Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Remarks on Functors in Lie algebras

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **60** (1976), n.5, p. 557–563.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_60_5_557_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Algebra. — *Remarks on Functors in Lie algebras* (*). Nota di LUIGI SERENA, presentata (**) dal Corrisp. G. ZAPPA.

RIASSUNTO. — In questa Nota si studiano i funtori definiti sulla classe della Algebre di Lie di dimensione finita su un campo algebricamente chiuso di caratteristica zero e si determinano quelli massimali e non coincidenti con il funtore universale sulle algebre di Lie risolubili oppure sulle algebre di Lie semisemplici.

In [2] Barnes and Gastineau-Hills introduced the notion of *functor* on the class of finite-dimensional (soluble) Lie algebras over some fixed field as a rule $\mathbf{F}$ selecting in every such Lie algebra L a set $\mathbf{F}(L)$ of subalgebras subject to the axioms:

α: If φ is a homomorphism of L and $F \in \mathbf{F}(L)$, then $F^{\varphi} \in \mathbf{F}(L^{\varphi})$;

β: If $L \in \mathbf{F}(L)$, then $\{L\} = \mathbf{F}(L)$;

γ: If $F \in \mathbf{F}(L)$, and F is contained in some subalgebra M of L, then $F \in \mathbf{F}(M)$.

If one restricts attention to Lie algebras over an algebraically closed field of characteristic O, then the results of [2] show that the only functors selecting only soluble subalgebras are the *zero functor* $\mathbf{O}$, the *Cartan functor* $\mathbf{C}$ which selects in every Lie algebra L the set $\mathbf{C}(L)$ of Cartan subalgebras, and the *Borel functor* selecting the set of all maximal soluble subalgebras of L. In non-soluble Lie algebras there are also other functors, for example the *Levi functor* $\mathbf{S}$ selecting in L the set $\mathbf{S}(L)$ of Levi (= maximal semi-simple) subalgebras. Of course one would like to obtain some sort of survey of the possible functors. Here we shall go a few steps in that direction.

The functors $\mathbf{C}$, $\mathbf{B}$ and $\mathbf{S}$ select in L a set of isomorphic subalgebras (in fact they are conjugate under the group Aut L). Thus one might hope that every functor is so well-behaved. We shall give an example showing that this hope is ill founded.

There are two natural partial orders on the set of all functors on the class of finite-dimensional Lie algebras: $\mathbf{F} \subset \mathbf{G}$ if and only if for every Lie algebra L and for every $F \in \mathbf{F}(L)$ there is a subalgebra $G \in \mathbf{G}(L)$ with $F \subseteq G$. More restrictively, we put $\mathbf{F} < \mathbf{G}$ if $\mathbf{F} \subset \mathbf{G}$ and if for every $G \in \mathbf{G}(L)$ there is an $F \in \mathbf{F}(L)$ with $F \subseteq G$. One would like to obtain a survey of the maximal functors (if they exist)—here we mean maximal distinct from $\mathbf{U}$, the universal functor, associating $\{L\} = \mathbf{U}(L)$ to L. Does every functor $\mathbf{F}$ lie below (in any of the two orderings some maximal functor?

(*) Eseguito nell'ambito dell'attività del G.N.S.A.G.A. del C.N.R.
(**) Nella seduta dell'8 maggio 1976.

In the search for answers to these questions we have found a few new types of functors which may be of some interest. All the maximal functors have been determined which do not coincide with the universal functor $\mathbf{U}$ on the soluble as well as the semi-simple Lie algebras, and every functor with this property lies below one of these. However, there exist functors $\mathbf{F} \neq \mathbf{U}$ coinciding with $\mathbf{U}$ on the soluble as well as the semi-simple Lie algebras. We have not been able to determine the maximal ones among these.

All Lie algebras considered will be finite-dimensional over some fixed algebraically closed field of characteristic zero. The standard fact used is that such an algebra L has the form $L = A + \mathbf{R}(L)$, where A is a Levi subalgebra (= maximal semi-simple) and $\mathbf{R}(L)$ is the maximal soluble ideal of L (see, for example [3]).

1. Some maximal functors

In a fixed simple Lie algebra S select a non-empty set $\mathbf{M}(S)$ of maximal subalgebras of S which is invariant under the group of all automorphisms of S. Observe that this defines the set $\mathbf{M}(T)$ for every algebra T isomorphic to S. We want to extend this selection to a rule $\mathbf{M}_S$ selecting a set of subalgebras in every Lie algebra L.

Definition. Let L be any Lie algebra and A any Levi subalgebra of L. The Lie algebra A decomposes into the direct sum $A = A_{\{S\}} \oplus A_{\{S\}'}$, where $A_{\{S\}}$ is a direct sum of copies of S and $A_{\{S\}'}$, a direct sum of simple subalgebras of A not isomorphic to S. If $A_S = (0)$, put $\mathbf{M}_S(L) = L$. If $A_{\{S\}} = \bigoplus_{i=1}^{n} S_i$, put

$$\mathbf{M}_S(L) = \left\{ \left(\bigoplus_{i=1}^{n} M_i \right) + A_{S'} + \mathbf{R}(L) \,;\, M_i \in \mathbf{M}(S_i) \right\}.$$

This definition does not depend on the choice of the Levi subalgebra A in L, since the Levi subalgebras of L are conjugate under the special automorphisms of L defined in terms of $\mathbf{R}(L)$ (see [3]).

Theorem 1. *For the simple Lie algebra* S *the rule* $\mathbf{M}_S$ *is a functor.*

Proof. Since homomorphism of L map Levi subalgebras of L to Levi subalgebras of the homomorphic image, it is clear that for every homomorphism φ of L one has $\mathbf{M}_S(L^{\varphi}) = (\mathbf{M}_S(L))^{\varphi}$. If $L \in \mathbf{M}_S(L)$, then L cannot have any composition factor isomorphic to S. By the definition of $\mathbf{M}_S$ one thus has $\mathbf{M}_S(L) = \{L\}$. If B is any subalgebra of L containing $M \in \mathbf{M}_S(L)$, then $B = (B \cap A_{\{S\}}) + A_{\{S\}'} + \mathbf{R}(L)$ for any Levi subalgebra A of L. Now

$$B \cap A_{\{S\}} = \bigoplus_{i=1}^{n} (B \cap S_i) \qquad \text{if} \quad M = \left(\bigoplus_{i=1}^{n} M_i \right) + A_{\{S\}'} + \mathbf{R}(L).$$

If now $B \cap S_i \supsetneqq M_i$, then $B \cap S_i = S_i$ and $M_i \in \mathbf{M}(S_i)$. Thus one has $M \in \mathbf{M}_S(B)$, and $\mathbf{M}_S$ is a functor.

In general, the simple Lie algebra S has maximal subalgebras which are non-isomorphic (see, for example, Dynkin [4]). Thus we see that the subalgebras of the Lie algebra L selected by the functor $\mathbf{M}_S$ need not be isomorphic.

COROLLARY 1. *If the functor* $\mathbf{F}$ *does not coincide with the universal functor* $\mathbf{U}$ *on the class of simple Lie algebras, then there is a simple Lie algebra* S *and a set* $\mathbf{M}(S)$ *of maximal subalgebras of* S *so that* $\mathbf{F} < \mathbf{M}_S$.

Proof. Since $\mathbf{F}$ does not coincide with $\mathbf{U}$ on the class of simple Lie algebras, there is a simple Lie algebra S with $S \notin \mathbf{F}(S)$. Put $\mathbf{M}(S)$ the set of all maximal subalgebras of S containing some $F \in \mathbf{F}(S)$. Then it follows from the definition of the functor $\mathbf{M}_S$ and from the homomorphism invariance of $\mathbf{F}(L)$ and $\mathbf{M}_S(L)$ that $\mathbf{F} < \mathbf{M}_S$.

The next statement is now pretty obvious, and will not be proved.

COROLLARY 2. *For the simple Lie algebra* S *the functor* $\mathbf{M}_S$ *is maximal with respect to the order relation* $<$, *it is also maximal with respect to the order relation* $\subset$ *if, and only if,* $\mathbf{M}(S)$ *is the set of all maximal subalgebras of* S.

There is a further remarkable property of the functor $\mathbf{M}_S$: for every Lie algebra L and for every ideal I of L, one has that $M \in \mathbf{M}_S(L)$ implies $I \cap M \in \mathbf{M}_S(I)$. Also the Levi functor $\mathbf{S}$ has this property. This property suggests the following definition.

DEFINITION. The functor $\mathbf{F}$ is called *ideal* (respectively, *radical*) if one has for every Lie algebra L and every $F \in \mathbf{F}(L)$ that $F \cap I \in \mathbf{F}(I)$ for every ideal I of L (respectively, $F \cap \mathbf{R}(L) \in \mathbf{F}(\mathbf{R}(L))$).

We now restate our results as a contrast and motivation for further considerations.

COROLLARY 3. *If the (ideal or radical) functor* $\mathbf{F}$ *is maximal with respect to the order relation* $<$, *and if* $\mathbf{F}$ *does not coincide with the universal functor* $\mathbf{U}$ *on the class of all simple Lie algebras, then* $\mathbf{F} = \mathbf{M}_S$ *for some simple Lie algebra* S.

LEMMA. *If the ideal functor* $\mathbf{F}$ *coincides with the universal functor* $\mathbf{U}$ *on the class of all simple Lie algebras, then it coincides with* $\mathbf{U}$ *on the class of all semi-simple Lie algebras. That is,* $\mathbf{S}$, *the Levi functor, satisfies* $\mathbf{S} < \mathbf{F}$.

Proof. If L is a semi-simple Lie algebra and $F \in \mathbf{F}(L)$, then one has $F \cap S \in \mathbf{F}(S)$ for every simple direct summand of L, since $\mathbf{F}$ is ideal. As $\mathbf{F}$ coincides with $\mathbf{U}$ on S, one has $F \cap S = S$. But then $F = L \in \mathbf{F}(L)$, and $\mathbf{F}$ coincides with $\mathbf{U}$ on L. If L is now an arbitrary Lie algebra and $F \in \mathbf{F}(L)$, then F must contain (or rather map onto) a Levi subalgebra of L, hence $\mathbf{S} < \mathbf{F}$.

THEOREM 2. *If the radical functor* $\mathbf{F} \neq \mathbf{U}$ *satisfies* $\mathbf{S} < \mathbf{F}$, *then either* $\mathbf{F} = \mathbf{S}$, *or* $\mathbf{F}$ *coincides with* C, *the Cartan functor on the class of all soluble Lie algebras.*

Proof. If $\mathbf{F} \neq \mathbf{S}$, the $\mathbf{F}$ cannot coincide with $\mathbf{O}$ on the class of all soluble Lie algebras. Barnes and Gastineau-Hills have shown that except for $\mathbf{O}$ the only functors on the class of all soluble Lie algebras are the Cartan functor C and the universal functor. If $\mathbf{F}$ coincides with $\mathbf{U}$ on the class of all soluble Lie algebras and if $F \in \mathbf{F}(L)$ for any Lie algebra L, then $F \cap \mathbf{R}(L) = \mathbf{R}(L)$. Since F contains a Levi subalgebra A by assumption, one has $F \supseteq A + \mathbf{R}(L) = L$. Hence $F = L$; and $\mathbf{F}$ coincides with U, contrary to our assumption. Thus, if $\mathbf{S} \neq \mathbf{F} \neq \mathbf{U}$, the functor $\mathbf{F}$ must coincide with the Cartan functor C on the class of all soluble Lie algebras.

Consider the hypothetical situation of Theorem 2, that is a radical functor $\mathbf{F}$ satisfying $\mathbf{S} < \mathbf{F}$, which coincides with C on the class of soluble Lie algebras. For any Lie algebra L let $F \in \mathbf{F}(L)$, then $F = C + A$, where C is a Cartan subalgebra of $\mathbf{R}(L)$ and A is a Levi subalgebra of L. Clearly, C is an ideal of F, and since C is its own idealiser in $\mathbf{R}(L)$, one has that $F = \{l \in L ; l \circ C \subseteq C\}$.

DEFINITION. The rule $\mathbf{I}$ selects in every Lie algebra L the set $\mathbf{I}(L)$ of idealisers in L of the Cartan subalgebras of $\mathbf{R}(L)$.

THEOREM 3. *The rule* $\mathbf{I}$ *is a radical functor satisfying* $\mathbf{S} < \mathbf{I}$.

Proof. By Barnes [1] one has $I + \mathbf{R}(L) = L$ for every Lie algebra L and every $I \in \mathbf{I}(L)$. Thus I must contain a Levi subalgebra of L. This shows $S < I$. If φ is a homomorphism of L, then $\mathbf{R}(L^\varphi) = (\mathbf{R}(L))^\varphi$, and Cartan subalgebras of $\mathbf{R}(L)$ are mapped to Cartan subalgebras of $\mathbf{R}(L^\varphi)$. Also the Levi subalgebras of L are mapped to those of L^φ. Since the Cartan subalgebra C^φ of the soluble Lie algebra $\mathbf{R}(L^\varphi)$ is its own idealiser in $\mathbf{R}(L^\varphi)$, it follows that the idealiser of C^φ in L^φ is of the form I^φ with $I \in \mathbf{I}(L)$. This shows the invariance of the rule $\mathbf{I}$ under homomorphisms. If $I \in \mathbf{I}(L)$ then $\mathbf{R}(L)$ idealises a Cartan subalgebra of $\mathbf{R}(L)$, thus $\mathbf{R}(L)$ is nilpotent and so its only Cartan subalgebra. Hence $\mathbf{I}(L) = \{L\}$. Let B be an intermediate subalgebra of $L : I \subseteq B \subseteq L$ for some $I \in \mathbf{I}(L)$; then $\mathbf{R}(B) = B \cap \mathbf{R}(L)$, and the Cartan subalgebra C of $\mathbf{R}(L)$ idealised by I is still a Cartan subalgebra of $\mathbf{R}(B)$. Thus $\mathbf{I}$ is a functor; clearly, it is radical.

COROLLARY. *The Levi functor* $\mathbf{S}$ *is the only ideal functor* $\mathbf{F} \neq \mathbf{U}$ *coinciding with* U *on the class of simple Lie algebras.*

Proof. Let $\mathbf{F}$ be such a functor. Since it coincides with $\mathbf{U}$ on the simple Lie algebras, the Lemma gives us that $\mathbf{S} < \mathbf{F}$. An ideal functor is in particular also radical. Thus Theorems 2 and 3 together yield that either $\mathbf{F} = \mathbf{S}$ or $\mathbf{F} = \mathbf{I}$. The ideal functor $\mathbf{F}$ defines an ideal functor on the class of all soluble Lie algebras, but there only the trivial functors $\mathbf{O}$ and $\mathbf{U}$ are ideal. Thus $\mathbf{F}$ cannot define the Cartan functor on the class of soluble Lie algebras. Hence $\mathbf{F} \neq \mathbf{I}$.

We have thus obtained a complete survey of all the maximal ideal functors and of those maximal radical functors which do not coincide with the universal functor **U** on the class of soluble as well as on the class of simple Lie algebras.

2. DIAGONAL FUNCTORS

If the functor **F** coincides with the universal functor **U** on the class of all simple Lie algebras, but not on the class of semi-simple Lie algebras, we shall call **F** a *diagonal functor*.

If **F** is a diagonal functor, then there is a simple Lie algebra such that **F** does not coincide with **U** on the class of all (finite) direct sums of copies of S. The reason for calling these functors *diagonal* will be apparent from the following result.

PROPOSITION. *If the functor* **F** *coincides with the universal functor* **U** *on the simple algebra* S, *but not on the class of all direct sums of copies of* S, *then one has for every* $L = \bigoplus_{i=1}^{n} S_i$, $S \simeq S_i$, *and for every* $F \in \mathbf{F}(L)$, *that* $F \simeq S$.

Proof. Let L be the direct sum of the minimal number m of copies of S, $L = \bigoplus_{i-1}^{m} S_i$, $S \simeq S_i$ so that $\mathbf{U}(L) \neq \mathbf{F}(L)$. The subalgebra $F \in \mathbf{F}(L)$ is a subdirect sum of the m copies of S. Let S_1 be an arbitrary minimal ideal of L. If the intersection $F \cap S_1 \neq (o)$, then $S_1 \subseteq F$. But then $F/S_1 \neq L/S_1$. On the other hand, $F/S_1 \in \mathbf{F}(L/S_1) = \mathbf{U}(L/S_1)$, by the minimality of L. These two statements contradict each other! Hence, for every minimal ideal S_1 of L one has $S_1 \cap F = (o)$. Minimality of L yields again that $S_1 + F = L$. But then F must contain a non-trivial ideal of L, unless $m = 2$ and F is a diagonal. This establishes in particular, the Proposition for the direct sum of two copies of S.

Suppose the Proposition has been proved for direct sums of fewer than n copies of S, and Let $L = \bigoplus_{i=1}^{n} S_i$. The subalgebra $F \in \mathbf{F}(L)$ is a subdirect sum of the S_i. Since $n > 2$, and $(F + S_i)/S_i$ is simple, by induction, one has that F is a direct sum of at most two copies of S. Hence, there is a minimal ideal S_1, say, of L with $F \cap S_1 = (o)$; and one obtains that $F \simeq (F + S_1)/S_1$ is simple.

Remark. For such a Lie algebra $L = \bigoplus_{i=1}^{n} S_i$ with $S \simeq S_i$ and for every $F \in \mathbf{F}(L)$ there are n isomorphisms $\varphi_i : S \to S_i$ so that $F = (S^{\varphi_1}, \cdots, s^{\varphi_n})$; $s \in S$, i.e. F is the *diagonal* of the S_i with respect to the isomorphisms $\{\varphi_i\}$.

DEFINITION. For the simple Lie algebra S we now define a *diagonal rule* $\mathbf{D}_S$ on the class of all Lie algebras. If L is a semi-simple Lie algebra, then $L = L_{\{S\}} \oplus L_{\{S\}'}$, where $L_{\{S\}}$ is the direct sum of the minimal ideals of L isomorphic to S and $L_{\{S\}}'$ is the direct sum of the minimal ideals of L not isomorphic to S; put

$$\mathbf{D}_S(L) = \{D + L_{\{S\}'}\ ;\quad D \text{ diagonal in } L_S\}.$$

For the arbitrary Lie algebra L choose a Levi subalgebra A and put

$$\mathbf{D}_S(L) = \{D + \mathbf{R}(L)\ ;\ D \in \mathbf{D}_S(A)\}.$$

THEOREM 4. *The rule* $\mathbf{D}_S$ *is a functor maximal with respect to the ordering* $\subset$. *For every diagonal functor* $\mathbf{F}$ *there is a simple Lie algebra* S *so that* $\mathbf{F} < \mathbf{D}_S$.

Proof. If $L \in \mathbf{D}_S(L)$, then—from the definition of $\mathbf{D}_S$—any Levi subalgebra of L can have at most one direct summand isomorphic to S. But in that case the definition of $\mathbf{D}_S$ yields $\{L\} = \mathbf{D}_S(L)$. If $D \in \mathbf{D}_S(L)$ and M is any intermediate subalgebra of $L : D \subseteq M \subseteq L$. Then one has $M = (A_{\{S\}'} + \mathbf{R}(L)) + (M \cap A_{\{S\}})$ for every Levi subalgebra A of L. Since $M \cap A_{\{S\}}$ contains a diagonal of $A_{\{S\}}$ (viz. $D \cap A_{\{S\}}$), the algebra $(M \cap A_{\{S\}})$ must be a direct sum of copies of S. And a diagonal of $A_{\{S\}}$ remains a diagonal of $M \cap A_{\{S\}}$. Thus $D \in \mathbf{D}_S(M)$.

If M is a Lie algebra with Levi subalgebra B and if φ is a homomorphism of L onto M such that $A^\varphi = B$, then clearly φ maps every diagonal of $A_{\{S\}}$ to one of $B_{\{S\}}$ and $B_{\{S\}'} + \mathbf{R}(M) = (A_{\{S\}'} + \mathbf{R}(L))^\varphi$. Hence for every subalgebra $D \in \mathbf{D}_S(L)$ one has $D^\varphi \in \mathbf{D}_S(M)$. Thus $\mathbf{D}_S$ is a functor.

That $\mathbf{D}_S$ is maximal with respect to the order relation $<$ is clear from the Proposition. Now let $\mathbf{F} \neq \mathbf{U}$ be a functor satisfying $\mathbf{D}_S \subseteq \mathbf{F}$. Since $\mathbf{D}_S$ coincides with $\mathbf{U}$ on the class of all Lie algebras without composition factor isomorphic to S, the axiomations of functors yields that there $\mathbf{F}$ also coincides with $\mathbf{U}$. On the direct sums of copies of S, however, the Proposition yields that $\mathbf{F}$ coincides with $\mathbf{D}_S$. Thus in the Lie algebra L the subalgebra $F \in \mathbf{F}(L)$ can differ from an element of $\mathbf{D}_S(L)$ at most in the intersection $F \cap \mathbf{R}(L)$. But that means $F + \mathbf{R}(L) \in \mathbf{D}(L)$. But now F and $F + \mathbf{R}(L)$ both are elements of $\mathbf{F}(L)$, hence of $\mathbf{F}(F + \mathbf{R}(L))$, and so—by axiom—$\mathbf{R}(L) \subseteq F$. This shows that $\mathbf{F} = \mathbf{D}_S$.

Remarks. 1) Observe that the functor $\mathbf{D}_S$ is radical. Thus, we now have obtained a complete survey of the maximal radical functors. 2) By modifying the definition of $\mathbf{D}_S$—essentially by replacing S by a set of simple Lie algebras—one may construct $2^{\aleph_0}$ distinct diagonal functors.

COROLLARY. *If the functor* $\mathbf{F}$ *does not coincide with* $\mathbf{U}$ *on the class of semi-simple Lie algebras, then there is a simple Lie algebra* S *and a set* $\mathbf{M}(S)$ *of maximal subalgebras of* S *such that either* $\mathbf{F} < \mathbf{M}_S$ *or* $\mathbf{F} < \mathbf{D}_S$.

If the functor $\mathbf{F}$ does not satisfy $\mathbf{F} < \mathbf{M}_S$ or $\mathbf{F} < \mathbf{D}_S$ for a suitable simple Lie algebra S and a set $\mathbf{M}(S)$ of maximal subalgebras of S, then $\mathbf{F}$ must coincide with the universal functor $\mathbf{U}$ on the soluble as well as on the semi-simple Lie algebras. Is there any such functor $\mathbf{F} \neq \mathbf{U}$? Is every such functor majorised by some maximal one? Describe the maximal functors in this class.

For the simple Lie algebra S let $\mathbf{K}_S$ be the rule which associates to every Lie algebra L the set $\mathbf{K}_S$ (L) of subalgebras of the form $A + K(S)$, where A is a Levi subalgebra of L with the decomposition $A = A_{\{S\}} + A_{\{S\}'}$ and K (S) is the annihilator of $A_{\{S\}}$ in $\mathbf{R}$ (L). It is not difficult to prove.

THEOREM 5. *The rule* $\mathbf{K}_S$ *is a functor which coincides with* $\mathbf{U}$ *on the soluble as well as the semi-simple Lie algebras.*

It seems likely that $\mathbf{K}_S$ is maximal with respect to $<$, but we have not been able to prove this. Replacing S by a set of simple Lie algebras one obtains similarly $2^{\aleph_0}$ distinct functors coinciding with $\mathbf{U}$ on the soluble as well as the semi-simple Lie algebras; but we do not know whether there are further essentially different functors in this class.

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