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On a formula of Ingleton and Scott

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Geometria algebrica. — *On a formula of Ingleton and Scott.* Nota di ISRAEL VAINSENCHER (*), presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Viene data una portata più larga alla formula (1.1) di Ingleton e Scott [5].

1. INTRODUCTION

Let V denote a smooth variety over an algebraically closed ground field of arbitrary characteristic. Let $V^* = P(\Omega^1_V)$ denote the bundle of tangent directions of V, and $s: V^* \to V$ the bundle map. Let L be an r-dimensional linear system on V, and M the associated invertible O_V-Module. Ingleton and Scott introduced in ([5], p. 365) a subvariety $\mathbf{L}$ of V^* parametrizing the tangent directions t such that either $s(t)$ lies in the base locus of L or t is formally tangent to every member of L passing through $s(t)$. Assuming L is *regular* (in the sense of p. 366, *loc. cit.*), they obtain a formula for the (classical) cohomology class l dual to $\mathbf{L}$,

$$l = \sum_{i=0}^{r} \binom{r+1}{i} s^* m^i v^{r-i}, \tag{1.1}$$

where m (resp. v) denotes the 1st. Chern class of M (resp. $O_{V^*}(1)$, the universal 1-quotient of $s^* \Omega^1_V$).

We prove here (1.1) holds already in the Chow ring of V^*, under the sole assumption that both $\mathbf{L}$ and the base locus B of L have the right codimension, namely, r and $r+1$ respectively. Our proof rests on the observation that $\mathbf{L}$ is naturally the scheme of zeros of a regular section of a certain vector bundle over V^*, whence it represents the top Chern class of that bundle in the Chow ring of V^* ([4], Cor. p. 153).

2. PRELIMINARIES

We borrow from ([1], 2.2. and 2.3) the definition and proposition below.

2.1. DEFINITION. *Let $p: Y \to S$ be a morphism of schemes, and let $u: A \to A'$ be an O_Y-homomorphism of O_Y-Modules. A closed sub-scheme of* S *is called the* scheme of zeros *of u in* S *if it has the universal property that a map $g: T \to S$ factors through it iff $u_T = g^*_Y(u)$ is equal to zero. (Here $g_Y: T \times_S Y \to Y$ is the pull-back).*

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2.2. PROPOSITION. *Let* $p : Y \to S$ *be a morphism of schemes, and let* $u : O_Y \to A$ *be a section of the* O_Y*-Module* A. *Assume* A *is quasi-coherent and* $p_*(A)$ *is locally free and its formation commutes with base change. Then the scheme of zeros of* u *in* S *exists (i.e., is representable by a closed subscheme of* S) *and is equal to the scheme of zeros of the adjoint* $u' : O_S \to p_*(A)$.

3. PRCOF OF (1.1)

We have an exact sequence of Chow groups,

$$A(B^*) \xrightarrow{i_*} A(V^*) \xrightarrow{j^*} A(V^* - B^*) \to 0,$$

([2], [3]), where $B = s^{-1}(B)$ (the pullback of the base locus), and i and j denote inclusion maps. Since the codimension of B^* in V^* is the same as that of B in V, namely $r+1$, the homomorphism j^* is an isomorphism in the relevant codimension r. Thus, it suffices to prove (1.1) after restriction to $V^* - B^*$. Therefore, we may assume B is empty. Now consider the diagram,

(3.1)
$$\begin{tikzcd}
 & & D^* \arrow[dl, "a"'] \arrow[dr, "b"] & \\
 & L\times V \supset D \arrow[dl] \arrow[dr, "q"] & & V^* \arrow[dl, "s"] \\
L & & V &
\end{tikzcd}$$

Here, D denotes the universal divisor of L (incidence correspondence), and D^* is the pullback of D to V^*. Since L has no base points, there is an exact sequence of O_V-Modules,

(3.2)
$$0 \to E \to O_V^{\oplus r+1} \to M \to 0.$$

It is well known that the dual epimorphism $O_V^{\oplus r+1} \to E^\vee$ induces the embedding of projective bundles over V,

$$D = P(E^\vee) \subset P(O_V^{\oplus r+1}) = L\times V.$$

On D^*, we have a diagram of sheaves,

(3.3)
$$\begin{tikzcd}
0 \arrow[r] & O_{D^*} \arrow[r] \arrow[dr, "u"'] & N_{D^*}\otimes a^*\,\Omega^1_{L\times V/L} \arrow[r] \arrow[d] & N_{D^*}\otimes a^*\,\Omega^1_{D/L} \arrow[r] & 0, \\
 & & N_{D^*}\otimes b^*\,O_{V^*}(1) & &
\end{tikzcd}$$

where the horizontal sequence is the pullback to D^* of the sequence of relative differentials and conormal sheaves of $D \subset L\times V$ over L, twisted by the normal line bundle N_D. The vertical map is the pullback to D^* of the universal 1-quotient of $s^*\,\Omega^1_V$, taking into account the identification

$$\Omega^1_{L\times V/L} \mid D = q^*\,\Omega^1_V.$$

Since $N_D = q^* M \otimes O_D(1)$, therefore $N_{D^*} = b^* s^* M \otimes O_{D^*}(1)$. Since D^* is a projective bundle over V^*, $R^1 b_*(O_{D^*}(1)) = 0$. Therefore, the sheaf

$$F = b_*(N_{D^*} \otimes b^*(O_{V^*}(1)))$$

is locally free and its formation commutes with base change ([6], p. 51). Let u' denote the section of F, adjoint of u in (3.3). By (2.2), the scheme of zeros of u' in V^* is equal to the scheme of zeros of u in V^*. However, looking at the restriction of (3.3.) to the fibre of b over a point t in V^* it is not hard to see that u_t is zero iff the cotangent direction corresponding to t is in fact a 1-quotient of the cotangent space of every member of L passing through $s(t)$. Thus, **L** is the scheme of zeros of a section of the bundle F. Equation (1.1) now follows from the lemma below.

4. LEMMA. *The rank of* F *is* r, *and its* r-*th Chern class equals the right hand side of* (1.1).

Proof. We have,

$$\begin{aligned} &\text{(projection formula)} & F &= O_{V^*}(1) \otimes s^* M \otimes b_* O_{D^*}(1) \\ &\text{(flat base change)} & &= O_{V^*}(1) \otimes s^* M \otimes s^* q_* O_D(1) \\ &\text{(Serre's theorem)} & &= O_{V^*}(1) \otimes s^* M \otimes s^* E^{\vee}. \end{aligned}$$

In view of (3.2), we get:

$$c_r F = c_r(O_{V^*}(1) \otimes s^* M^{\oplus r+1})$$

which is (1.1), by standard properties of Chern classes.

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