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On polarities of symmetric semi partial geometries

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Geometrie finite. — *On polarities of symmetric semi partial geometries.* Nota di INGRID DEBROEY e JOSEPH A. THAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Dopo aver data la definizione delle geometrie semiparziali, ed in particolare di quelle simmetriche, si danno esempi di geometrie semiparziali che non sono geometrie parziali. Si studiano poi le polarità nelle geometrie semiparziali simmetriche e di esse si forniscono esempi.

1. INTRODUCTION

A semi partial geometry is a finite incidence structure $S = (P, B, I)$ for which the following properties are satisfied:

(i) any point is incident with $u + 1$ $(u \geq 1)$ lines and two distinct points are incident with at most one line;

(ii) any line is incident with $s + 1$ $(s \geq 1)$ points and two distinct lines are incident with at most one point;

(iii) if two points are not collinear, then there are α $(\alpha > 0)$ points collinear with both;

(iv) if a point x and a line L are not incident, then there are 0 or t $(t \geq 1)$ points x_i and respectively 0 or t lines L_i such that $x \mathrm{I} L_i \mathrm{I} x_i \mathrm{I} L$.

For any point x and any line L which are not incident, we define $\langle x, L\rangle$ to be the number of points incident with L and collinear with x.

It is clear that $t \leq \min(u + 1, s + 1)$. If $t = s + 1$, any two points are collinear, and S is a 2-design. For the remainder of this paper we suppose that this is not the case, and so $t \leq \min(u + 1, s)$.

If $|P| = v$ and $|B| = b$, then $b = v(u + 1)/(s + 1)$, where

$$v = 1 + (u + 1) s (1 + u(s - t + 1)/\alpha) \; [4].$$

Now we consider the graph G whose vertices are the points of S and where two distinct vertices are adjacent if and only if the corresponding points are collinear. The adjacency matrix of this graph is denoted by A. If we define $D = (u(t - 1) + s - 1 - \alpha)^2 + 4((u + 1)s - \alpha)$, then A has eigenvalues $c_0 = (u + 1)s$, $c_1 = (u(t - 1) + s - 1 - \alpha + \sqrt{D})/2$ and $c_2 = (u(t - 1) + s - 1 - \alpha - \sqrt{D})/2$, with resp. multiplicities $m_0 = 1$,

$$m_1 = (-(u+1)s - ((v - 1)(u(t - 1) + s - 1 - \alpha)/2) + (v - 1)\sqrt{D}/2)/\sqrt{D}$$

and

$$m_2 = ((u+1)s + ((v - 1)(u(t - 1) + s - 1 - \alpha)/2) + (v - 1)\sqrt{D}/2)/\sqrt{D} \; [4].$$

(*) Nella seduta del 14 maggio 1977.

We also remark that D always is a square, except for the case $u = s = t = = \alpha = 1$ where $D = 5$ (and then S is a pentagon) [4].

The semi partial geometry $S = (P, B, I)$ is called symmetric if $u = s = n$, i.e. if $v = b = 1 + (n+1)n(1 + n(n-t+1)/\alpha)$. In this case we have

$$D = (nt - 1 - \alpha)^2 + 4(n(n+1) - \alpha) \quad , \quad c_0 = n(n+1) \quad ,$$

$$c_1 = (nt - 1 - \alpha + \sqrt{D})/2 \quad , \quad c_2 = (nt - 1 - \alpha - \sqrt{D})/2 \quad , \quad m_0 = 1,$$

$$m_1 = (-(n+1)n - ((v-1)(nt-1-\alpha)/2) + (v-1)\sqrt{D}/2)/\sqrt{D}$$

and

$$m_2 = (n(n+1) + ((v-1)(nt-1-\alpha)/2) + (v-1)\sqrt{D}/2)/\sqrt{D}\,.$$

Symmetric partial geometries are studied in [8] (a partial geometry is a semi partial geometry for which $(u+1)t = \alpha$).

2. Examples of symmetric semi partial geometries which are not partial geometries

2.1. Let G be a graph with valency $r\,(> 1)$, girth 5, and the minimal number $1 + r^2$ of vertices [1]. Then necessarily $r \in \{2, 3, 7, 57\}$ [1]. If $r = 2$, G is isomorphic to the pentagon; if $r = 3$, G is isomorphic to the Petersen graph [1]; if $r = 7$, G is isomorphic to the graph of Hoffman and Singleton [7]; for $r = 57$ it is not known whether or not such a graph exists [1].

Now we define P to be the set of vertices of G, and B to be the set $\{C_x \| x \in P\}$, with $C_x = \{y \in P \| y \sim x\}$. If I is the natural incidence relation, then $S = (P, B, I)$ is a semi partial geometry with parameters $u = s = t = r - 1$ and $\alpha = (r-1)^2$. In particular, if $r = 2$, S is isomorphic to the pentagon; if $r = 3$, S is isomorphic to the Desargues configuration [4].

2.2. Define P to be the set of lines of PG$(4, q)$, B to be the set of planes of PG$(4, q)$, and I to be the inclusion relation.

Then $S = (P, B, I)$ is a semi partial geometry with parameters $u = s = = q^2 + q$, $t = q + 1$ and $\alpha = (q+1)^2$ [4].

3. Polarities

3.1. THEOREM. *Let π be a polarity of the symmetric semi partial geometry $S = (P, B, I)$ with parameters n, t and α. If δ is the number of absolute points of π, then*

$$\delta = n + 1 + s_1\sqrt{(nt + 2n + 1 - \alpha + \sqrt{D})/2} + s_2\sqrt{(nt + 2n + 1 - \alpha - \sqrt{D})/2}\,,$$

with $s_1 \equiv m_1 \pmod 2$ and $s_2 \equiv m_2 \pmod 2$.

Proof. Suppose $P = \{x_1, \cdots, x_v\}$ and $B = \{L_1, \cdots, L_v\}$ with $x_i^\pi = L_i$ $(i = 1, \cdots v)$. Then the incidence matrix Q of $S = (P, B, I)$ is symmetric.

Remark that $n+1$ is an eigenvalue of Q and that $Q^2 = A + (n+1) I_v$, where I_v is the identity matrix of order v. From this it follows that the eigenvalues of Q are given by

$$n+1 \quad , \quad \sqrt{(nt+2n+1-\alpha+\sqrt{D})/2} \quad , \quad -\sqrt{(nt+2n+1-\alpha+\sqrt{D})/2} \, ,$$

$$\sqrt{(nt+2n+1-\alpha-\sqrt{D})/2}$$

and

$$-\sqrt{(nt+2n+1-\alpha-\sqrt{D})/2}$$

with resp. multiplicities $1, s_1^1, s_1^2, s_2^1$ and s_2^2, where $s_1^1 + s_1^2 = m_1$ and $s_2^1 + s_2^2 = m_2$. As the number of absolute points of π is equal to tr Q, we have

$$\delta = n + 1 + (s_1^1 - s_1^2)\sqrt{(nt+2n+1-\alpha+\sqrt{D})/2} + $$
$$+ (s_2^1 - s_2^2)\sqrt{(nt+2n+1-\alpha-\sqrt{D})/2}$$

which proves the theorem.

Remarks. If we define δ_1 to be the number of non-absolute points x for which $\langle x, x^\pi\rangle = t$, then we get in an analogous way that

$$\delta_1 = (n^2(n+1)/t) + (s_{11}\sqrt{A_1 + B_1\sqrt{D}}/t\sqrt{2}) + (s_{12}\sqrt{A_1 - B_1\sqrt{D}}/t\sqrt{2}) \, ,$$

$$\delta + \delta_1 = ((n^2+t)(n+1)/t) + (s_{21}\sqrt{A_2 + B_2\sqrt{D}}/t\sqrt{2}) + (s_{22}\sqrt{A_2 - B_2\sqrt{D}}/t\sqrt{2})$$

and

$$\delta - \delta_1 = (-(n^2-t)(n+1)/t) + (s_{31}\sqrt{A_3 + B_3\sqrt{D}}/t\sqrt{2}) + (s_{32}\sqrt{A_3 - B_3\sqrt{D}}/t\sqrt{2}),$$

with

$$s_{i1} \equiv m_1 \pmod 2 \; (i = 1, 2, 3) \, , \; s_{i2} \equiv m_2 \pmod 2 \; (i = 1, 2, 3) \, ,$$

$$A_1 = n^3 t^3 - n^3 t^2 + 2 n^3 t - 3\alpha n^2 t^2 + 2\alpha n^2 t - 2\alpha n^2 - 2 n^2 t^2 +$$
$$+ 3 n^2 t + 3\alpha^2 nt - \alpha^2 n + \alpha nt - \alpha n + nt - \alpha^3 + \alpha^2 \, ,$$

$$B_1 = n^2 t^2 - n^2 t - 2\alpha nt + \alpha n - nt + \alpha^2 \, ,$$

$$A_2 = A_1 + 2 n^2 t^3 - 4\alpha nt^2 + nt^3 + 2\alpha^2 t - \alpha t^2 - 2\alpha t + t^2$$

$$B_2 = B_1 + 2 nt^2 - 2\alpha t + t^2 \, ,$$

$$A_3 = A_1 - 2 n^2 t^3 + 4\alpha nt^2 + nt^3 + 4 nt^2 - 2\alpha^2 t - \alpha t^2 + 2\alpha t + t^2$$

and

$$B_3 = B_1 - 2 nt^2 + 2\alpha t + t^2 \, .$$

3.2. THEOREM. *Let π be a polarity of the symmetric semi partial geometry* $S = (P, B, I)$ *with parameters n, t and α. If the number δ of absolute points of π is different from 0, then $\delta \geq n+1$ if t is even and $\delta \geq n + 1 - n/t$ if t is odd. Moreover, if for any line* $L \in B$ $\delta(L)$ *denotes the number of absolute* *points incident with* L, *then for any non-absolute line* L $\langle L^\pi, L\rangle = 0$ *implies* $\delta(L) = 0$, *and* $\langle L^\pi, L\rangle = t$ *implies* $\delta(L) \leq t$ *and* $\delta(L) \equiv t$ (*mod* 2).

Proof. Let L be a non-absolute line such that $\langle x, L\rangle = 0$, with $x = L^\pi$. Suppose there exists an absolute point y incident with L. Then x and y are incident with y^π, a contradiction.

Let L be a non-absolute line such that $\langle x, L\rangle = t$, with $x = L^\pi$. Denote by $x_1, \cdots, x_t$ and $L_1, \cdots, L_t$ the points and lines of S defined by $x \,\mathrm{I} L_i \,\mathrm{I}\, x_i \,\mathrm{I} L$. It is clear that the mapping $\sigma : \{x_1, \cdots, x_t\} \to \{x_1, \cdots, x_t\}$, $x_i \to y_i$, where $y_i \,\mathrm{I} L$ and $y_i \,\mathrm{I}\, x_i^\pi$, is a permutation of order 2 whose fixed points are precisely the absolute points of π incident with L. And consequently $\delta(L) \leq t$ and $\delta(L) \equiv t$ (mod 2).

Now we consider the case where t is even. Suppose $\delta > 0$ and let x be an absolute point of π. Any line M different from $L = x^\pi$ and incident with x is non-absolute. Moreover $\langle M^\pi, M\rangle = t$, and so $\delta(M) \equiv t$ (mod 2) $\equiv 0$ (mod 2). So $\delta(M) \geq 1$ implies $\delta(M) \geq 2$ and thus $\delta \geq n + 1$.

Now we consider the case where t is odd. Suppose $\delta > 0$ and let x be a non-absolute point for which $\langle x, L\rangle = t$ with $L = x^\pi$ (if M is an absolute line and $x \,\mathrm{I} M$, $x \neq M^\pi$, then $\langle x, L\rangle = t$ where $L = x^\pi$). For any line M_i ($i = 1, \cdots, t$) incident with x and concurrent with L, we have $\delta(M_i) = 1$ if M_i is an absolute line, and $\delta(M_i) \equiv t$ (mod 2) (and so $\delta(M_i) \geq 1$) if M_i is a non-absolute line. So the number of absolute points of π collinear with x is at least t. Now, let y be an absolute point of π and let $M = y^\pi$. Then any point $x \neq y$ and incident with M is non-absolute and $\langle x, x^\pi\rangle = t$. So x is collinear with at least $t - 1$ absolute points different from y. Since each absolute point different from y is collinear with at most t non-absolute points of M, we get

$$\delta \geq (n(t-1)/t) + 1 = n + 1 - n/t .$$

COROLLARY 1. *Let* π *be a polarity of the symmetric semi partial geometry* $S = (P, B, I)$ *with parameters* n, t *and* α. *If* $0 < \delta \leq t$, *then* $t = \delta = 1$, *and so* S *is a symmetric partial quadrangle* [3] (*a partial quadrangle is a semi partial geometry for which* $t = 1$).

Proof. Let $0 < \delta \leq t$. If t is even, then $\delta \geq n + 1$ and so $t = n + 1$, a contradiction (see 1). Hence t is odd. From the proof of the preceding theorem there follows that $\delta \geq t$, and consequently $\delta = t$. So each absolute point different from the absolute point y, is collinear with the n non-absolute points of $M = y^\pi$. Hence $t = 1$ or $t = n$ (since $t \neq n + 1$). Now we suppose that $t = n$. From [4] follows that S is a (symmetric) net or the pentagon. It is easy to show that for the pentagon $\delta \in \{0, 2\}$ (see 4).

Moreover, if S is a net and $\delta > 0$, then $\delta \geq n + 1$ [8] and so $t = n + 1$, a contradiction. We conclude that S is a symmetric partial quadrangle.

COROLLARY 2. $\delta \leq vt/(n + t)$.

Proof. On each absolute line there is exactly one absolute point, and on each non-absolute line there are at most t absolute points, so $\delta \leq (\delta + (v - \delta)\, t)/(n + 1)$, which proves that $\delta \leq vt/(n + t)$.

Remark that if $S = (P , B , I)$ is a symmetric generalized quadrangle, equality holds [8] (a generalized quadrangle is a partial geometry for which $t = 1$).

3.3. THEOREM. *Let π be a polarity of the symmetric semi partial geometry $S = (P , B , I)$ with parameters n , t and α. If the number of absolute points of π is denoted by δ, then*

$$\delta \leq \frac{v\,(1 + \sqrt{(nt + 2n + 1 - \alpha + \sqrt{D})/2})}{n + 1 + \sqrt{(nt + 2n + 1 - \alpha + \sqrt{D})/2}} .$$

Proof. Let $x_1 , \cdots , x_\delta$ be the absolute points of π, let

$P = \{x_1 , \cdots , x_v\}$ and let $B = \{L_1 , \cdots , L_v\}$, where $x_i^\pi = L_i\ (i = 1 , \cdots , v)$.

Then the corresponding symmetric incidence matrix Q of S is of the form

$$Q = \begin{bmatrix} I_\delta & U \\ U^T & V \end{bmatrix}, \quad \text{with } I_\delta = [\delta_{ij}]_{\substack{1\leq i\leq\delta \\ 1\leq j\leq\delta}}$$

the identity matrix of order δ

$$U = [u_{ij}]_{\substack{1\leq i\leq\delta \\ 1\leq j\leq v-\delta}} \quad , \quad V = [v_{ij}]_{\substack{1\leq i\leq v-\delta \\ 1\leq j\leq v-\delta}} \quad \text{and } V = V^T .$$

Now we put

$$d_{11} = \left(\sum_{i=1}^{\delta}\sum_{j=1}^{\delta}\delta_{ij}\right)\Big/\delta \quad , \quad d_{12} = \left(\sum_{i=1}^{\delta}\sum_{j=1}^{v-\delta}u_{ij}\right)\Big/\delta ,$$

$$d_{21} = \left(\sum_{i=1}^{v-\delta}\sum_{j=1}^{\delta}u_{ji}\right)\Big/(v-\delta) \quad , \quad d_{22} = \left(\sum_{i=1}^{v-\delta}\sum_{j=1}^{v-\delta}v_{ij}\right)\Big/(v-\delta) .$$

If $\lambda_1 , \lambda_2 , \cdots , \lambda_v$, with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_v$, are the eigenvalues of Q and if μ_1 , μ_2, with $\mu_1 \leq \mu_2$, are the eigenvalues of $R = [d_{ij}]_{1\leq i,j\leq 2}$, then a result of Sims (page 144 of [6]) gives us $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_v$. As

$$\lambda_1 \geq -\sqrt{(nt + 2n + 1 - \alpha + \sqrt{D})/2} \quad \text{and} \quad \lambda_v = n + 1 ,$$

there follows that

$$-\sqrt{(nt + 2n + 1 - \alpha + \sqrt{D})/2} \leq \mu_1 \leq \mu_2 \leq n + 1 .$$

Now we determine the eigenvalues μ_1 and μ_2 of R. We see immediately that $d_{11} + d_{12} = d_{21} + d_{22} = n + 1$, $d_{11} = 1$ and $(v - \delta)\, d_{21} = \delta d_{12}$.

There results that $d_{11} = 1$, $d_{12} = n$, $d_{21} = \delta n/(v - \delta)$ and $d_{22} = n + 1 - (\delta n/(v - \delta))$.

Consequently, the eigenvalues of R are $\mu_1 = 1 - (\delta n/(v - \delta))$ and $\mu_2 = n + 1$.

So we have

$$-\sqrt{(nt+2n+1-\alpha+\sqrt{D})/2} \leq 1-(\delta n/(v-\delta)).$$

Finally we obtain

$$\delta \leq \frac{v\,(1+\sqrt{(nt+2n+1-\alpha+\sqrt{D})/2}}{n+1+\sqrt{(nt+2n+1-\alpha+\sqrt{D})/2}}\,.$$

COROLLARY:

$$\delta \leq \min\left(\frac{vt}{n+t}, \frac{v\,(1+\sqrt{(nt+2n+1-\alpha+\sqrt{D})/2})}{n+1+\sqrt{(nt+2n+1-\alpha+\sqrt{D})/2}}\right).$$

4. EXAMPLES

(For examples and particular cases of polarities of symmetric partial geometries see [8]).

4.1. Define $P=\{x_1, x_2, x_3, x_4, x_5\}$, $B=\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_1\}\}$ and I the natural incidence relation. Then $S=(P, B, I)$ is a symmetric semi partial geometry with parameters $n=t=\alpha=1$ (remark that S is isomorphic to the pentagon).

Now, let φ_0 and φ_1 be the mappings from P onto B defined by $x_1^{\varphi_0} = \{x_3, x_4\}$, $x_2^{\varphi_0}=\{x_4, x_5\}$, $x_3^{\varphi_0}=\{x_5, x_1\}$, $x_4^{\varphi_0}=\{x_1, x_2\}$, $x_5^{\varphi_0}=\{x_2, x_3\}$, and $x_1^{\varphi_1}=\{x_3, x_4\}$, $x_2^{\varphi_1}=\{x_2, x_3\}$, $x_3^{\varphi_1}=\{x_1, x_2\}$, $x_4^{\varphi_1}=\{x_1, x_5\}$, $x_5^{\varphi_1} = \{x_4, x_5\}$. The mapping φ_0 (resp. φ_1) induces a polarity π_0 (resp. π_1) onto S. It is easy to see that up to an automorphism π_0 and π_1 are the only polarities of S. For π_0 we have $\delta=0$, and for π_1 there holds $\delta=2\ (=n+1)$.

4.2. Define $P=\{u, x, x_1, x_2, y, y_1, y_2, z, z_1, z_2\}$, $B=\{\{u, x_1, x_2\}, \{u, y_1, y_2\}, \{u, z_1, z_2\}, \{x_1, z_1, y\}, \{y_1, x_1, z\}, \{z_1, y_1, x\}, \{x_2, z_2, y\}, \{y_2, x_2, z\}, \{z_2, y_2, x\}, \{x, y, z\}\}$, and I the natural incidence relation. Then $S=(P, B, I)$ is a symmetric semi partial geometry with parameters $n=t=2$ and $\alpha=4$ (S is the Desargues configuration). Now, let φ, φ_0 and φ_1 be the mappings from P onto B defined by $u^{\varphi}=\{x, y, z\}$, $x_1^{\varphi}=\{z_2, y_2, x\}$, $x_2^{\varphi}=\{y_1, z_1, x\}$, $y_1^{\varphi}=\{z_2, x_2, y\}$, $y_2^{\varphi}=\{x_1, z_1, y\}$, $z_1^{\varphi}=\{x_2, y_2, z\}$, $z_2^{\varphi}=\{x_1, y_1, z\}$, $x^{\varphi}=\{x_1, x_2, u\}$, $y^{\varphi}=\{y_1, y_2, u\}$, $z^{\varphi}=\{z_1, z_2, u\}$, by $u^{\varphi_0}=\{y_1, z_1, x\}$, $x^{\varphi_0}=\{u, x_1, x_2\}$, $x_1^{\varphi_0}=\{y_2, z_2, x\}$, $x_2^{\varphi_0}=\{x, y, z\}$, $y^{\varphi_0}=\{x_2, y_2, z\}$, $y_1^{\varphi_0}=\{u, z_1, z_2\}$, $y_2^{\varphi_0}=\{x_1, z_1, y\}$, $z^{\varphi_0}=\{x_2, z_2, y\}$, $z_1^{\varphi_0}=\{u, y_1, y_2\}$, $z_2^{\varphi_0}=\{x_1, y_1, z\}$ and by $u^{\varphi_1}=\{u, y_1, y_2\}$, $y_1^{\varphi_1}=\{u, x_1, x_2\}$, $y_2^{\varphi_1}=\{u, z_1, z_2\}$, $x_1^{\varphi_1}=\{y_1, x_1, z\}$, $x_2^{\varphi_1}=\{y_1, z_1, x\}$, $z_1^{\varphi_1}=\{x_2, y_2, z\}$, $z_2^{\varphi_1}=\{y_2, z_2, x\}$, $x^{\varphi_1}=\{z_2, x_2, y\}$, $z^{\varphi_1}=\{x_1, z_1, y\}$, $y^{\varphi_1}=\{x, z, y\}$. The mapping φ (resp. φ_0, φ_1) induces a polarity π (resp. π_0, π_1) onto S. It is easy to see that up to an automorphism π, π_0 and π_1 are the only polarities of S. For π and π_0 there holds $\delta=0$ and for π_1 there holds $\delta=4$ (from 3. there follows that $\delta=0$ or $3\leq\delta\leq 5$).

4.3. Let $S = (P, B, I)$ be the symmetric semi partial geometry with parameters $n = t = 6$, $\alpha = 36$ (see 2.1). From 3. there follows that $7 \leq \delta \leq \leq 20$ or $\delta = 0$. If $x \in P$, then there is exactly one element $L \in B$ for which $\langle x, L \rangle = 0$. Then by defining x^π to be L, we obtain a polarity of S for which $\delta = 0$.

4.4. Define P to be the set of lines of PG $(4, q)$, B to be the set of planes of PG $(4, q)$, and I to be the inclusion relation. Then $S = (P, B, I)$ is a symmetric semi partial geometry with parameters $n = q^2 + q$, $t = q + 1$ and $\alpha = (q+1)^2$ [4]. If π is a polarity of PG $(4, q)$, then π induces a polarity π_0 of S, and conversely. It is clear that the absolute points of π_0 are the totally isotropic lines with respect to π. So the number of absolute points is given by $(q\sqrt{q}+1)(q^2\sqrt{q}+1)$ if π is a unitary polarity and by q^3+q^2+q+1 if π is an orthogonal polarity [5].

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