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Geometria algebrica. — *Geometrical theory of the vanishing of theta-functions for complex algebraic curves.* Nota II di FEDERICO GAETA, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — La classica teoria riemanniana dell'annullamento di $\theta_c \circ \int_P : C \to \mathbf{C}$ (12) diventa la discussione algebrico-geometrica delle intersezioni del divisore D_c rappresentato da $\theta(u+c) = 0$ in $J(C)$ (dipendente da $c+L$ variabile nella Jacobiana duale $\tilde{J}_{p-1}(C)$) (cfr. Nota precedente (13)) con la superficie fissa $\Sigma = \alpha(C \times C)$ dove $\alpha : C \times C \to J(C)$ è definita da $\alpha(x, y) = |y - x|$. Tutta la discussione dipende dall'indice di specialità $s\,|L_{p-1}|$ di una classe d'equivalenza lineare $|L_{p-1}|$ di grado $p-1$ variabile in $J_{p-1}(C)$ (cfr. § 3); tutti i teoremi classici s'interpretano agevolmente sull'imagine $\beta_d(\Sigma)$ per la dilatazione β_d indotta da $J(C) \to \mathbf{P}^N$ $(N = d^p - 1)$ definita dalle θ^d $(d \geq 2)$ annullantisi in O.

INTRODUCTION

The classical vanishing theory of $\theta_c \circ \int_P : C \to \mathbf{C}$ (14) depends on the point $c + L$ (15) of the torus $\mathbf{C}^p/L$ but it depends also very strongly on the origin of integration P (16). If we allow free variation of both ends of integration x, y, the theory is transformed in a standard intersection problem of two subvarieties Σ and D_c of the Jacobian $J(C)$ of C. Σ is the surface image

(*) Nella seduta dell'11 gennaio 1975.

(12) Cfr. ENRIQUES-CHISINI, *Teoria geometrica delle equazioni...*, Vol. IV, *Funzioni ellittiche e abeliane*, Capitolo III, § 3, p. 178.

(13) This paper is the continuation of our previous preparatory Note *Dual Jacobians and correspondences* (p, p) of valency -1. « Rend. Acc. Lincei », *58* (1), 1975. The notations, and the numeration of paragraphs, definition formulas and footnotes continue from the previous paper.

(14) $\theta \circ \int_P$ indicates for short $\theta\left(\cdots, \int_P^x \omega_j, \cdots\right)$ where the ω_j $(j = 1, 2, \cdots, p)$ are a normalized basis of holomorphic differentials with normalized Riemann matrix $(\tau\, \mathbf{1}_p)$ $\tau \in G_p$ (Siegel's p-upper half-plane) and $\mathbf{1}_p$ the $p \times p$ unit matrix). θ is defined in $H_1(C, \mathbf{C})$. Cfr. A. WEIL, *Varietés kählériennes*, « Act. Sci. Ind. », 1267, Hermann Paris 1958.

(15) L, *the lattice of periods*, is generated by the $2p$ columns of $(\tau\, \mathbf{1}_p)$.

(16) Cfr. LEWITTES' paper: *Riemann surfaces and the theta function*, « Acta mathematica », *111* (1964), 37-61 for a careful classification of the rôle of the base point $P \in C$. The most recent report of the classical subject is RAUCH-FARKAS, *Theta functions with applications to Riemann surfaces*, Williams and Wilkins, Co. Baltimore 1974; cfr. also C. L. SIEGEL, *Topics in complex function theory*, Vol. II, Interscience J. Wiley, 1971, Ch. IV, § 10, where the reader can find the complete references. The « Bible » is still KRAZER's, *Thetafunktionen*, Leipzig (1903), photocopied by Chelsea, N. Y. in 1970.

of $C \times C$ by the maps α, $\int = \lambda \circ \alpha$ appearing in the commutative diagram:

$$\begin{array}{ccc} C \times C & \xrightarrow{\alpha} & J_0(C) \\ & \searrow^{\int} & \downarrow \lambda \\ & & \mathbf{C}^p/L \end{array} \tag{14}$$

where $\alpha(x, y) = |y - x|$ $((x, y) \in C \times C)$ and the vertical arrow λ is the isomorphism of $J(C)$ with the complex torus $\mathbf{C}^p/L$ given by Abel's Theorem [17]. D_c is the irreducible divisor of $J(C)$ characterized by $\theta(u + c) = 0$ [18]. The discussion has a slight complication due to the fact that Σ (the reduced model of $C \times C$, cfr. § 4) has a singular point at the origin $0 = \alpha(\Delta)$; Δ *is recovered with a " blowing up "* $\beta_d : \Sigma \to \mathbf{P}^N$ $(N = d^p)$ *with theta functions of any fixed order* d (≥ 2) *vanishing at zero.* We need to distinguish the two cases: $d \geq 3$ and $d = 2$. For $d \geq 3$, $\beta_d(\Sigma)$ *is a non singular model of the square* $C \times C$. In the case $d = 2$ $\beta_2(\Sigma)$ *is a non singular model of the symmetric square* $C^{(2)}$. In both cases Δ is recovered as the diagonal (resp. as the branch curve) of $\beta_d(\Sigma)$ $(d \geq 3)$ (of $\beta_2(\Sigma)$). Our discussion on correspondences (cfr. *loc. cit.* in [13]) enables us to summarize the vanishing theory in the statement of Theorem 2 below, and then to characterize geometrically the natural origin $|0_{p-1}|$ of $J_{p-1}(C)$ corresponding to the " vector of Riemann constants " [19] in terms of B. Segre's " first covariant of immersion " of the pair (Σ, R), $\Sigma \subset R$ [20] where R is the *Riemann divisor* of $J(C)$ (cfr. § 5). $|0_{p-1}|$ is a well-defined distinguished half-canonical divisor class on C such that $|p_1^{-1}(0_{p-1}) + p_2^{-1}|0_{p-1}| + \Delta|$ is Severi's functional equivalence [21] of Σ in the virtual intersection $\Sigma \cdot R$. Let $\theta_R(u) = 0$ be any first order theta representing the Riemann divisor $\theta(u + K(P)) = 0$ [22]. R is the natural

(17) The vertical map λ of the diagram (11) is defined mod. L by $D \mapsto \left(\int_\gamma \omega_1, \cdots, \int_\gamma \omega_p\right)$ where γ is any differentiable 1-chain on C such that $\partial\gamma = 0$. Abel's Theorem is equivalent to the exactness of the sequence $0 \to \mathscr{L}_0(C) \to \mathrm{Div}_0(C) \to \mathbf{C}^p/L \to 0$.

(18) Classically $\theta : \mathbf{C}^p \to \mathbf{C}$ is identified with the standard theta series in p complex variables attached to $(\tau\, \mathbf{1}_p)$. Later on we shall assume that the θ's are defined deductively from the « Appell-Humbert Theorem » as a natural analytic tool to define divisors on $\mathbf{C}^p/L$. Cfr. WEIL, *loc. cit.* in [14].

(19) Cfr. RAUCH-FARKAS' book, p. 161. The distinction between $J(C)$ and its dual $\tilde{J}(C)$ helps to clarify the rôle of both base points. « The origin » of $J_0(C)$ is clear. The natural origin for $\tilde{J}(C)$ is the *Riemann theta divisor*, represented by $\{cl(E_{p-1}) - cl(H_{p-1})\} \cdot (E_{p-1} \in C^{p-1})$ is a half-canonical class depending on the " marking of the Riemann surface C " (choice of a canonical basis in $H_1(C; \mathbf{Z})$ in J_0 i.e. by $C^{(p-1)}/\equiv$ in J_{p-1}. Cfr. § 6.

(20) Cfr. B. SEGRE, *Dilatazioni e varietà canoniche*, « Annali di Matematica », *54* (1954), 139–155.

(21) This functional equivalence, originally defined by Severi using a natural limit process, can be interpreted today as the first Chern class (in the « Chow ring » of Σ) of the vector bundle obtained from Σ by blowing up of Σ in R. Cfr. B. SEGRE, *loc. cit.*

(22) Cfr. footnotes [16], [18] $K(P)$ is the vector of Riemann constants depending on the integration origin P.

origin in $\tilde{J}(C)$, thus in our notations $\theta_c(u)$ is a theta of characteristic $c + L \in \mathbf{C}^p/L$ representing $\theta_R(u + c) = 0$ [23].

THEOREM 2. *The intersection cycle* $D_c \cdot \Sigma$ (when it is defined) *blows up* (*for* $d \geq 3$) *as a correspondence of type* $(p, p; -1)$ *of the Jacobian type* (*cfr. Definition* 5). $D_c \cdot \Sigma$ *satisfies a linear equivalence*

$$\beta_d(D_c \cdot \Sigma) \equiv p_1^{-1}(\tilde{L}_{p-1}) + p_2^{-1}(L_{p-1}) + \Delta \tag{15}$$

in $\beta_d(\Sigma)$, *where* L_{p-1} *corresponds bijectively to* $c + L \in \mathbf{C}^p/L$, i.e. $|L_{p-1}|$ *belongs to our* J_{p-1} *model of the Jacobian* $J(C)$ (cfr. § 2).

The discussion depends entirely on the specialty index s *of* $|L_{p-1}|$. *In the case* $s = 0$, $|L_{p-1}|$ *is not effective and conversely. Then* $D_c \cdot \Sigma$ *exists and it is a fix-point free correspondence* (*satisfying* (15)), i.e. *its graph does not meet the diagonal. For* $s = 1$, (15) *becomes an equality in which* $L_{p-1}, \tilde{L}_{p-1}$ *are well determined positive divisors of* C, *complementary with respect to the canonical series*: $\beta_d(D_c \cdot \Sigma) = p_1^{-1}(\tilde{L}_{p-1}) + p_2^{-1}(L_{p-1}) + \Delta$. *In the case* $s = 1$, $\theta_c(0) \neq 0$; *for* $s = 1$ θ_c *vanishes with multiplicity equal to one in zero.*

In the case $s \geq 2$ *both complementary complete linear systems* $|L_{p-1}|$, $|K - L_{p-1}|$ *appearing in* (15) *have dimension* $s - 1 > 0$. *The effective divisors of the l.e.c. of the right hand side of* (15) *form a complete linear system of positive dimension. Then* $D_c \supset \Sigma$, *thus* $D_c \cdot \Sigma$ *does not exist in the strict sense, but Severi's functional equivalence of* Σ *in* $D_c \cdot \Sigma$ *exists and it is still given by* (15). *Moreover* θ_c *vanishes at zero with multiplicity equal to* s. *The leading term* $\Sigma(\partial^s \theta_c / \partial u_1^{j_1} \partial u_2^{j_2} \cdots \partial u_p^{j_p}) \partial u_1^{j_1} u_2^{j_2} \cdots u_p^{j_p}$ *of the Taylor expansion of* θ *at zero is not identically zero. Thus it represents a hypersurface of order* s *containing the infinitesimal model* Γ *of* C *at zero* [24]. (Cfr. § 4).

If the divisor D_c *of* J(C) *is symmetric with respect to the origin then everything in the previous discussion becomes symmetric, because* Σ *is also symmetric*; *e.g. for* $s = 0, 1$ *the intersection cycle* $D_c \cdot \Sigma$ *is a symmetric correspondence, for* $s \geq 2$ *the linear system of* (15) *is also symmetric, thus* $|L_{p-1}| = |K - L_{p-1}|$ i.e. $|L_{p-1}|$ *is half-canonical. If the origin of* $J_{p-1}(C)$ *is chosen to be half-canonical* (*for instance for the canonical choice*) *then the symmetry with respect to the origin is represented faithfully by the map* $|L_{p-1}| \mapsto |K - L_{p-1}|$. *In this way we can interpret geometrically the discussion of the first order thetas with half-integer characteristics* (D_c *is symmetric iff* θ_c *is even or odd*). θ_c *is*

(23) We shall prove Theorem 2 without choosing any origin and later on, as a consequence the mentioned will appear as the natural origin, for $\tilde{J}_{p-1}(C)$.

(24) The infinitesimal neighborhood of zero in Σ can be identified naturally with a canonical model of (in the standard sense) which appears also canonically in the construction. It helps to understand Andreotti-Mayer's success in finding extra relations among the τ_{ij} of τ (cfr. footnote [13]) expressed by the vanishing of the theta nulls and their partial derivatives at the origin. Cfr. ANDREOTTI-MAYER, *On period relations for Abelian integrals on algebraic curves*, « Annali della Scuola Normale Superiore di Pisa », *21* (2), 1967.

even or odd according to the parity of the specialty index $s(L_{p-1})$ [25]. *For* $s=0$, $\theta_c(0)\neq 0$, *for* $s=1$ θ_c *vanishes simply at zero and* $\beta_d(D_c\cdot\Sigma)$ *degenerates in the form* $\Delta + p_1(H_{p-1}) + p_2(H_{p-1})$ *where* H_{p-1} *is half-canonical, effective and linearly isolated* ($s(H_{p-1})=1$).

The importance of the vanishing theory of $\theta_c\int_P$ is well-known [26]. The last updating we know is Lewittes' report (cfr. footnote [16]). The Author is preparing a new report [*], from the previously described, algebraic geometric point of view, in which we use the projective non singular embeddings of J(C) constructed with the d^p-dimensional complex vector space of theta functions of order d [27] (i.e. attached to ($\tau\,\mathbf{1}_p$), for any $d\geq 3$ and the Kummer-Wirtinger double covering of J(C) constructed with the second order theta functions [28]. In this Note we summarize the report as follows: § 4 deals with the construction of Σ, showing that the infinitesimal curve Γ (cfr. Introduction) is a canonical model of C in the projective space $\mathbf{P}^{p-1}$ of tangent lines to $J_0(C)$ at zero. In § 5 we outline the proof of Theorem 2. Finally in § 6 we study more closely the choice of the origin in $J_{p-1}(0)$ which was not needed in the proof.

4. Σ AS A MODEL OF THE REDUCED PRODUCT $(C\times C)_\Delta$ [29] AND THE INFINITESIMAL MODEL Γ ON J(C)

If C *is not hyperelliptic the restriction of* α (*or* $\int$) *to* $C\times C-\Delta$ *is injective*; besides $\alpha(\Delta)=\int(\Delta)=0$; in other words: Σ *is a model of the reduced product* $(C\times C)_\Delta$ (cfr. footnotes [6], [29]). As a consequence *both irrational pencils of genus* p $\{x\times C\}_{c\in C}$, $\{C\times y\}_{x\in C}$ *of mutually unisecant curves are represented in* Σ *by pencils of birational models of* C *all passing through zero.*

(25) The use of "half integer characteristics" $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}=\begin{bmatrix}\varepsilon_1\ \varepsilon_i\cdots\varepsilon_j\\ \varepsilon'_1\ \varepsilon'_2\cdots\varepsilon_p\end{bmatrix}$ (cfr. RAUCH-FARKAS; *loc. cit.*) cannot be made in an intrinsic geometric discussion before establishing an isomorphism $J_{p-1}\approx\mathbf{C}^p/L$, equivalent to the choice of an origin in J_{p-1} (Cfr. § 6). Such canonical choice will be justified after discussing the case $s\geq 2$ directly in J_{p-1} (cfr. footnote [20]).

(26) RIEMANN's theory on the vanishing of the theta function has not the same legendary prestige as the theory of the $\zeta(s)\cdots$ but it still inspires lyric prose as this: "... *We now come to the pièce de résistance: the theorem of theorems perhaps the crowning glory of Riemann's; creations in this area (or perhaps any area of his creations) the Riemann vanishing theorem*", Cfr. RAUCH-FARKAS, *loc. cit.*, p. 170.

(27) For a proof cfr. SIEGEL, *Moduln Abelscher Funk ionen* Nachrichten der Akademie der Wiss. Gesammelte Abhandjungen, vol. III, Springer, 1966.

(28) WIRTINGER, *Untersuchungen über Thetafunktionen*, Teubner (1895).

(29) We call *reduced product* $(C\times C)_\Delta$ of C (and *reduced symmetric square* $C^{(2)}_\Delta$) the surface obtained from $C\times C$ (resp. from $C^{(2)}$) by collapsing Δ to a single point. The fact that $(C\times C)_\Delta$ or $C^{(2)}_\Delta$ are algebraic follows from the α-map itself. Another proof was given in § 3 by means of a direct study of the complete linear systems of divisor in $C\times C$. Full details will appear in a third forthcoming Lincei Note

(*) GAETA, *Riemann's theta vanishing as a correspondence problem*, to appear in the "Contributi del Centro Linceo interdisciplinare per le Scienze matematiche e loro applicazioni".

Moreover the images of C_x $(= x \times C)$ *and* C^x $(= C \times x)$ *are tangent at zero.* The point zero is a singularity of Σ of a very simple type that can be removed by " blowing up " with the thetas of any order $d\,(\geq 3)$ vanishing at zero. *We shall assume in the sequel that* C *is not hyperelliptic.* This implies $p \geq 3$. Let us see how to construct the " infinitesimal model " Γ of C. The maps $x \mapsto |x - P|$, $x \mapsto |P - x|$ for arbitrarily fixed P are always injective. Their images are the curves C_p, C^p. Let t_p be the tangent line to C_p at zero. When we move P in C the image of C by the map $P \mapsto t_P$ is a conical algebraic surface Cone (Γ) of vertex zero in the tangent vector space $T_0(J)$. We call Γ the corresponding projection in the projective space $\mathbf{P}^{p-1}$ of tangent lines to J at zero.

Γ is a canonical model in the standard sense (30). It is constructed " canonically " as well in the sense that no arbitrary privileged points of C are used in the construction (31).

The non singular model of J (C) obtained with the θ^d $(d \geq 3)$ contains a projective model $\Sigma^{(d)}$ of Σ in $\mathbf{P}^N$ $(N = d^p - 1)$; $\Sigma^{(d)}$ has still the same singularity at zero as Σ.

A similar fact happens for $d = 2$: the Wirtinger variety contains a model $\Sigma^{(2)}$ of $C_\Delta^{(2)}$ in $(2^d - 1)$-projective space. In both cases the " blowing up " of zero by means of all the hyperplanes through zero (which amounts to a projection from zero to a hyperplane in the ambient space) allows us to recover Δ as a subvariety of the non singular projective model $\Sigma^{(d,0)}$ $(d \geq 3)$ (resp. $\Sigma^{(2,0)}$) are non-singular projective models of $C \times C$ (respectively $C^{(2)}$) containing a non singular model of the diagonal Δ (resp. of the branch curve Δ). The explicit construction can be easily described as follows: Let $\theta_1, \theta_2, \cdots, \theta_M$ be a basis of the complex vector space of dimension $d^p - 1$ consisting of all the theta functions of order d vanishing at zero. The parametric equations

$$z_j = \theta_j\left(\cdots, \int_x^y \omega_j, \cdots\right) = f_j + \cdots \qquad j = 1, 2, \cdots, M \tag{16}$$

for $(x, y) \in C \times C$ describe $\Sigma^{(d,0)}$ in $(d^p - 2)$-projective space $(d \geq 2)$, where the z_j are homogeneous coordinates of the image point of (x, y) $(d \geq 3)$ or the image point of the divisor $x + y$ in the case $d = 2$. In the case $x \neq y$ the meaning of (16) (which is always well-defined) is clear. For $x = y$ the

(30) I.e. associated to a global section of the cotangent bundle. Explicity: if $v_x \in T_x(C)$ is a non zero tangent vector to C at x then $\omega_1(v_x), \omega_2(v_x), \cdots, \omega_p(v_x)$ to the homogeneous coordinates of the canonical embedding of C defined by the ω-basis: $C \to \mathbf{P}^{p-1}$ is injective iff C is not hyperelliptic. The curve Γ appears also in Andreotti-Mayers's (cfr. *loc. cit.*, p. 206) using a kind of " GAUSS map " in a particular C_P, i.e. by translating to zero the tangent line to C_P in a variable $Q \in C_P$ with the unique torus translation $Q \mapsto 0$. It is trivial to check that Γ is identical with the curve defined in the text without fixing any P in C.

(31) On the contrary, the embeddings C_P depend on an embarrassing choice of the arbitrary point $P \in C$.

leading terms f_j ($j = 1, 2, \cdots, M$) of the Taylor expansions of the thetas at zero (remember that all vanish at zero) define the image of the diagonal point (x, x) (cfr. § 6). In the case $d \geq 3$ the f_j are linear forms which can be identified with holomorphic differentials on C, thus the image of Δ is a canonical model of C.

In the case $d = 2$ the f_j are quadratic differentials on C, thus the image of Δ is a bicanonical model of C.

5. OUTLINE OF THE PROOF

The "Theorem of theorems" (cfr. footnote [26]) can be updated by showing directly that R is an irreducible divisor of J(C) containing O and symmetric respect to zero, such that its corresponding alternating form $A : L \times L \to \mathbf{Z}$ (cfr. Weil, *loc. cit.* in [14]) has its elementary divisors equal to one). The representations of R in the models J_0, J_{p-1}, were discussed in footnote [19]. We need the J_{p-1} representation (cfr. § 2). It is convenient to choose as a temporary origin any half-canonical $O_{p-1} > 0$. Then the isomorphism $J_{p-1} \xrightarrow{j} \mathbf{C}^p/L$ is uniquely defined by $|L_{p-1}| \xrightarrow{j} \left(\cdots, \int_\gamma \omega_j, \cdots\right)$ where $\partial\gamma = L_{p-1} - O_{p-1}$. Then Riemann's "Theorem of theorems" becomes $\theta_R(j(L_{p-1} - O_{p-1})) = 0$, iff L_{p-1} is *effective* (where $\theta_R(u) = 0$ represents R i.e. $\theta_R(j(L_{p-1} - O_{p-1})) = 0$ *iff there exist a positive* $E_{p-1} \in \mathrm{Div}_{p-1}(C)$ *such that* $L_{p-1} \equiv E_{p-1}$. Let $T_c(R)$ be the divisor of J (C) represented by $\theta_R(u + c) = 0$. The discussion on the vanishing of $\theta_R\left(c + \int_x^y\right) = 0$ can be done directly in J_{p-1} using only the "Theorem of theorems", without any further use of the "numerical torus [32]" $\mathbf{C}^p/L$. Let us distinguish the two cases $s = s(L_{p-1}) = 0, 1$ (the only ones where $D_c \cdot \Sigma$ exists) from those $s \geq 2$ equivalent to $D_c \supset \Sigma$. Case $s = 0$: Then $\theta_R\left(c + \int_x^y\right) = 0$ iff $\beta_d(x, y) \in \beta_d(\Sigma)$ ($d \geq 3$) is a pair of the following fix-point free correspondence Ω of indices $(p, p; -1)$ of the Jacobian type (cfr. § 3) $(x, y) \in \Omega$ iff there exists a positive E_{p-1} such that $L_{p-1} + y - x \equiv E_{p-1}$. But because of Riemann-Roch's Theorem there exists at least one positive $D_p \equiv L_{p-1} + y$ with $D_p - y \not> 0$ (because $s(L_{p-1}) = 0$. This D_p is non special (and as a consequence it is unique) because otherwise $L_{p-1} \equiv \tilde{D}_p - y > 0$ for some $\tilde{D}_p \equiv D_p$. This

(32) However we cannot do that here for lack of space. We call the attention however the proof of the case $s = 0$ gives an independent algebraic-geometric verification of the fact that the theta function has p zeros on the torus. The usual Riemann's original proof uses the logarithmic indicator of the theta.

shows that the correspondence $\Omega^{-1}: y \mapsto x$ is different form $\beta_d(\Sigma)$ thus the second index of Ω is equal to p. Ω is of the Jacobian type (cfr. § 3) because if $\tilde{L}_{p-1} \equiv K - L_{p-1}$ we have $|\tilde{L}_{p-1} - O_{p-1}| + |\tilde{L}_{p-1} - O_{p-1}| \equiv 0$ and because of the symmetry of R with respect to $\theta_R(c+u) = 0 \Longleftrightarrow \theta_R(-c-u) = 0$. Thus, the first index of the correspondence is also equal to p. The valency of Ω is equal to -1 because $D_p(y) - y \equiv L_{p-1}$ for every $y \in C$. Summarizing: Ω is defined by $y \mapsto D_p(y)$ where $D_p(y)$ is the unique positive divisor satisfying the previous linear equivalence. *The graph of* Ω *does not meet the diagonal*, otherwise we would have $D_z = z + E_{p-1}$ with $E_{p-1} > 0$, $E_{p-1} \equiv L_{p-1}$ in contradiction with $s(L_{p-1}) = 0$. Passing from $\beta_d(\Sigma)$ to Σ we have: in the case $s = 0$, $\theta_c(0) \neq 0$.

Case $s = 1$. The previous discussion becomes sharper because there is a *unique* positive $E_{p-1} \equiv L_{p-1}$ and for every y, we have $y \mapsto y + E_{p-1}$. On the other hand $s(E_{p-1}) = 1$ implies the existence of a unique positive complementary $\tilde{E}_{p-1}: E_{p-1} + \tilde{E}_{p-1} \equiv K$. The previous analysis leads to the the more precise *equality* $\Omega = p_1^{-1}(\tilde{E}_{p-1}) + p_2^{-1}(E_{p-1}) + \Delta$. In other words in the case $s = 1$, the intersection cycle still exists in $\beta_d(\Sigma)$ and it is a correspondence of the Jacobian type (cfr. Note I) degenerate in the diagonal and the respective projections of two linearly isolated $(p-1)$-divisors complementary with respect to the canonical series. Coming back to Σ and D_c we see that in the case $s = 1$, $\theta_R(c+0) = 0$ vanishes with multiplicity equal to one at zero. Moreover the differential $(d\,\theta_R(c+u))_{u=0}$ at the origin cuts Cone (Γ) in $p-1$ generators, obtained from the inverse image of E_{p-1} by the blowing up β_d.

The case $s \geq 2$. Then the previous discussions fail because for any pair $(x, y) \in C \times C$ we can find some positive E_{p-1} satisfying the previous linear equivalence, thus $\Sigma \subset D_c$ or equivalently: $\theta_R(c+u)$ *vanishes identically* in Σ. However a more accurate analysis shows that the, still meaningful! linear system $|p_1^{-1}(L_{p-1}) + p_2^{-1}(L_{p-1}) + \Delta|$ represents Severi's functional equivalence in the virtual intersection $D_c \cdot \Sigma$. We shall give the lenghty details in the report.

The classical fact that the multiplicity of $\theta_R(u+c) = 0$ at zero is equal to s at zero can be established also algebraic-geometrically with the following interesting complement.

The leading term φ *of order* s (cfr. § 4), (which is not identically zero) *represents a hypersurface of order* s *in* $\mathbf{P}^{p-1}$ *containing the canonical curve* Γ [33]. In fact, if if we write the local expressions of the ω_j in a local coordinate systems $\omega_j = \psi_d\, dz$ and we substitute these values in the Taylor expansion of θ_R at the origin: $\theta_R = \varphi_s + \varphi_{s+1}$ and then we divide by dz^s we see that φ_s vanishes at any non zero tangent vector at zero to any C_p (or C^p): *This is equivalent to the identical vanishing of* φ_s *in* Γ, q.e.d.

(33) This result is stated for $s = 2$ in ANDREOTTI-MAYER, *loc. cit.*

6. THE NATURAL ORIGINS OF J_{p-1} (C)

The previous discussion, without fixing any particular origin (among the half-canonical $| H_{p-1} |$) can be applied to the case R, Σ. Since $R \supset \Sigma$ the functional equivalence of Σ in R exists and it has the type $p_1^{-1}(o_{p-1}) + p_2^{-1}(o_{p-1}) + \Delta$ (with $2 o_{p-1} \equiv K$) and $| o_{p-1} |$ well defined with speciality index $s \geq 2$ (34). This $| o_{p-1} |$ is the origin implicitly used in " the vector of Riemann constants ".

(34) The case $s = 2$ should be considered as " generic ". The $| o_{p-1} |$ is a complete g^1_{p-1} attached to Andreotti-Mayer's quadric of rank 4 through Γ. We shall see in the report that the apparent " uniqueness " of the origin in RIEMANN's comutations depend on the fact that we use a " marked RIEMANN surface ", i.e. we add a cammical homology basis in $H_1(C ; \mathbf{Z})$ as an additional structure.