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# Convergence of Fourier series of almost periodic functions with values in Banach spaces

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**Analisi matematica.** — *Convergence of Fourier series of almost periodic functions with values in Banach spaces* (*). Nota di BOLIS BASIT, presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — La presente Nota concerne la convergenza delle serie di Fourier di funzioni q.p. a valori in uno spazio di Banach. Si assegna una condizione necessaria e sufficiente perché una serie debolmente incondizionatamente convergente lo sia fortemente. Si dimostra inoltre che in certi spazi di Banach $(C(Q), L_p(a, b), l_p)$ se i coefficienti di Fourier sono positivi, allora la serie di Fourier è fortemente incondizionatamente convergente.

In this paper we are concerned with the convergence of Fourier series of almost periodic (ap) functions with values in Banach spaces. Precisely, we are going to discuss two classical results concerning absolute convergence of the Fourier series of ap functions. The first of these results [1] states that if all the coefficients of the Fourier series $\sum_{n=1}^{\infty} a_n e^{i\lambda_n t}$ of the ap function $f: R \to \mathbf{C}$ are positive then the series $\sum_{n=1}^{\infty} a_n$ is convergent. The second result states that if the Fourier exponents of the ap function $f: R \to \mathbf{C}$ are linearly independent [1], then the Fourier series associated with $f(x)$ is uniformly convergent. We generalize these two results to the case of ap functions with values in Banach spaces.

NOTATIONS AND DEFINITIONS. B will denote a Banach space and $B^*$ will denote its conjugate. We put $B_1 = C(Q)$, $B_2 = L_p(a, b)$ and $B_3 = l_p$ $(p \geq 1)$, where C (Q) denotes the Banach space of continuous functions on a compact Hausdorff topological space Q. In this paper we shall deal only with continuous functions.

DEFINITION 1. The function $f: R \to B$ is said to be ap if for each $\varepsilon > 0$ the set $\{\tau : \sup_{t \in R} \| f(t+\tau) - f(t) \| < \varepsilon, \tau \in R\}$ is relatively dense.

DEFINITION 2. The function $f: R \to B$ is said to be weakly almost periodic (wap) if $x^*(f(t))$ is ap for each $x^* \in B^*$.

DEFINITION 3. The function $f: R \to B$ is said to be almost automorphic (*aa*) if for each $\varepsilon > 0$ and each bounded interval K the set $C_\varepsilon(K) = \{\tau : \sup_{t \in K} \| f(t+\tau) - f(t) \| < \varepsilon, \tau \in R\}$ is relatively dense; moreover,

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there exists $\delta(\varepsilon, K) > 0$, K′, where $K \supset K'$, such that $C_\delta(K)' - C_\delta(K') \subset C_\varepsilon(K)$.

The aim of this paper is to prove the following.

THEOREM 1. *If all the coefficients of the Fourier series* $\sum_{n=0}^{\infty} a_n e^{i\lambda_n t}$ *of the ap function* $f: R \to B_i$ $(i = 1, 2, 3)$ *are positive, then* $\sum_{n=0}^{\infty} a_n$ *is strongly unconditionally convergent (suc).*

*Proof.* 1) *The case* $B_1 = C(Q)$.

For each $s_0 \in Q$ the function $x^*_{s_0}(x) = x(s_0)$, $x \in C(Q)$ is an element of $C^*(Q)$. Therefore, $\sum_{n=1}^{\infty} a_n(s_0) e^{i\lambda_n t}$ is the Fourier series of the ap function $x^*_{s_0}(f(t))$. Hence $\sum_{n=0}^{\infty} a_n(s_0)$ is a convergent series. This implies that the series

$$\sum_{n=1}^{\infty} a_n(s) \quad , \quad s \in Q , \tag{1}$$

converges pointwise to $f(0)$. By Dini's theorem the series (1) converges uniformly to $f(0)$. This means that the series $\sum_{n=1}^{\infty} a_n$ is strongly convergent. Since all the coefficients $\{a_n(s)\}$ are positive, we obtain that the series $\sum_{n=1}^{\infty} a_n$ is suc.

2) *The case* $B_2 = L_p(a, b)$, $(p \geq 1)$.

Let $x^* \in L^*_p(a, b)$ be positive functional; this means that there exists a positive function $\psi(t)$, $\psi \in L_q(a, b)$, such that

$$x^*(\varphi) = \int_a^b \varphi(t)\,\psi(t)\,dt \quad , \quad \varphi \in L_p(a, b). \tag{2}$$

Consider the Fourier series $\sum_{n=0}^{\infty} x^*(a_n) e^{i\lambda_n t}$ of the ap function $x^*(f)$. Using (2) we obtain that $x^*(a_n) \geq 0$, $n \geq 1$. Therefore, the series $\sum_{n=1}^{\infty} x^*(a_n)$ is convergent. Since each functional $z^*$ of $L_p(a, b)$ can be written in the form $z^* = x^*_1 - x^*_2 + i(x^*_3 - x^*_4)$, where $x^*_i$ $(i = 1, 2, 3, 4)$ are positive functionals of $L^*_p(a, b)$, we conclude tat the series $\sum_{n=1}^{\infty} z^*(a_n)$ is absolutely convergent for each $z^* \in L^*_p(a, b)$. Since $L_p(a, b)$, $(p \geq 1)$, is weakly sequentially complete, we obtain that the series $\sum_{n=1}^{\infty} a_n$ is suc (Ref. [2], p. 61).

3) *The case* $B_3 = l_p\,(p \geq 1)$

This case can be proved as in Case 2.

Notice that Theorem 1 holds true in the case of wap functions with Fourier coefficients belonging to $B_i\,(i = 1, 2, 3)$. This leads to the following.

Suppose that $f: R \to B$ is a wap function with Fourier coefficients belonging to B (see Ref. [3], p. 43) and with weakly unconditionally convergent (wuc) Fourier series $\sum_{n=1}^{\infty} a_n e^{i\lambda_n t}$. One asks whether or not the series $\sum_{n=1}^{\infty} a_n$ is suc. The answer to this question is the following.

THEOREM 2. *The series* $\sum_{n=1}^{\infty} a_n$ *is suc iff* B *does not contain a subspace which is isomorphic and isometric to the space* $c_0\,(B \not\supset c_0)$.

*Proof.* If $B \not\supset c_0$, then every wuc series is suc [4].

Necessity can be proved by the following.

*Example 1.* Consider the function $f: R \to c_0$, $f(t) = \sum_{n=1}^{\infty} \sin t/n\, e_n$, $e_1 = (1, 0, 0, \cdots)$, $e_2 = (0, 1, 0, \cdots), \cdots$ One can prove that this function is wap and its Fourier series is wuc, but the series $\sum_{n=1}^{\infty} e_n$ is not suc.

Now if the Fourier exponents $\{\lambda_n\}$ of wap functions $f: R \to B$ with Fourier coefficients belonging to B are linearly independent, the Fourier series $\sum_{n=1}^{\infty} a_n e^{i\lambda_n t}$ associated with $f$ is wuc. One asks whether or not the series $\sum_{n=1}^{\infty} a_n$ is suc. The answer is given by Theorem 3.

THEOREM 3. *The series* $\sum_{n=1}^{\infty} a_n$ *is suc iff* $B \not\supset c_0$.

*Proof.* We only need to prove the necessity which follows from Example 2.

*Example 2.* Consider the function $f: R \to c$ ($c$-space of convergent sequences),

$$f(t) = \sum_{n=1}^{\infty} e^{i\lambda_n t} e_n\,,$$

where the Fourier exponents $\{\lambda_n\}$ are linearly independent and $\lambda_n \to 0$, $n \to \infty$.

It is easy to verify that $f(t)$ is a wap function with wuc Fourier series, but the series $\sum_{n=1}^{\infty} e_n$ is not suc.

*Remarks.* 1) Theorem 1 holds true for $B = C_b(X)$, where $C_b(X)$ is the Banach space of continuous bounded functions on a topological space X. To prove that, we only need to consider Gel'fand's representation (Ref. [5], p. 30) of the Banach algebra $C_b(X)$.

2) Theorem 1 holds true in the case of almost automorphic functions [6] with Fourier coefficients belonging to the strong convex hull of its range.

3) If $\{\alpha_n\}$ is any sequence of reals, then the set of sequences $\{\xi : \xi = \{e^{i\frac{\alpha_n}{n}t}\}, t \in \mathrm{R}\}$ is not fundamental in $m$ (space of bounded sequences).

*Proof.* The function $f : \mathrm{R} \to c$, $f(t) = \sum_{n=1}^{\infty} e^{i\frac{\alpha_n}{n}t} e_n$ is wap with wuc Fourier series. If the set $\{\xi : \xi = \{e^{i\frac{\alpha_n}{n}t}\}, t \in \mathrm{R}\}$ is fundamental in $m$, then arguing as in Lemma 3.2.1. of Ref [2] (p. 61), we obtain $\sum_{n=1}^{\infty} e_n$ is suc. This a contradiction which proves the remark.

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