Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Valter Franceschini, Sandro Graffi, Sergio Levoni

On the Eigenvalues of the Bounded Harmonic Oscillator

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.2, p. 205–214.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_2_205_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Fisica matematica. — *On the Eigenvalues of the Bounded Harmonic Oscillator* (*). Nota di VALTER FRANCESCHINI, SANDRO GRAFFI e SERGIO LEVONI, presentata (**) dal Corrisp. G. FICHERA.

RIASSUNTO. — Il metodo degli invarianti ortogonali di Fichera viene applicato al problema di autovalori per l'equazione di Schrödinger per l'oscillatore armonico limitato in meccanica quantistica. In tal modo viene ottenuto un procedimento per l'approssimazione di ogni autovalore che conferma in modo rigoroso, e migliora numericamente, precedenti calcoli compiuti da altri Autori usando differenti metodi.

1. INTRODUCTION

This paper deals with a rigorous treatment of the computation of the eigenvalues of the quantum mechanical system known as bounded harmonic oscillator. By this we mean a harmonic oscillator placed in the center of an infinitely high potential well of length L, $0 < L < \infty$. The Hamiltonian of such a system reads:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 z^2 + V(z) \tag{1.1}$$

where m is the mass of the oscillator and ω its frequency, and:

$$V(z) = \begin{cases} 0 \, , \, |z| < \dfrac{L}{2} \\ \infty \, , \, |z| \geq \dfrac{L}{2} \, . \end{cases} \tag{1.2}$$

Putting:

$$x = \left(\frac{m\omega}{\hbar}\right)^{1/2} z \quad , \quad \lambda = \frac{2E}{\hbar\omega} \quad , \quad l = \left(\frac{m\omega}{\hbar}\right)^{1/2} L \, , \tag{1.3}$$

the Schrödinger equation leads to the following Sturm-Liouville problem:

$$\left(-\frac{d^2}{dx^2} + x^2\right) \psi(x) = \lambda\psi(x) \quad , \quad \psi\left(-\frac{l}{2}\right) = \psi\left(\frac{l}{2}\right) = 0 \, . \tag{1.4}$$

Such a problem, although very simple, is not exactly solvable: whence the necessity of approximate computations, already performed by several Authors, [1], [2], [3], [4], [5], [6], in view of the importance of the present system in an astrophysical problem, [1], and also in various problems of theoretical physics, such as the magnetic properties of metallic solids and the anharmonic effects in crystal solids.

(*) This work was supported by the " Gruppo Nazionale per la Fisica Matematica del C.N.R. ".

(**) Nella seduta dell'8 febbraio 1975.

The approximate methods so far employed (perturbation theory, [3], WKB methods, [5], [6], approximate solutions of differential equations, [1], [2], [4], [5], [6]), although numerically effective, are not completely satisfactory from a rigorous standpoint. Therefore in this paper the problem is treated in the light of the methods for rigorously computing the eigenvalues of differential operators: for any eigenvalue we will compute a non-increasing sequence of upper bounds together with a non-decreasing one of lower bounds. Both sequences, in addition, do converge to the exact eigenvalue. The upper bounds sequence is given, as usual, by the standard Rayleigh-Ritz method. The much more delicate problem of computing the lower bounds sequence is solved through an application of the recent, and already classical, orthogonal invariants method of Fichera [7], [8], [9]. The numerical analysis, as it will clearly appear later on, essentially confirms the numerical results so far obtained through the other methods and in addition shows that the present computation is not only rigorous but also the most effective numerically so far performed.

In the next section we treat the application of the Rayleigh-Ritz method and of the Fichera one to the problem under discussion, and in section 3 the numerical results are presented and discussed.

2. Approximation of the eigenvalues

As already mentioned, our purpose is to compute the eigenvalues λ_k, $k = 0, 1, \cdots$, of the following Sturm-Liouville problem:

$$\psi''(x) + (\lambda - x^2)\,\psi(x) = 0 \tag{2.1}$$

with the boundary conditions:

$$\psi\left(-\frac{l}{2}\right) = \psi\left(\frac{l}{2}\right) = 0 \,. \tag{2.2}$$

Let H be the Hilbert space $L^2(-l/2 , l/2)$. As it is well known, the regular Sturm-Liouville problem (2.1)-(2.2) can be realized as a strictly positive self-adjoint operator in H, with spectrum consisting only of simple eigenvalues. If we indicate with A such operator, our eigenvalue problem can be rewritten in the more abstract form:

$$A\psi = \lambda\psi \quad , \quad \psi \in D(A) \tag{2.3}$$

where D (A) is the domain of the operator A.

Let us begin our considerations on the eigenvalues by remarking that the subspace $L^2_+(-l/2 , l/2)$ and $L^2_-(-l/2 , l/2)$ reduce the operator A. Here $L^2_+(-l/2 , l/2)$ is the subspace of $L^2(-l/2 , l/2)$ formed by the functions even with respect to $x = 0$, and $L^2_-(-l/2 , l/2)$ is the subspace formed by the functions odd with respect to $x = 0$. This follows from the fact that the projection operator P_+ onto $L^2_+(-l/2, l/2)$ commutes with A, as it is easy to see, and the same is true for P_-, projection operator onto $L^2_-(-l/2, l/2)$. Hence the eigenvalue problem (2.3) separates out in the following two problems:

$$(2.4) \qquad A\psi = \lambda\psi \quad , \quad \psi \in D(A)_+ \quad ; \quad A\psi = \lambda\psi \quad , \quad \psi \in D(A)_-$$

where in the first equation A is intended as its part in $L^2_+(-l/2, l/2)$ and $D(A)_+ = P_+ D(A)$, and in the second one is intended as its part in $L^2_-(-l/2, l/2)$, and $D(A)_- = P_- D(A)$. The first equation yields of course the even eigenvalues, $\lambda_0, \lambda_2, \cdots$, and the second one the odd ones, $\lambda_1, \lambda_3, \cdots$ As repeatedly emphasized, for any eigenvalue we will give a non-decreasing sequence of lower bounds and a non-increasing one of upper bounds, both converging to the eigenvalue.

As usual, the upper bounds are obtained through the classical Rayleigh-Ritz method, [10], which we proceed now to apply to our case. According to the standard procedure, we have to compute the eigenvalues of the following $N \times N$ real and symmetric matrix:

$$(2.5) \qquad A_{i,k} = (\varphi_i , A\varphi_k)_{i,k=0,1,\cdots,N-1} ,$$

$\varphi_0, \varphi_1, \cdots, \varphi_{N-1}$ being N orthonormal vectors belonging to D (A). It is well known that the eigenvalues $\lambda_k^{(N)}, k = 0, 1, \cdots, N-1$ of (2.5) are upper bounds for the first N eigenvalues of A, and that the sequence $\lambda_k^{(N)}$ is non-increasing: $\lambda_k^{(N)} \geq \lambda_k^{(N+1)}$ for any k. In addition, if the system $\{A\varphi_k\}_{k=0}^{\infty}$, is complete in H, one has, on the basis of our hypothesis on A:

$$(2.6) \qquad \lim_{N\to\infty} \lambda_k^{(N)} = \lambda_k , \qquad k = 0, 1, 2, \cdots .$$

In our case, let us first consider the eigenvalue problem in $L^2_+(-l/2, l/2)$, i.e. the even eigenvalues of A. Introduce the following complete orthonormal system in $L^2_+(-l/2, l/2)$:

$$(2.7) \qquad \{\varphi_n\} = \left\{\sqrt{\frac{2}{l}} \cos \frac{(2h+1)\pi}{l} x\right\} , \qquad h = 0, 1, 2, \cdots .$$

We have of course $\varphi_h \in D(A)_+$ for any h, so that the corresponding Rayleigh-Ritz matrix is given by:

$$(2.8) \quad A^+_{i,k} = (\varphi_i , A\varphi_k) = \frac{2}{l} \int_{-l/2}^{l/2} \cos \frac{(2i+1)\pi}{l} x \left(-\frac{d^2}{dx^2} + x^2\right) \cos \frac{(2k+1)\pi}{l} x \, dx .$$

One obtains easily:

$$(2.9) \qquad A^+_{i,k} = \begin{cases} (-1)^{i+k} \dfrac{l^2}{2\pi^2} \left[\dfrac{1}{(i-k)^2} - \dfrac{1}{(i+k+1)^2}\right] , & i \neq k \\[2ex] \dfrac{l^2}{12} + \dfrac{(2k+1)^2 \pi^2}{l^2} - \dfrac{l^2}{2\pi^2 (2k+1)^2} , & i = k \end{cases}$$

$i, k = 0, 1, 2, \cdots$

Since, as it is well known, [1], the system $\{A\varphi_h\}$ is complete in $L^2_+(-l/2, l/2)$, we have:

$$\lim_{N\to\infty} \lambda_k^{(N)} = \lambda_{2k}, \qquad k = 0, 1, 2, \cdots. \tag{2.10}$$

Here $\lambda_k^{(N)}$, $k = 0, 1, \cdots, N-1$, are the eigenvalues of the matrix $A^+_{i,k}$, $i, k = 0, 1, \cdots, N-1$.

In a completely analogous way, choosing in $L^2_-(-l/2, l/2)$ the orthonormal complete set given by:

$$\{\varphi_h\} = \left\{\sqrt{\frac{2}{l}} \sin \frac{2(h+1)\pi}{l} x\right\} \qquad h = 0, 1, \cdots \tag{2.11}$$

and putting:

$$A^-_{i,k} = \frac{2}{l} \int_{-l/2}^{l/2} \sin \frac{2(i+1)\pi}{l} x \left(-\frac{d^2}{dx^2} + x^2\right) \sin \frac{2(k+1)\pi}{l} x \, dx, \tag{2.12}$$

we get:

$$A^-_{i,k} = \begin{cases} (-1)^{i+k} \dfrac{l^2}{2\pi^2} \left[\dfrac{1}{(i-k)^2} - \dfrac{1}{(i+k+2)^2}\right], & i \neq k \\[2ex] \dfrac{l^2}{12} + \dfrac{4(k+1)^2\pi^2}{l^2} - \dfrac{l^2}{8(k+1)^2\pi^2}, & i = k \end{cases} \tag{2.13}$$

$i, k = 0, 1, \cdots$. Here again, since $\{A\varphi_h\}$ is complete in $L^2_-(-l/2, l/2)$, we have:

$$\lim_{N\to\infty} \lambda_k^{(N)} = \lambda_{2k-1}, \qquad k = 0, 1, 2, \cdots, \tag{2.14}$$

where $\lambda_k^{(N)}$, $k = 1, 2, \cdots, N$, are the eigenvalues of the matrix A^-_{ik}, $i, k = 1, 2, \cdots, N$.

We proceed now to obtain the sequence of lower bounds. As it is well known, this is the most delicate problem in eigenvalues calculations. Here, as repeatedly mentioned, use will be made of the orthogonal invariants method of Fichera. We give here only some essential notions of such a method, strictly necessary in what follows, referring the reader to [7], [8], [9], for a complete treatment.

Let A be a self-adjoint operator in a separable Hilbert space H, strictly positive with compact resolvent, so that A has a pure point spectrum consisting of the eigenvalues $\lambda_k = \mu_k^{-1}$, where μ_k, $k = 0, 1, \cdots$ are the eigenvalues of $G = A^{-1}$. Now, $\{v_k\}_{k=1}^{\infty}$ being any orthonormal complete set in H, put:

$$\mathscr{I}_s^n(G) = \frac{1}{S!} \sum_{h_1 \cdots h_s} G^{(n)}(v_{h_1}, \cdots, v_{h_s}) \tag{2.15}$$

where the sum is extended over all possible ways of choosing s positive integers, G^n is the n-th power of G and:

$$\mathrm{G}^{(n)}(v_1, \cdots, v_s) = \det(\mathrm{G}^n v_i, v_j)_{i,j=1,2,\cdots,s}. \tag{2.16}$$

$\mathscr{I}^n_s(\mathrm{G})$ exists, and in addition does not depend on the particular orthonormal set in (2.15), if and only if G^n belongs to the trace class. If this is the case, one has:

$$\mathscr{I}^n_s(\mathrm{G}) = \sum_{h_1 < h_2 < \cdots h_s} \mu^n_{h_1} \mu^n_{h_2} \cdots \mu^n_{h_s}. \tag{2.17}$$

Let now W_ν be the ν dimensional subspace of H spanned by the vectors $v_1, \cdots, v_\nu$, P_ν the orthogonal projection operator from H onto W_ν, $w^{(\nu)}_k$ an eigenvector corresponding to the eigenvalue $\mu^{(\nu)}_k$ of the operator $P_\nu G P_\nu$ and $W_{\nu,k}$ the subspace of W_ν orthogonal to $w^{(\nu)}_k$. For $s > 0$ put:

$$\sigma^{(\nu)}_k = \left\{ \frac{\mathscr{I}^n_s(\mathrm{G}) - \mathscr{I}^n_s(\mathrm{P}_\nu \mathrm{G} \mathrm{P}_\nu)}{\mathscr{I}^n_{s-1}(\mathrm{P}_{\nu,k}\,\mathrm{G}\mathrm{P}_{\nu,k})} + [\mu^{(\nu)}_k]^n \right\}^{1/n} \tag{2.18}$$

where $P_{\nu,k}$ is the orthogonal projection from H onto $W_{\nu,k}$. Then the following relations hold:

$$\sigma^{(\nu)}_k \geq \sigma^{(\nu+1)}_k \geq \cdots \geq \mu_k \quad , \quad \lim_{\nu \to \infty} \sigma^{(\nu)}_k = \mu_k. \tag{2.19}$$

Thus by considering $[\sigma^{(\nu)}_k]^{-1}$ we obtain a non-decreasing sequence of lower bounds to λ_k, converging to the eigenvalue as $\nu \to \infty$, $k = 0, 1, \cdots$.

In our case use will be made of (2.18) with $s = 1$, $n = 2$, i.e. of the orthogonal invariant $\mathscr{I}^2_1$. Now $\mathscr{I}^n_0(\mathrm{G}) = 1$, so that formulae (2.17) e (2.18) show that the lower bounds for the eigenvalues $\lambda_k = \mu_k^{-1}$ take the following form, involving only the orthogonal invariant $\mathscr{I}^2_1(\mathrm{G})$:

$$\lambda_k > \left\{ [\lambda^{(n)}_k]^{-2} + \mathscr{I}^2_1(\mathrm{G}) - \sum_{h=1}^{n} [\lambda^{(n)}_h]^{-2} \right\}^{-1/2} = [\sigma^{(n)}_k]^{-1}. \tag{2.20}$$

Here $\lambda^{(n)}_k = [\mu^{(n)}_k]^{-1}$ are of course the n-th upper bounds to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ obtained by means of the Rayleigh-Ritz method through the orthonormal complete set $\{v_i\}$. (2.20) shows that once computed the sequence of the Rayleigh-Ritz upper bounds, the Fichera lower bounds sequence is known when $\mathscr{I}^2_1$ is known.

To find an explicit expression for $\mathscr{I}^2_1$ in our case, we will first rewrite our Sturm-Liouville problem into a Fredholm integral equation, thus finding explicitly the operator G. Then it will be possible to apply the explicit formulae of Fichera for the orthogonal invariants of integral operators, which in our case read:

$$\mathscr{I}^n_s(\mathrm{G}) = \frac{1}{s!} \int_{-l/2}^{l/2} \cdots \int_{-l/2}^{l/2} f(x^{(1)}, x^{(2)}, \cdots, x^{(s)})\, \mathrm{d}x^{(1)}\, \mathrm{d}x^{(2)} \cdots \mathrm{d}x^{(s)} \tag{2.21}$$

where $f(x^{(1)}, \cdots, x^{(s)}) = \det\{K(x^{(i)}, x^{(j)})\}$, $i, j = 1, \cdots, s$, $K(x, y)$ being the integral kernel of the operator G^n:

$$G^n u = \int_{-l/2}^{l/2} K(x, y)\, u(y)\, dy \quad , \quad u \in L^2\left(-\frac{l}{2}, \frac{l}{2}\right) \tag{2.22}$$

i.e. the n-th iterated kernel of G.

Consider therefore again the eigenvalue problem (2.3). According to the standard procedure, the problem can be rewritten in the following integral form:

$$\psi(x) = (\lambda + 1) \int_{-l/2}^{l/2} G(x, y)\, \psi(y)\, dy \tag{2.23}$$

Here $G(x, y)$ is the Green function of the differential operator $\psi''(x) - (x^2 + 1)\psi(x) = 0$ with the boundary conditions (2.2), and it is given by:

$$G(x, y) = \begin{cases} C e^{\frac{x^2+y^2}{2}} \displaystyle\int_{-l/2}^{x} e^{-\xi^2}\, d\xi \int_{y}^{l/2} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \le x \le y \le \frac{l}{2} \\ C e^{\frac{x^2+y^2}{2}} \displaystyle\int_{-l/2}^{y} e^{-\xi^2}\, d\xi \int_{x}^{l/2} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \le y \le x \le \frac{l}{2} \end{cases} \tag{2.24}$$

where

$$C = \left[\int_{-l/2}^{l/2} e^{-\xi^2}\, d\xi\right]^{-1}. \tag{2.25}$$

It is easy to check, as a matter of fact, that (2.24) is the Green function of the differential operator under discussion. Furthermore, putting:

$$\begin{cases} G_0(x, y) = \frac{1}{4}\left[G(x, y) + G(x, -y) + G(-x, y) + G(-x, -y)\right] \\ G_1(x, y) = \frac{1}{4}\left[G(x, y) - G(x, -y) - G(-x, y) + G(-x, -y)\right] \end{cases} \tag{2.26}$$

problem (2.23) separates out in the following two problems:

$$\psi(x) = (\lambda + 1) \int_{-l/2}^{l/2} G_0(x, y)\, \psi(y)\, dy \quad , \quad \psi \in L^2_+\left(-\frac{l}{2}, \frac{l}{2}\right) \tag{2.27}$$

$$\psi(x) = (\lambda + 1) \int_{-l/2}^{l/2} G_1(x, y)\, \psi(y)\, dy \quad , \quad \psi \in L^2_-\left(-\frac{l}{2}, \frac{l}{2}\right) \tag{2.28}$$

$G_0(x, y)$ and $G_1(x, y)$ being of course the Green function of the parts of A in $L^2_+(-l/2, l/2)$ and $L^2_-(-l/2, l/2)$, respectively. For $G_0(x, y)$ one easily finds the following explicit form:

$$
(2.29) \qquad G_0(x, y) = \begin{cases}
\frac{1}{2} e^{\frac{x^2+y^2}{2}} \int\limits_y^{l/2} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \leq x \leq y \leq \frac{l}{2} \\
 & \text{and } -\frac{l}{2} \leq -y \leq x \leq \frac{l}{2} \\
\frac{1}{2} e^{\frac{x^2+y^2}{2}} \int\limits_{-l/2}^{x} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \leq x \leq y \leq \frac{l}{2} \\
 & \text{and } -\frac{l}{2} \leq x \leq -y \leq \frac{l}{2} \\
\frac{1}{2} e^{\frac{x^2+y^2}{2}} \int\limits_{-l/2}^{y} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \leq y \leq x \leq \frac{l}{2} \\
 & \text{and } -\frac{l}{2} \leq x \leq -y \leq \frac{l}{2} \\
\frac{1}{2} e^{\frac{x^2+y^2}{2}} \int\limits_{x}^{l/2} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \leq y \leq x \leq \frac{l}{2} \\
 & \text{and } -\frac{l}{2} \leq -y \leq x \leq \frac{l}{2} \,.
\end{cases}
$$

By application of (2.21) we easily get, in the subspace $L^2_-(-l/2, l/2)$, the explicit formula for the orthogonal invariant ${}^0\mathcal{I}_1^2$:

$$
(2.30) \qquad {}^0\mathcal{I}_1^2 = 2 \int\limits_0^{l/2} \left[e^{x^2} \left(\int\limits_x^{l/2} e^{-\xi^2}\, d\xi \right)^2 \int\limits_0^x e^{-\xi^2}\, d\xi \right] dx.
$$

In a completely analogous way one has in $L^2_-(-l/2, l/2)$:

$$
(2.31) \qquad G_1(x, y) = \begin{cases}
\frac{C}{2} e^{\frac{x^2+y^2}{2}} \int\limits_y^{l/2} e^{-\xi^2}\, d\xi \int\limits_{-x}^{x} e^{-\zeta^2}\, d\xi \quad , & -\frac{l}{2} \leq x \leq y \leq \frac{l}{2} \\
 & \text{and } -\frac{l}{2} \leq -y \leq x \leq \frac{l}{2} \\
-\frac{C}{2} e^{\frac{x^2+y^2}{2}} \int\limits_{-l/2}^{x} e^{-\xi^2}\, d\xi \int\limits_{-y}^{y} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \leq x \leq y \leq \frac{l}{2} \\
 & \text{and } -\frac{l}{2} \leq x \leq -y \leq \frac{l}{2} \\
-\frac{C}{2} e^{\frac{x^2+y^2}{2}} \int\limits_{-l/2}^{y} e^{-\xi^2}\, d\xi \int\limits_{-x}^{x} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \leq y \leq x \leq \frac{l}{2} \\
 & \text{and } -\frac{l}{2} \leq x \leq -y \leq \frac{l}{2} \\
\frac{C}{2} e^{\frac{x^2+y^2}{2}} \int\limits_{x}^{l/2} e^{-\xi^2}\, d\xi \int\limits_{-y}^{y} e^{-\xi^2}\, d\xi \quad , & -\frac{l}{2} \leq y \leq x \leq \frac{l}{2} \\
 & \text{and } -\frac{l}{2} \leq -y \leq x \leq \frac{l}{2}
\end{cases}
$$

and:

$$(2.32)\qquad {}^{1}\mathcal{I}_1^2 = 8\,\mathrm{C}^2 \int_0^{l/2} \left[e^{-x^2} \left(\int_x^{l/2} e^{-\xi^2} \mathrm{d}\xi \right)^2 \int_0^x e^{t^2} \left(\int_0^t e^{-\xi^2} \mathrm{d}\xi \right)^2 \mathrm{d}t \right] \mathrm{d}x \,.$$

3. Numerical results

In this section we give the numerical results concerning the first 20 eigenvalues. The approximate values obtained for these eigenvalues can be read in Tables I-IV: Table I corresponds to $l = 1$, Table II to $l = 2$, Table III to $l = 4$, Table IV to $l = 8$. In these tables the upper bound is that given by the Rayleigh-Ritz method for $\mathrm{N} = 50$: i.e. the infinite matrices (2-9) and (2-13) have been truncated at $\mathrm{N} = 50$. The lower bound is that obtained through the corresponding formulae (2.20). The orthogonal invariant ${}^{0}\mathcal{I}_1^2$ and ${}^{1}\mathcal{I}_1^2$ have been computed numerically starting from their explicit expression (2-30) and (2-32)(1).

TABLE I ($l = 1$)

Even eigenvalues			Odd eigenvalues		
	Lower bounds (Fichera)	Upper bounds (Rayleigh-Ritz)		Lower bounds (Fichera)	Upper bounds (Rayleigh-Ritz)
λ_0	9.9022575	9.9022587	λ_1	39.549013	39.549069
λ_2	88.9035	88.9042	λ_3	157.9905	157.9939
λ_4	246.808	246.822	λ_5	355.350	355.388
λ_6	483.59	483.70	λ_7	631.52	631.74
λ_8	799.08	799.53	λ_9	986.2	987.1
λ_{10}	1192.8	1194.4	λ_{11}	1418.9	1421.4
λ_{12}	1664.0	1668.1	λ_{13}	1928.5	1934.6
λ_{14}	2211	2221	λ_{15}	2513	2527
λ_{16}	2832	2853	λ_{17}	3171	3198
λ_{18}	3524	3564	λ_{19}	3897	3948

TABLE II ($l = 2$)

Even eigenvalues			Odd eigenvalues		
	Lower bounds (Fichera)	Upper bounds (Rayleigh-Ritz)		Lower bounds (Fichera)	Upper bounds (Rayleigh-Ritz)
λ_0	2.5969190	2.5969197	λ_1	10.151145	10.151165
λ_2	22.51747	22.51766	λ_3	39.7984	39.7994
λ_4	62.007	60.011	λ_5	89.144	89.155
λ_6	121.207	121.223	λ_7	158.19	158.25
λ_8	200.07	200.20	λ_9	246.86	247.08
λ_{10}	298.51	298.89	λ_{11}	355.03	355.64
λ_{12}	416.3	417.4	λ_{13}	482.4	484.0
λ_{14}	553.1	555.5	λ_{15}	628.6	632.0
λ_{16}	708.4	713.5	λ_{17}	793.0	799.8
λ_{18}	881.5	891.1	λ_{19}	974.7	987.3

(1) The lower bounds of the present paper can be further improved taking advantage of a suggestion of Dr. C. Cassisa and Dr. R. Ambrosetti. The improved bounds will appear in a paper by V. Franceschini, in press on the "Atti Sem. Mat. Fis. Un. Modena", *24*, Issue 1, 1975. For istance, for $l = 1$ one has: $9.902258640 < \lambda_0 < 9.902258648$. It is a pleasure to thank Dr. C. Cassisa and Dr. R. Ambrosetti for their interesting remark.

TABLE III ($l = 4$)

Even eigenvalues			Odd eigenvalues		
	Lower bounds (Fichera)	Upper bounds (Rayleigh-Ritz)		Lower bounds (Fichera)	Upper bounds (Rayleigh-Ritz)
λ_0	1.0749204	1.0749225	λ_1	3.529613	3.529633
λ_2	6.79947	6.79958	λ_3	11.1688	11.1693
λ_4	16.7365	16.7378	λ_5	23.5268	23.5300
λ_6	31.544	31.553	λ_7	40.791	40.808
λ_8	51.263	51.295	λ_9	62.959	63.016
λ_{10}	75.870	75.970	λ_{11}	89.99	90.16
λ_{12}	105.31	105.58	λ_{13}	121.8	122.3
λ_{14}	139.5	140.2	λ_{15}	158.3	159.3
λ_{16}	178.3	179.7	λ_{17}	199.4	201.2
λ_{18}	221.5	224.1	λ_{19}	244.8	248.1

TABLE IV ($l = 8$)

Even eigenvalues			Odd eigenvalues		
	Lower bounds (Fichera)	Upper bounds (Rayleigh-Ritz)		Lower bounds (Fichera)	Upper bounds (Rayleigh-Ritz)
λ_0	0.999973	1.000001	λ_1	2.99981	3.00003
λ_2	4.99964	5.00041	λ_3	7.0016	7.0034
λ_4	9.0157	9.0193	λ_5	11.072	11.079
λ_6	13.232	13.243	λ_7	15.571	15.588
λ_8	18.157	18.183	λ_9	21.031	21.068
λ_{10}	24.205	24.262	λ_{11}	27.68	27.77
λ_{12}	31.47	31.60	λ_{13}	35.56	35.74
λ_{14}	39.93	40.19	λ_{15}	44.61	44.95
λ_{16}	49.56	50.02	λ_{17}	54.80	55.41
λ_{18}	60.27	61.10	λ_{19}	66.05	67.11

In the following Table V we report the numerical results obtained in [5] for the first eigenvalues for $l = 2, 4$, comparing them with those obtained in the present paper.

TABLE V

	$l = 2$		$l = 4$	
	[5]	Present paper	[5]	Present paper
λ_0	2.596	2.5969190–2.5969197	1.075	1.0749204–1.0749225
λ_1	10.15	10.151145–10.151165	3.529	3.529613–3.529633
λ_2	22.52	22.51747–22.51766	6.799	6.79947–6.79958
λ_3	39.80	39.7984–39.7994		

In the following Table VI we compare our results with those obtained in [6] again through an approximate solution of the differential equation. Here the lowest (ground state) eigenvalue is examined.

TABLE VI

l	λ_0 in [6]	λ_0 in present paper
1	9.90225	$9.9022575 < \lambda_0 < 9.9022587$
2	2.59691	$2.5969190 < \lambda_0 < 2.5969197$
4	1.07492	$1.0749204 < \lambda_0 < 1.0749225$

As is well known, the eigenvalues of the present problem converge monotically downward to the eigenvalues of the harmonic oscillator given by $\mu_n = 2n + 1$, $n = 0, 1, 2, \cdots$

Such a convergence is so fast that already for $l = 8$ it can be seen from Table IV that μ_0 is a better lower bound for λ_0 then that reported, as far as the lowest eigenvalue is concerned. Of course this is still not true for higher eigenvalues.

Let us conclude by remarking that Chandrasekhar in 1943 [1] obtained, for λ_1 ($l = 8$), trough a very simple ingenious approximation, the value: $\lambda_1 = 3.0026$, to be compared with: $3.00000 < \lambda_1 < 3.00003$.

REFERENCES

[1] S. CHANDRASEKHAR (1943) – *Dynamical Friction*, II, « Astr. Jour. », 263–273.

[2] F. C. AULUCK and D. S. KOTHARI (1945) – *The Quantum mechanics of a bounded linear harmonic oscillator*, « Proc. Cambridge Phyl. Soc. », *41*, 175–179.

[3] J. S. BAIJAL and K. K. SINGH (1955) – *The energy levels and the transition probabilities for a bounded linear harmonic oscillator*, « Progr. Theor. Phys. », *14*, 214–224.

[4] P. DEAN (1966) – *The constrained quantum mechanical harmonic oscillator*, « Proc. Cambridge Phyl. Soc. », *62*, 277–286.

[5] R. VAWTER (1968) – *Effects of finite boundaries on a one-dimensional harmonic oscillator*, « Phys. Rev. », *174*, 749–757.

[6] R. VAWTER (1973) – *Energy eigenvalues of a bounded centrally located harmonic oscillator*, « Jour. Math. Phys. », *14*, 1864–1870.

[7] G. FICHERA (1965) – *Linear elliptic differential systems and eigenvalue Problem*, « Lecture Notes in Math. », *8*, Springer–Verlag, Berlin–Heidelberg–New York.

[8] G. FICHERA (1973) – *Abstract and numerical aspects of eigenvalue theory*, « Dep. Math. », Un. of Alberta, Echmonton, Alberta, Canada.

[9] G. FICHERA (1974) – *Metodi e risultati concernenti l'analisi numerica e qualitativa*, « Mem. Acc. Naz. dei Lincei », ser. VIII, *12* (1).

[10] See, e.g., the treatment of Ref. [7], [8], [9] above.

[11] S. G. MIKHLIN and K. L. SMOLITSKIY (1967) – *Approximate solutions of differential and integral equations*, Elsevier, New York, London, Amsterdam, Chapt. 3, §2, 8.