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## Holomorphy on surjective limits of locally convex spaces

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**Analisi funzionale.** — *Holomorphy on surjective limits of locally convex spaces* (*). Nota di PAUL D. BERNER, presentata (**) dal Corrisp. G. CIMMINO.

RIASSUNTO. — Limiti suriettivi di spazi vettoriali complessi localmente convessi sono stati un utile strumento per la olomorfia a infinite dimensioni. La principale proprietà adoperata è il fatto che una funzione olomorfa di un sottoinsieme aperto di un limite suriettivo si fattorizza mediante proiezioni del limite suriettivo. Noi mostriamo che limiti suriettivi aperti hanno anche una proprietà di fattorizzazione globale per le funzioni olomorfe. Noi utilizziamo questa proprietà più forte per studiare topologie localmente convesse su spazi di funzioni olomorfe e per studiare il problema di Levi per domini distesi.

## INTRODUCTION

Surjective limits of locally convex complex topological vector spaces (l.c.s.) have been a useful tool for infinite dimensional holomorphy (see [1], [2] and [3]). The main property used is the fact that a holomorphic function of an open subset of a surjective limit locally factors through some projection map of the surjective limit. We show that open surjective limits also have a global factorization property for holomorphic functions. We use this stronger property to study locally convex topologies on some spaces of holomorphic functions and to study the Levi problem for domains spread.

## DEFINITIONS

An l.c.s. E is called an *open surjective limit* of l.c. spaces $\{E_\alpha\}_{\alpha \in A}$ if $\{A, \geq\}$ is a directed indexing set and for each $\alpha \in A$ and $\beta \in A$, $\beta \geq \alpha$, there are open and continuous linear surjective maps $\pi_\alpha : E \to E_\alpha$ and $\pi_{\alpha\beta} : E_\beta \to E_\alpha$ such that $\{E_\alpha, \pi_\alpha, \pi_{\alpha\beta}\}_{\beta \geq \alpha}$ forms a projective system and the topology on E is the projective limit topology $E = \varprojlim_{\alpha \in A} (E_\alpha, \pi_\alpha, \pi_{\alpha\beta})_{\beta \geq \alpha}$.

*Example.* The space of distributions over $\mathbf{R}$, $\mathscr{D}'(\mathbf{R})$ is an open surjective limit of the Silva spaces $\{\mathscr{E}'((-n, n))\}_{n \in \mathbf{N}}$.

Given $\Omega$, a domain spread over an l.c.s., $H(\Omega)$ will denote the space of holomorphic functions from $\Omega$ to $\mathbf{C}$.

Let $E = \varprojlim_{\alpha \in A} (E_\alpha, \pi_\alpha, \pi_{\alpha\beta})_{\beta \geq \alpha}$ denote a fixed open surjective limit, and let $\Omega$ be a fixed domain spread over E with local homeomorphism $\varphi : \Omega \to E$.

(*) These results are contained in the Author's 1974 University of Rochester Ph. D. dissertation supervised by Professor Leopoldo Nachbin and written with the guidance of dott. Sean Dineen.

(**) Nella seduta del 10 giugno 1976.

Let $f \in H(\Omega)$ and $\alpha \in A$. $f$ is said to *factor locally through* $\pi_\alpha$, if $\Omega$ is covered by charts, W, such that $f \circ (\varphi|_W)^{-1} = g \circ \pi_{\alpha|\varphi(W)}$ for some $g \in H(\pi_\alpha(\varphi(W)))$. $L_\alpha(\Omega)$ will denote the set of all $f \in H(\Omega)$ which factor locally through $\pi_\alpha$.

PROPOSITION 1. (See [2] or [3]): $H(\Omega) = \bigcup_{\alpha \in A} L_\alpha(\Omega)$.

There is an example in [5] of a connected open subset U of E and an $f \in H(U)$ which factors locally through each $\pi_\alpha$ and yet there does not exist any (single valued) function $h : \pi_\alpha(U) \to \mathbf{C}$ satisfying: $f = h \circ \pi_{\alpha|U}$ for any $\alpha \in A$. We can, however, write $f$ in the form $f = h \circ J_\alpha$ where $h$ is defined on a domain spread over $E_\alpha$ and $J_\alpha$ is locally the same as $\pi_\alpha$ as follows:

THEOREM 1. *If $\Omega$ is a connected domain spread over E with local homeomorphism $\varphi$, then there is a cofinal subset A, $A_\Omega$, such that for each $\alpha \in A_\Omega$ there exists a domain $\Omega_\alpha$ spread over $E_\alpha$ with local homeomorphism $\varphi_\alpha$ and a continuous surjective map $J_\alpha : \Omega \to \Omega_\alpha$ satisfying*:

(1) $\varphi_\alpha \circ J_\alpha = \pi_\alpha \circ \varphi$,

(2) *For all $f \in L_\alpha(\Omega)$ there exists an $f_\alpha \in H(\Omega_\alpha)$ such that $f = f_\alpha \circ J_{\alpha'}$*

(3) *If $\Sigma_\alpha$ is a connected domain spread over $E_\alpha$ and $K_\alpha : \Omega \to \Sigma_\alpha$ is a continuous surjective map satisfying the corresponding conditions* (1) *and* (2) *then there exists a unique morphism of domains* $M : \Sigma_\alpha \to \Omega_\alpha$ *such that* $M \circ K_\alpha = J_\alpha$.

(4) *If $\beta \in A$ and $\beta \geq \alpha$, there exists a continuous map $J_{\alpha\beta} : \Omega_\beta \to \Omega_\alpha$ such that $J_\alpha = J_{\alpha\beta} \circ J_\beta$.*

*Furthermore*: (a) *$\Omega$ is pseudo-convex iff $\Omega_\alpha$ is pseudo-convex*;

(b) *$\Omega$ is an $L_\alpha(\Omega)$–domain of holomorphy if $\Omega_\alpha$ is a domain of holomorphy*;

(c) *$\Omega$ is a domain of existence for some $f \in L_\alpha(\Omega)$ iff $\Omega_\alpha$ is a domain of existence.*

## APPLICATIONS

It is clear from Theorem 1, that if, over each $E_\alpha$, pseudo-convex domains coincide with domains of existence, then the same is true for domains spread over E. Hence we have that if the Levi-problem always has a solution in a collection of l.c. spaces, then it has a solution in any open surjective limit of those spaces. See [6] for the usefulness of this result.

Given a connected domain spread over E, $\Omega$, each of the maps $J_\alpha$ [resp. $J_{\alpha\beta}$, $\beta \geq \alpha$], $\alpha \in A_\Omega$, induces an injection $J^\alpha$ [resp. $J^{\alpha\beta}$] : $f \in H(\Omega_\alpha) \mapsto f \circ J_\alpha \in H(\Omega)$ [resp. $f \circ J_{\alpha\beta} \in H(\Omega_\beta)$]. Theorem 1 implies that $H(\Omega) = \bigcup_{\alpha \in A_\Omega} J^\alpha H(\Omega_\alpha)$, and $\{H(\Omega_\alpha), J^{\alpha\beta} \mid \alpha, \beta \in A_\Omega, \beta \geq \alpha\}$ forms an inductive system for the compact open topologies $\mathscr{T}_0$. Hence we may define an inductive limit topology $\mathscr{T}_1$ on $H(\Omega)$ by:

$$(H(\Omega), \mathscr{T}_1) \equiv \varinjlim_{\alpha \in A_\Omega} ((H(\Omega), \mathscr{T}_0), J^\alpha, J^{\alpha\beta})_{\beta \geq \alpha}.$$

If E is also a *compact* open surjective limit (that is for all $\alpha \in A$, each compact subset of $E_\alpha$ is the image under $\pi_\alpha$ of a compact subset of E) then $\mathscr{T}_1$ is a strict inductive limit of closed subspaces $\{J^\alpha H(\Omega_\alpha), \mathscr{T}_0\}_{\alpha \in A_\Omega}$. Our example $\mathscr{D}'(\mathbf{R})$ is a compact open surjective limit of Silva spaces.

THEOREM 2. *Let* E *be a compact open surjective limit of Silva spaces with indexing set* $A = \mathbf{N}$. *Let* $\Omega$ *be a connected domain spread over* E, *then*:

(a) $(H(\Omega), \mathscr{T}_I)$ *is a strict* (LF)-*Montel space*,

(b) $\mathscr{X} \subset H(\Omega)$ *is* $\mathscr{T}_I$-*bounded iff for some* $\alpha \in A_\Omega$,
$\mathscr{X} \subset J^\alpha_* H(\Omega_\alpha)$ *and is* $\mathscr{T}_0$-*bounded there, and*

(c) $\mathscr{T}_I = \mathscr{T}_\delta$ on $H(\Omega)$. (See [4] for a definition of the $\mathscr{T}_\delta$ topology).

Parts (a) and (b) follow from our above remarks and following proposition:

PROPOSITION 2. *If* $\Omega$ *is a connected domain spread over a Silva-space then* $(H(\Omega), \mathscr{T}_0)$ *is a Fréchet Montel space.*

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