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Pairs of linear connections and Banach-Lie groups

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Geometria differenziale. — *Pairs of linear connections and Banach–Lie groups.* Nota di AUREL BEJANCU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Usando una coppia di connessioni lineari su una varietà di Banach, si ottengono condizioni necessarie e sufficienti per l'esistenza di una struttura locale di gruppo di Lie–Banach.

We have found necessary and sufficient conditions for the existence of a local structure of Banach-Lie group induced by a linear connection on a Banach manifold [1]. The purpose of the present Note is to establish such conditions using a pair of linear connections on a Banach manifold.

Let M be a C^∞–manifold on a real Banach space $\mathbf{E}$ and (U, φ) be a local chart on M. The study which follows will be made on the domain of this local chart. Let ∇ be a linear connection on M and Γ be its function of connection on U [1], [2]. Denote by (∇', Γ') the linear connection transposed to the linear connection (∇, Γ), that is

$$\Gamma' : U \times \mathbf{E} \times \mathbf{E} \to \mathbf{E} \quad , \quad \Gamma'(x ; u , v) = \Gamma(x ; v , u) \qquad \forall x \in U , u , v \in \mathbf{E} .$$

Let $(\overset{(a)}{\nabla}, \overset{(b)}{\nabla})$ be a pair of linear connections on M. Define:

1) *the deformation tensor*

$$T_{(a,b)} : U \times \mathbf{E} \times \mathbf{E} \to \mathbf{E} \quad , \quad T_{(a,b)}(x ; u , v) = \overset{(a)}{\Gamma}(x ; u , v) - \overset{(b)}{\Gamma}(x ; u , v) ; \tag{1}$$

2) *the curvature tensor*

$$R_{(a,b;c)} : U \times \mathbf{E} \times \mathbf{E} \times \mathbf{E} \to \mathbf{E} ,$$

$$\begin{aligned} R_{(a,b;c)}(x ; u , v , w) &= \left(\frac{\partial \overset{(a)}{\Gamma}}{\partial x}\right)(x ; v , w)(u) - \left(\frac{\partial \overset{(b)}{\Gamma}}{\partial x}\right)(x ; u , w)(v) \\ &+ \overset{(b)}{\Gamma}(x ; u , \overset{(a)}{\Gamma}(x ; v , w)) - \overset{(a)}{\Gamma}(x ; v , \overset{(b)}{\Gamma}(x ; u , w)) - T_{(a,b)}(x ; \overset{(c)}{\Gamma}(x ; u , v) , w) , \end{aligned} \tag{2}$$

where $c = a , b , a' , b'$.

In what follows, $L_p(\mathbf{E})$ will denote the Banach space of p-linear continuous operators from $\mathbf{E} \times \mathbf{E} \times \cdots \times \mathbf{E}$ (p times) to $\mathbf{E}$.

DEFINITION. Let $f : U \to L_p(\mathbf{E})$ be a C^1–differentiable mapping and $u \in \mathbf{E}$. The covariant derivative of the mapping f with respect to u, induced

(*) Nella seduta dell'8 gennaio 1977.

by the linear connection (∇, Γ), is the mapping

$$\nabla_u f : \mathrm{U} \to \mathrm{L}_p(\mathbf{E}) ,$$

$$(\nabla_u f)(x ; u_1, \cdots, u_p) = \left(\frac{\partial f}{\partial x}\right)(x ; u_1, \cdots, u_p)(u) + \tag{3}$$

$$+ \Gamma(x ; u, f(x ; u_1, \cdots, u_p)) - \sum_{i=1}^{p} f(x ; u_1, \cdots, u_{i-1}, \Gamma(x ; u, u_i), \cdots, u_p),$$

$$\forall x \in \mathrm{U} , u_i \in \mathbf{E} \ (1 \leq i \leq p) .$$

The linear connection $\overset{(o)}{\nabla}$ will be associated to $\overset{(a)}{\nabla}$, and its function of connection is defined as follows

$$\overset{(0)}{\Gamma} : \mathrm{U} \times \mathbf{E} \times \mathbf{E} \to \mathbf{E} \quad , \quad \overset{(0)}{\Gamma}(x ; u , v) = \tfrac{1}{2}[\overset{(a)}{\Gamma}(x ; u , v) + \overset{(a)}{\Gamma}(x ; v , u)]$$

$$\forall x \in \mathrm{U} , u , v \in \mathbf{E} .$$

The usual curvature tensor of the linear connection $\overset{(a)}{\nabla}$ (resp. $\overset{(0)}{\nabla}$) will be denoted by R (resp. L). Now, we replace $\overset{(a)}{\Gamma}$ (resp. $\overset{(a')}{\Gamma}$) by $\overset{(0)}{\Gamma} + 2\,\mathrm{T}_{(a,a')}$ (resp. $\overset{(0)}{\Gamma} - 2\,\mathrm{T}_{(a,a')}$) and the curvature tensor $\mathrm{R}_{(a,a';a)}$ for the pair of linear connections $(\overset{a}{\nabla}, \overset{a'}{\nabla})$ takes the form

$$\begin{aligned} \mathrm{R}_{(a,a';a)}(x ; u , v , w) &= \mathrm{L}(x ; u , v , w) + 2\,[(\nabla_u \mathrm{T}_{(a,a')})(x ; v , w) + \\ &+ (\nabla_v \mathrm{T}_{(a,a')})(x ; u , w)] + 4\,[\mathrm{T}_{(a,a')}(x ; v , \mathrm{T}_{(a,a')}(x ; u , w)) - \\ &- \mathrm{T}_{(a,a')}(x ; u , \mathrm{T}_{(a,a')}(x ; v , w))] + 8\,\mathrm{T}_{(a,a')}(x ; \mathrm{T}_{(a,a')}(x ; u , v) , w) . \end{aligned} \tag{4}$$

Now, we can state

THEOREM 1. *All the curvature tensors for the pair of linear connections* $(\overset{(a)}{\nabla}, \overset{(a')}{\nabla})$ *are linear combinations with constant coefficients of tensors* $\mathrm{L}, \mathrm{T}_{(a,a')}$, $\mathrm{T}_{(a,a')}(x) \circ [\mathrm{T}_{(a,a')}(x) \times \mathrm{I}]$, *where* I *is the identity on* $\mathbf{E}$.

The proof follows from (4) and from the form of the other curvature tensor:

$$\mathrm{R}_{(a,a';a')}(x ; u , v , w) = \mathrm{R}_{(a,a';a)}(x ; u , v , w) + 16\,\mathrm{T}_{(a,a')}(x ; \mathrm{T}_{(a,a')}(x ; u , v) , w) .$$

From (4) and Bianchi's identities for a linear connection on a Banach manifold [2] we get

$$\begin{aligned} &\mathrm{R}_{(a,a';a)}(x ; u , v , w) + \mathrm{R}_{(a,a';a)}(x ; v , u , w) = \\ &= 4\,[(\overset{(0)}{\nabla}_u \mathrm{T}_{(a,a')})(x ; v , w) + (\nabla_v \mathrm{T}_{(a,a')})(x ; u , w)] , \end{aligned} \tag{5}$$

$$\sum_{\text{cycl}}^{(u,v,w)} \{\mathrm{R}_{(a,a';a)}(x ; u , v , w)\} = 16 \sum_{\text{cycl}}^{(u,v,w)} \{\mathrm{T}_{(a,a')}(x ; \mathrm{T}_{(a,a')}(x ; u , v) , w)\} . \tag{6}$$

Suppose $R_{(a,a';a)} = 0$ and from (5) and (6) we have

$$L(x; u, v, w) = -4\, T_{(a,a')}(x; T_{(a,a')}(x; u, v), w). \tag{7}$$

If, in addition, $R = 0$, then $\overset{(0)}{\nabla} T_{(a,a')} = 0$.

Conversely, assume that (7) and $\overset{(0)}{\nabla} T_{(a,a')} = 0$ hold true. From this assumption and (4) we get $R_{(a,a';a)} = R = 0$. In this way, by Theorem 5 in [1] we have

THEOREM 2. *The pair of linear connections* $(\overset{(a)}{\nabla}, \overset{(a')}{\nabla})$ *induces a local structure of Banach-Lie group such that* (+) *and* (—) *Cartan's connections are the connections* $\overset{(a)}{\nabla}$ *and* $\overset{(a')}{\nabla}$ *if, and only if, one of the following condition is fulfilled:*

1) $R_{(a,a';a)} = R = 0$;

2) $R_{(a,a';a')}(x; u, v, w) = 16\, T_{(a,a')}(x; T_{(a,a')}(x; u, v), w)$, $R = 0$.

By computations similar to the previous ones for the pair of connections $(\overset{(a)}{\nabla}, \overset{(a')}{\nabla})$, we have the curvature tensors for the pair of connections $(\overset{(a)}{\nabla}, \overset{(0)}{\nabla})$

$$R_{(a,0;a)}(x; u, v, w) = L(x; u, v, w) + 2\,(\overset{(0)}{\nabla}_u T_{(a,a')})(x; v, w) - \\ - 4\, T_{(a,a')}(x; T_{(a,a')}(x; u, v), w),$$

$$R_{(a,0;a')}(x; u, v, w) = L(x; u, v, w) + 2\,(\overset{(0)}{\nabla}_u T_{(a,a')})(x; v, w) + \\ + 4\, T_{(a,a')}(x; T_{(a,a')}(x; u, v), w),$$

$$R_{(a,0;0)} = [R_{(a,0;a)} + R_{(a,0;a')}]/2,$$

These formulas lead us to state the following two theorems.

THEOREM 3. *All the curvature tensors for the pair of connections* $(\overset{(a)}{\nabla}, \overset{(0)}{\nabla})$ *are linear combinations with constant coefficients of tensors* $L, T_{(a,a')}$ *and* $T_{(a,a')}(x) \circ [T_{(a,a')}(x) \times I]$.

THEOREM 4. *The pair of linear connections* $(\overset{(a)}{\nabla}, \overset{(0)}{\nabla})$ *induces a local structure of Banach-Lie group on* M, *such that* (+) *and* (0)*-Cartan's connections are the connections* $\overset{(a)}{\nabla}$ *and* $\overset{(0)}{\nabla}$ *if, and only if, one of the following conditions is verified*

1) $R_{(a,0;a)}(x; u, v, w) = -8\, T_{(a,a')}(x; T_{(a,a')}(x; u, v), w)$, $\overset{(0)}{\nabla} T_{(a,a')} = 0$;

2) $R_{(a,0;a')} = \overset{(0)}{\nabla}_u T_{(a,a')} = 0$, $\quad \forall u \in \mathbf{E}$;

3) $R_{(a,0;0)}(x; u, v, w) = L(x; u, v, w) = -4\, T_{(a,a')}(x; T_{(a,a')}(x; u, v), w)$.

The curvature tensors for a pair of linear connections on a differentiable manifold of finite dimension have been introduced by L. A. Santaló [3].

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