Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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## Dual Jacobians and correspondences (p, p) of valency -1

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **57** (1974), n.6, p. 542–547.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_6_542_0>

**Geometria algebrica.** — *Dual Jacobians and correspondences* $(p, p)$ *of valency* $-1$. Nota I di FEDERICO GAETA, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — La varietà abeliana $\tilde{J}(C)$ duale (1) della Jacobiana J(C) di una curva algebrica di genere $p > 2$ è rappresentata naturalmente da corrispondenze di indici $(p, p)$ con valenza $-1$ sul prodotto $C \times C$ (2).

## INTRODUCTION

The vanishing theory of the function $\Theta_c \circ \int_P : C \to \mathbf{C}$ with an arbitrarily fixed origin of integration $P \in C$ is replaced successfully with the correspondent vanishing theory of $\Theta_c \circ \int : C \times C \to \mathbf{C}$, i.e. $(x, y) \mapsto \theta\left(\cdots, \int_x^y \omega_j, \cdots\right)$ *with both* $x, y$, *variable in* C (3); $(\omega_1, \omega_2, \cdots, \omega_p$ is a basis of holomorphic differentials). Algebraic-geometrically speaking *this problem is equivalent to the discussion of the intersection of two subvarieties* $\Sigma$, $D_c$ where $\Sigma$ is the image of $C \times C$ by the map $C \times C \xrightarrow{\alpha} J(C)$ in the Jacobian J(C) (4) of C defined by $\alpha(x, y,) = x - y$ and $D_c$ is the principal polar divisor of J(C) characterized by $\theta(u + c) = 0$.

In a second Lincei Note (5) we shall prove that *if the intersection* $D_c \cdot \Sigma$ *exists it is represented* (in a suitable way (6)) *by a correspondence of indices*

(*) Nella seduta del 14 dicembre 1974.

(1) The principal polar divisors of a polarized Abelian veriety A are parametrized by another polarized Abelian variety $\tilde{A}$, the dual of A. Since the Jacobian has all the elementary divisors equal to one $\tilde{J}(C)$ is isomorphic with J(C) but we need to distinguish both carefully. $\tilde{J}(C)$ represents the divisors $\theta(u + c) = 0$, where $\theta(u)$ is the first order theta and $c$ (defined modulo the lattice of periods L) is variable.

(2) Cfr. SEVERI, *Trattato di Geometria algebrica*, Zanichelli, Bologna (1926), Ch. VI, pp. 170–284, for an algebraic treatment of the correspondences with valency. We need also the prior transcendental treatment given by HURWITZ, *Über algebraische Korrespondenzen und das verallgemeinerte Korrespondenzenprinzip*, «Math. Annalen», 3 d., *28* (1887), S. 561–585.

(3) C indicates simultaneously the irreducible, algebraic, non-singular complex curve of genus $p$ and its Riemann surface. Later on we shall assume that C is not hyperelliptic.

(4) As a pure abstract group J(C) is defined by the exact sequence $0 \to \mathscr{L}_0(C) \to \mathrm{Div}_0(C) \to J(C) \to 0$ where $\mathrm{Div}_0(C)$ is the submodule of divisors of degree zero of the divisor group Div(C) (additively written in this paper). $\mathscr{L}_0(C)$ is the subgroup of *principal divisors* $(f)$ («zeroes minus poles» of $f$) attached to a (non identically zero) rational function $f$ over C. Moreover J(C) has a well known structure as an Abelian variety of dimension $p$. We say that two divisors F, G of C are linearly equivalent and we write $F \equiv G$ iff $F - G \in \mathscr{L}_0(C)$.

(5) F. GAETA, *Geometrical theory of the vanishing of theta functions for complex algebraic curves*, «Rend. Acc. Naz. Lincei», *58* (1), 1975.

(6) We prove in (3) that if C is not hyperelliptic $\Sigma$ is obtained from $C \times C$ by collapsing the diagonal $\Delta$ to one point 0; 0 is singular for $\Sigma$ and *its infinitesimal neighborhood is an inte-*

$(p, p)$ *and valency* $-1$; *in the case* $\Sigma \subset D_c$ Severi's *functional equivalence of* $\Sigma$ *in* $\Sigma \cdot D_c$ *is represented by a linear system of correspondences of the same type* $(p, p; -1)$. *Conversely any such correspondence on* C *can be obtained in this way.*

*These results enable us to reconstruct completely the discussion of the vanishing of the theta.* In order to make a clear exposition we decided to study first these correspondences $(p, p; -1)$ (in the present paper) as a natural model of $\tilde{J}(C)$ (or of $J(C)$). Cfr. § 3.

## 1. RECALL ON CORRESPONDENCES WITH VALENCY

A divisor $E$ ($\Longleftrightarrow$ 1-cycle) on the surface $C \times C$ is the graph of an algebraic correspondence of indices $\alpha, \beta$ iff $\alpha = \deg E(x)$, $\beta = \deg E^{-1}(y)$ whenever $x \mapsto E(x)$, $y \mapsto E^{-1}(y)$ are defined by:

$$(1) \qquad E(x) = p_2((x \times C) \cdot E) \quad , \quad E^{-1}(y) = p_1((C \times y) \cdot E)$$

where $p_1, p_2$ are the projections $C \times C \to C$.

E should be distinguished from the inverse correspondence $E^{-1}$ = the image of E by $S : (x, y) \mapsto (y, x)$. If E is irreducible the intersection cycles $(x \times C) \cdot E$ and $(C \times y) \cdot E$ are defined for arbitrary $x, y \in C$. We shall assume E to be irreducible for the sake of simplicity (7).

DEFINITION 1. We say that E has valency $v (\in \mathbf{Z}, v \gtreqless 0)$ iff there exists a linear equivalence class $D(E) = D + \mathscr{L}(C)$ in C such that for any $x \in C$ we have

$$(2) \qquad vx + E(x) \equiv D.$$

The integer $v$ is uniquely determined by E whenever $p > 0$.

DEFINITION 2. We say that an irreducible correspondence E on C has a valency iff (2) holds for a suitable $v$, in other words iff for any pair of points $x, x' \in C$ we have $vx + E(x) \equiv vx' + E(x')$.

If E has a valency $v$, $E^{-1}$ has also a valency and moreover $v(E) = v(E^{-1})$. Hurwitz proved (cfr. *loc. cit.*) that if C has general moduli any $(\alpha, \beta)$-correspondence on C has a valency $v$. This property implies that $(X, Y, \Delta)$ *with* $X = x \times C$, $Y = C \times y$ and $\Delta = \{(x, y) \in C \times C \mid x = y\}$ *form a minimal basis for the algebraic equivalence* ($\equiv$) (8) on $C \times C$. In other words, for any given

*resting infinitesimal model of* C *lying in the tangent vector space* $T_0(J)$ at 0. The singularity at 0 can be removed by "blowing up" with theta functions of any order $d \geq 2$ vanishing at 0. For $d \geq 3$ we recover $\Delta$ as a curve on a projective model of $C \times C$. For $d = 2$ we recover $\Delta$ as well as the branch curve of the symmetric square $C^{(2)}$ of C.

(7) We leave the easy task of extending the statements to the general case.

(8) The *algebraic equivalence* (indicated by $\equiv$) was treated by Severi and several other mathematicians in a long series of papers. The last recent global formalization seems to be CHEVALLEY, *Anneaux de Chow*, Sem. E.N.S., Paris, $A \equiv B \Rightarrow A \equiv A$ but not conversely ($A, B \in \operatorname{Div}(C \times C)$). The distinction of $\equiv$ and $\equiv$ is important because $C \times C$ is an irregular surface.

divisor E on $C \times C$ there exist integers $r, s, t$ such that:

$$E \equiv rX + sY + t\Delta. \tag{3}$$

The integers $r, s, t$ are uniquely determined by E; they are related to $v$ by $r = \beta + v$, $s = \alpha + v$, $t = -v$ because of certain well known intersection numbers [9], i.e. we can rewrite (3) as

$$E \equiv (\beta + v) X + (\alpha + v) Y - v\Delta \tag{4}$$

Severi (*loc. cit.*) proved the existence of correspondences with any prescribed valency $v (\gtreqless 0)$ on any curve of genus $p > 0$.

Let us interpret the previous result and find an obvious converse property:

PROPOSITION 1. *Any correspondence* E *with valency* $v$ *satisfies a linear equivalence of type*:

$$E \equiv p_1^{-1}(B) + p_2^{-1}(A) - v\Delta \tag{5}$$

*where* B, A, *are divisors of* C *of degree* $r = \beta + v$, $s = \alpha + v$.

Conversely: Let A, B, be any pair of divisors of C. Then (5) defines a linear equivalence class $E + \mathcal{L}_0(C \times C)$ of corespondences whith valency $v$.

DEFINITION 3. We say that the linear equivalence class (l.e.c.) $|E|$ in $C \times C$ is defined by the ordered pair $(|A|, |B|)$ of l.e.c. in C and that $|E|$ determines $(|A|, |B|)$.

PROPOSITION 2. *The coincidence divisor* $E \cdot \Delta$ (*intersection cycle of* E *and* $\Delta$) *satisfies the linear equivalence*

$$E \cdot \Delta \equiv A + B + vK \quad [10] \tag{6}$$

*where* K *is a canonical divisor of* C. *The degree deg* $(E \cdot \Delta)$ *is given by the* Cayley-Brill-Hurwitz *formula*:

$$\deg (p_1(E \cdot \Delta)) = \deg (E \cdot \Delta) = \alpha + \beta + 2pv \tag{7}$$

*giving the "number of fixed points of the correspondence", see loc. cit. in* (3).

*Remark.* We are particularly interested in the correspondences E with $\alpha = \beta$ and "without fixed points" (precisely those with deg $(\Delta \cdot E) = 0$) because the "blown up" intersections of a $D_c$ with $\Sigma$ (cfr. Introduction) have this property. However, this property is not characteristic. We need the stronger requirement of next

(9) $[X, X] = [Y, Y] = 0$ ; $[X, \Delta] = [Y, \Delta] = 1$, $[\Delta, \Delta] = 2 - 2p$.

(10) We used the well-known property that if a virtual $\Delta'$ is linearly equivalent to $\Delta$ in $C \times C$, then $[\Delta', \Delta]$ is an *anticanonical divisor* $-K$, where $|K|$ is the canonical equivalence class in $\Delta$.

DEFINITION 4. *A correspondence E on* $C \times C$ *is said to be of the Jacobian type iff* E *has both indices equal to* $p$ *and the two attached l.e.c.* $|A|, |B|$ *of* Definition 3 *are complementary with respect to the canonical class* $|K| : A + B \equiv K$ *in* C.

Obvious consequences of Definition 4 are the following two propositions:

PROPOSITION 3. *A correspondence* E *of the Jacobian type on* $C \times C$ *has the following properties*:

1) E *satisfies the algebraic equivalence* (see footnote [8]):

$$E \equiv (p - 1) X + (p - 1) Y + \Delta \tag{8}$$

2) *The intersection divisor* $E \cdot \Delta$ *is linearly equivalent to zero in* $\Delta$;

3) *The "number of fixed points" of* E *is equal to zero*;

4) *The valency of* E *is equal to* $-1$ (cfr. (7)).

PROPOSITION 4. *There exists a* $(1-1)$ – *mapping* $j$ *between the set* $\mathscr{L}_{p-1}(C)$ *of l.e.c. of divisors of degree* $p - 1$ *in* C *and the set* $\tilde{J}(C)$ *of l.e.c. of* $(p, p; -1)$ – *correspondences of the Jacobian type* (cfr. Definition. 5); $j$ *is defined by:*

$$|L_{p-1}| \longleftrightarrow |p_2^{-1}(L_{p-1}) + p_1^{-1}(K - L_{p-1}) + \Delta| \tag{9}$$

*where* $|L_{p-1}| \in \mathscr{L}_{p-1}(C)$ *and* $|p_2^{-1}(L_{p-1}) + p_1^{-1}(K - L_{p-1}) + \Delta|$ *lies in* $C \times C$.

DEFINITION 5. A l.e.c. $|E|$ of correspondences of the Jacobian type (an $|L_{p-1} \in \mathscr{L}_{p-1}$) define $|L_{p-1}|$ (resp. $|E|$) iff they correspond to each other as described in Proposition 4.

COROLLARY. *If* $|E| \in \tilde{J}(C)$ *and it is attached to* $L_{p-1}$, *then the inverse* S(E) *is also of the Jacobian type and attached to* $|K - L_{p-1}|$. In particular:

*A symmetric correspondence* E *of the Jacobian type is attached to a half-canonical* $|H_{p-1}|$ (*i.e.* $2 H_{p-1} \equiv K$); *conversely any half-canonical l.e.c.* $H_{p-1}$ *defines a symmetric correspondence of the Jacobian type.*

## 2. MODELS $J_n(C)$ $(n \gtreqless 0)$ OF THE JACOBIAN

Let $D_n(C)$ be the coset of Div (C) mod $\text{Div}_0(C)$ consisting of all the divisors of degree $n : D_n(C) = \{G \in \text{Div}(C), \deg(G) = n\}$.

*The quotient set* $D_n(C)/\mathscr{L}_0(C) = J_n(C)$ *is equipotent with* $J_0(C) = D_0(C)/\mathscr{L}_0(C)$, as we can see using the bijection $J_n \longleftrightarrow J_0$ defined by $G \longleftrightarrow G - O_n$ ($O_n$ arbitrarily fixed origin in $D_n(C)$).

DEFINITION. 6. *We call* $J_n(C)$ *the* $n^{th}$*-model of the abstract Jacobian.*

The choice of $n$ is a matter of mathematical taste depending on the kind of structure that we want to endow J(C) with. For instance $J_0(C)$ shows the abelian group structure of J(C) very well.

The classical model $J_p(C)$ is good because any l.e.c. $G_p + \mathscr{L}_0(C)$ of degree $p$ contains at least one effective divisor $E_p$. Moreover this effective $E_p$ is "almost always" unique; precisely $E_p$ *is unique iff the specialty index of* $G_p + \mathscr{L}_0(C)$ *is equal to zero*. Thus we can identify $J_p(C)$ with the quotient set $C^{(p)}/\equiv$ where $C^{(p)}$ is the set of all effective $p$-divisors ($\Longleftrightarrow$ the $p$-fold symmetric power $C^{(p)}$ of C).

It is well-known and easy to show that $C^{(p)}$ is a projective algebraic variety, and from this fact we can deduce that $C^{(p)}/\equiv$ is also a projective algebraic variety, which can be built with theta functions of any fixed degree $d \geq 3$.

We are going to show in § 3 that *the model* $J_{p-1}(C)$ *deserves a particular attention also* in spite of the fact that a *non special class* $G_{p-1} + \mathscr{L}_0(C)$ *contains no effective divisors at all* (Riemann-Roch Theorem). We are going to see that *we can attach bijectively to any* point of $J_{p-1}(C)$ *a linear equivalence class of correspondences* $(p, p; -1)$ *of the Jacobian type* (cfr. Definition 4) *which contains "almost always" a unique effective correspondence.*

## 3. The model $J_{p-1}(C)$ of the Jacobian

Proposition 4 shows how we can establish a bijection (9) which gives an alternative definition of $J_{p-1}(C)$; But in Proposition 4 we obtain a linear equivalence class of correspondences rather that a unique one. We can sharpen this result by proving next Theorem.

THEOREM 1. *Any non special l.e.c.* $|L_{p-1}| \in J_{p-1}(C)$ *defines a unique correspondence* E *of the Jacobian type* (*cfr.* Definition 5) *of indices* $p$, $p$ *and valency* $-1$. *The graph of* E *does not meet the diagonal* $\Delta$.

The proof consist in an explicit construction of $x \mapsto E(x)$ but before proving the Theorem let us state and prove the following lemmas:

LEMMA 1. $A(p-1)$ - *l.e.c.* $|L_{p-1}|$ *is non special iff* $|L_{p-1}|$ *does not contain any effective divisor* [11].

In fact if $E_{p-1} > 0$ belongs *to* $|L_{p-1}|$, $i(E_{p-1}) \geq 1$ and $i(E_{p-1}) \geq 1$ implies $\dim |L_{p-1}| = -1 + i \geq 0$ (R-R Theorem again).

LEMMA 2. *Let $x$ be any point of* C. *Let* $|L_{p-1}|$ *be any non special l.e.c. of degree* $p-1$. *Then the l.e.c.* $|x + L_{p-1}|$ *of degree* $p$ *contains a unique effective divisor which does not contain* $x$.

In fact $|x + L_{p-1}|$ contains at last one effective divisor. If there are two $|x + L_{p-1}|$ defines an special $g_p^i$ $(i > 0)$ and there is al least one $\bar{G}_p$ of the $g_p^i$ containing $x$, thus $\bar{G}_p - x$ is effective against Lemma 1.

(11) We recall that a divisor $D = \sum_{I \in \mathscr{I}} n_I I$ on an algebraic variety V is an element of the free module over **Z** generated by the of irreducible subvarieties of V of codimension one. D is called *positive* $\Longleftrightarrow$ *effective* iff $D \neq 0$ and $n_I \geq 0$ for every $I \in \mathscr{I}$. Of course $A > B$ is equivalent to $A - B > 0$. To recall that a divisor D is not necessarily effective we say that it is virtual.

*Proof of the Theorem.* Let E be the correspondence $x \mapsto E(x)$, where $E(x)$ is the unique $G_p \equiv x + L_{p-1}$ constructed before. $E(x)$ is well-defined for any $x$ and $E(x) - x$ cannot be effective $\Longleftrightarrow$ *the graph of* $E(x)$ *does not meet the diagonal.*

In order to prove that E has type $(p, p; -1)$ it is convenient to rephrase the definition of E as follows:

*The pair* $(x, y) \in C \times C$ *belongs to* E *iff there exists an effective* $E_{p-1} \in \mathrm{Div}_{p-1}(C)$ *such that*

$$y + E_{p-1} - x \equiv L_{p-1}. \tag{10}$$

If K is a canonical divisor we have:

$$x + (K - E_{p-1}) - y \equiv K - L_{p-1}, \tag{11}$$

where $K - E_{p-1}$ contains also an effective $E'_{p-1}$. Thus E has indices $(p, p)$ and $L_{p-1} + (K - L_{p-1}) \equiv K$ is trivially true thus E *has the Jacobian Type.* This completes the proof.

We know already from § 1) that the converse is true:

*Let* E *be a* $(p, p; -1)$ *correspondence of the Jacobian type attached to* $(L_{p-1}, K - L_{p-1})$ (cfr. Definition 4) *such that graph* $E \cap \Delta = \varnothing$. *Then* $|L_{p-1}|$ *has specialty index* $= 0$.

Otherwise we could construct a degenerate model $L_{p-1} + \tilde{L}_{p-1} + \Delta$ with both $L_{p-1}$, $\tilde{L}_{p-1} (\equiv K - L_{p-1})$ effective against the hypothesis graph $E \cap \Delta = \varnothing$.

*Discussion of the case* $s(L_{p-1}) > 0$. In the previous case $s(L_{p-1}) = 0$ to equivalent to " $|L_{p-1}|$ does not contain any $L_{p-1} > 0$)" there exist a *unique effective* correspondence E linearly equivalent to $p_2^{-1}(L_{p-1}) + p_1^{-1}(K - L_{p-1}) + \Delta$ and it is not difficult to prove that E is irreducible. For any $s > 0$, $s - 1 = \dim |K - L_{p-1}| = \dim |L_{p-1}|$. In particular *for* $s = 1$ *there exists a unique effective* $E_{p-1} \equiv L_{p-1}$; then the previous construction still defines *the unique effective correspondence* $\Delta + p_2^{-1}(E_{p-1}) + p_1^{-1}(\tilde{E}_{p-1})$ where $\tilde{E}_{p-1} > 0$ is uniquely determined as the rest of $E_{p-1}$ with respect to the unique canonical divisor $K > E_{p-1}$

$$K = E_{p-1} + \tilde{E}_{p-1}. \tag{12}$$

For $s > 1$ $|L_{p-1}|$ is an infinite dimensional $g_{p-1}^{s-1}$, then there is not a unique effective E but we can select a natural representative, as follows: for a generic $Q \in C$ there exist a unique $E_{p-1-s} > 0$ such that $sQ + E_{p-1-s} \equiv L_{p-1}$ then the $E_{p-s}$ describe a well-defined effective correspondence $\tilde{E}$ such that

$$\tilde{E} + (s-1)\Delta \equiv p_1^{-1}(L_{p-1}) + p_2^{-1}(K - L_{p-1}) + \Delta \tag{13}$$ [12].

The justification of the name *dual Jacobians* given to the set $J_{p-1}(C)$ of linear equivalence classes $L_{p-1} + \mathscr{L}_0(C)$ of degree $p - 1$ appears in the second note (*loc. cit.* in [5]) because of the map (9).

*These correspondences parametrize naturally the intersection of* $\Sigma$ *with the first order theta-divisors* $D_c: \theta(c + u) = 0$ *and* $c/L$ *describes the torus* $\mathbf{C}^p/L \approx J(C)$.

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