long circular cylindrical duct with an *internal*, supersonic, perfectly conducting gas flow, a magnetic field (with $\mathbf{U} \,\|\, \mathbf{H}_0$) being also present, the determination of the unsteady magnetoaerodynamic pressure expression leads to the governing equation (8). Its solution, subjected to the condition (11) and the finiteness condition at $x_3 = 0$, yields for what concerns $\mathring{\rho}_0(s, x_3)$ and $\mathring{\mathrm{P}}\,|_{\bar{x}_3=1}$ expressions similar to those given by (12) and (13), respectively, where $\mathrm{K}_n(\zeta)$ is to be replaced by $\mathrm{I}_n(\zeta)$ (the modified Bessel function of the first kind).

For reasons already discussed, two asymptotic evaluations of $\varphi(\breve{\zeta}) \equiv \mathrm{I}_n(\breve{\zeta})/(\breve{\zeta}\mathrm{I}'_n(\breve{\zeta}))$ shall be used, i.e.: 1) $\varphi(\breve{\zeta}) \cong (n^2 - \breve{\zeta}^2)^{-\frac{1}{2}}$ (see e.g. [4]) and

ii) $\varphi(\breve{\zeta}) \cong n^{-1}$, obtained respectively under similar conditions as for $\Psi(\breve{\zeta})$. Within the two above mentioned evaluations, one gets the pressure $P_-^{(i,ii)}$ formally expressed as $P_-^{(i,ii)} = -P_+^{(i,ii)}$, where $P_+^{(i,ii)}$ denotes the asymptotic pressure obtained for the external flow conditions, in the case of the evaluations i) or ii) (expressed by Eq. (17) and (19), respectively). The above given result valid for the first-order approximation theories only, is also met in the classical case of the non-conducting gas flow (see e.g. [4]). In the case of the *simultaneous* internal and external flow of a supersonic ideally conducting gas with $U_+ = U_- = U$ and $\mathbf{U} \parallel \mathbf{H}_0$, within the above considered approximate theories i) and ii) one gets:

$$P_t^{(i,ii)} = P_+^{(i,ii)} - P_-^{(i,ii)} = 2\, P_+^{(i,ii)} .$$

Concerning the applicability ranges of the various magnetoaerodynamic approximate theories in the flutter analysis of cylindrical shells of finite length, the conclusions drawn in the classical case in [10–13] deserve well to be continued in the present case, too.

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