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Teorie combinatorie. — *Some arithmetical problems on the use of the balance.* Nota (*) del Socio BENIAMINO SEGRE.

RIASSUNTO. — Una questione generale di notevole importanza pratica e teorica, ma che non mi consta sia mai stata studiata sistematicamente, è quella che segue. Dati due insiemi N , R ed un'applicazione σ del primo nel secondo, si vogliano dedurre certe peculiarità di N in relazione a σ da un minimo di informazioni relative al modo come σ opera su certi sottoinsiemi di N opportunamente scelti.

Un caso assai semplice, tanto da sembrare a prima giunta banale, è quello in cui l'insieme N risulti finito ed R sia il campo reale. Allora N consta di un numero n (intero positivo) di elementi od «oggetti» a, il numero reale $\sigma(a)$ potrà dirsi il «peso» di a e, più generalmente, ogni sottoinsieme A di N sarà dotato di un «peso»

$$\sigma(\mathrm{A}) = \sum_{a \in \mathrm{A}} \sigma(a) .$$

Se poi A e B denotano due sottoinsiemi qualsiansi di N, sussiste manifestamente una ed una sola delle relazioni

$$\sigma(\mathrm{A}) > \sigma(\mathrm{B}) \quad , \quad \sigma(\mathrm{A}) = \sigma(\mathrm{B}) \quad , \quad \sigma(\mathrm{A}) < \sigma(\mathrm{B}) ;$$

ebbene, l'informazione relativa ad A , B specificante quale di tali relazioni risulta verificata può in pratica ottenersi con l'uso di una bilancia, e si dirà quindi fornita mediante una «pesata».

La presente Nota si occupa del particolare problema di disporre gli elementi di N in *una successione a cui corrispondano pesi* non decrescenti, effettuando un numero minimo di convenienti pesate, con l'eventuale premessa di altre informazioni relative a σ. Si ottengono al riguardo i risultati enunciati dai teoremi I–V (rispettivamente stabiliti nei nn. 3, 5, 6, 7, 10); si veggano altresì le congetture dei nn. 4, 8. Le considerazioni qui svolte si prestano ad approfondimenti ed estensioni molteplici che potranno formare oggetto di ulteriori ricerche.

PRELIMINARY REMARKS

1. If $n \geq 1$, $p \geq 2$ denote any two integers, we will indicate by $\mathrm{ilog}_p\, n$ the whole part of the logarithm of n with respect to the base p, that is, the integer (≥ 0)

$$k = \mathrm{ilog}_p\, n$$

such that

$$p^k \leq n \leq p^{k+1} - 1 .$$

It is clear that, in defining k in this way, $k + 1$ will be exactly the number of digits ($0, 1, \cdots, p-1$) required for representing n in the numeration to the base p.

(*) Presentata nella seduta del 19 giugno 1973.

Moreover, putting for brevity

$$n' = n - 1 \quad , \quad k' = \mathrm{ilog}_p\, n',$$

the relation $k' = k - 1$ holds if, and only if, n is a power of p (with a whole-number exponent). In every other case it is found that $k' = k$.

2. If N is a set of $n\,(\geq 1)$ objects (undifferentiated in appearance) and a *weight* $\sigma(a)$ is attributed to each of them, every subset A of N will have a *weight* $\sigma(\mathrm{A})$ given by

$$\sigma(\mathrm{A}) = \sum_{a \in \mathrm{A}} \sigma(a)\,.$$

A balance is an instrument that enables the weights of any two subsets A, B of N to be compared, and hence capable of establishing which of the three relations

$$\sigma(\mathrm{A}) > \sigma(\mathrm{B}) \quad , \quad \sigma(\mathrm{A}) = \sigma(\mathrm{B}) \quad , \quad \sigma(\mathrm{A}) < \sigma(\mathrm{B})$$

holds. A comparison of this kind will therefore be briefly termed a *weighing*.

It is clear that $n(n-1)/2$ weighings enable the weights of the n objects of N to be compared, taken two by two, and hence show how these objects can be arranged in a succession of non-decreasing weights. However, this same purpose can be achieved more economically in the way that we shall now show.

THE GENERAL CASE

3. Let us begin with the case in which we have no preliminary information about the weights of the n objects of N, so that *a priori* these weights may be all different. We will then prove

THEOREM I.—*Using a balance, it is possible to arrange n given objects (undifferentiated in appearance) in a succession of non-decreasing weights, by performing a number of weighings not greater than*

$$\theta(n) = n\,\mathrm{ilog}_2\, n - 2^{\mathrm{ilog}_2 n + 1} + n + 1\,. \tag{1}$$

With reference to no. 1, expression (1) at once gives $\theta(1) = 0$, $\theta(2) = 1$, so that the theorem obviously holds for $n = 1$ and for $n = 2$. We can therefore assume $n \geq 3$ and prove the assumption by induction with respect to n.

Putting for brevity

$$n' = n - 1$$

and again taking account of no. 1, (1) now gives the equality

$$\theta(n) = \theta(n') + \mathrm{ilog}_2\, n' + 1. \tag{2}$$

Having chosen one, a, of the n given objects, on the basis of the assumed induction the remaining $n - 1 = n'$ objects can be arranged in a succession of objects

$$a_1, a_2, \cdots, a_{n'} \tag{3}$$

having non-decreasing weights by performing $\theta(n')$ weighings, at the most. Since $n' = n - 1 \geq 2$, we can assume

$$n' = 2r + \varepsilon \quad \text{with } r \text{ a positive integer and } \varepsilon = 0, 1. \tag{4}$$

That being stated, a and $a_{r+\varepsilon}$ are weighed against each other. If these two objects were found to be of equal weight, our purpose would be achieved by inserting a in the succession (3) immediately before or immediately after $a_{r+\varepsilon}$, having thus performed, at the most, a total of $\theta(n') + 1$ weighings, a number that—by virtue of (2)—does not exceed $\theta(n)$.

If, on the other hand, the objects a and $a_{r+\varepsilon}$ are not of equal weight, according as to whether the weight of a is less or greater than that of $a_{r+\varepsilon}$ we insert a in the first or in the second of the following successions

$$a_1, a_2, \cdots, a_{r-1+\varepsilon}, \tag{3'}$$

$$a_{r+1+\varepsilon}, a_{r+2+\varepsilon}, \cdots, a_{n'} = a_{2r+\varepsilon}. \tag{3''}$$

It may be noted that each of these successions comprises r elements, at the most, and that, by virtue of (4),

$$\operatorname{ilog}_2 r = \operatorname{ilog}_2 n' - 1 .$$

If it should be the case that $\operatorname{ilog}_2 r = 0$, and hence $r = 1$, our original purpose would be achieved with, at the most, one further weighing. At any rate, it would then be achieved by repeating—$\operatorname{ilog}_2 n' + 1$ times, at the most—the procedure that has led us from (3) to the determined case of (3′) or (3″).

Taking account of the (at most) $\theta(n')$ weighings by which the n' objects distinct from a have been arranged in succession (3), the equality (2) shows that, in this way, a total of $\theta(n)$ weighings have been performed, at the most, thus achieving our purpose. And this proves theor. I.

4. Simple considerations of a combinatorial nature show that for the first few values of n the result expressed by theor. I cannot be improved. *It may be conjectured that this is true for every value of* n: i.e., that $\theta(n) - 1$ cannot be written in place of $\theta(n)$ in (1); but any possible proof of this fact is certainly rather complex, both because it is not granted that the new version of theor. I must be established by complete induction with respect to n and since we cannot exclude the possibility that it may involve weighings in which two or more objects are placed in each pan of the balance.

The result reached in no. 3 can naturally be improved—and even to a very great extent—in a case where we have preliminary information on the weights of the n objects: for example, if we know that some of them are equal. The simplest case, in which it is known that there are $n - 1$ equal weights (not assigned to particular objects), will be studied more thoroughly in the remainder of this Note.

THE CASE OF ONE SINGLE WEIGHT IN EXCESS

5. Let us consider a set N of n (≥ 2) objects, of which we only know that $n - 1$ of them (not identified separately) are of equal weight, while the remaining one, x (unknown), is of greater weight (the procedure would be exactly similar in the case where it was known that x was of lesser weight). We then have to *identify this object x by performing a small number h of weighings.*

The object x can be identified immediately by choosing one of the objects of N at random and comparing its weight with those of the remaining $n - 1$ objects, which will involve $n - 1$ weighings. However, as soon as it is assumed that $n \geq 3$ the number of weighings can be reduced considerably, as it is specified by the following

THEOREM II.—*For determining the object x of the set* N *that is of greater weight it is sufficient to perform h weighings, where h denotes the natural number defined by*

$$3^h \geq n > 3^{h-1}. \tag{5}$$

In order to prove this theorem let us distinguish two cases, according as to whether the equality sign in (5) does or does not hold.

In a case in which

$$n = 3^h, \qquad \text{and hence} \quad h = \mathrm{ilog}_3 n \,, \tag{6}$$

we can subdivide N into three subsets A , B , C, disjoint by twos, each consisting of 3^{h-1} objects. With one weighing, A and B are compared; then

$$\text{if} \quad \sigma(\mathrm{A}) = \sigma(\mathrm{B}) \,, \qquad \text{necessarily} \quad x \in \mathrm{C} \,,$$

whereas if the subsets A and B are of different weights the object x is in the one of greater weight. If $h = 1$ (and hence $3^{h-1} = 1$), x is thus identified with a single weighing, which proves the theorem in the conditions (6) and $h = 1$. If $h > 1$, according to the above argument the theorem immediately follows under case (6) on the basis of a complete induction with respect to h.

If the equality sign in (5) does not hold, by virtue of no. 1 we have

$$\mathrm{ilog}_3 n = h - 1 \,,$$

and hence the number n can be written in the form

$$n = \sum_{i=1}^{h} \alpha_i \, 3^{h-i} \tag{7}$$

with $\alpha_i = 0, 1, 2$, $\alpha_1 \neq 0$, and with the exclusion of h-pla

$$(8) \qquad \alpha_1 = 1 \quad \text{together (if } h \geq 2) \quad \text{with} \quad \alpha_2 = \cdots = \alpha_h = 0$$

(corresponding to which we already know that $h - 1$ weighings are sufficient).

We observe that if $h = 1$ necessarily $n = 2$ and x is determined with $n - 1 = 1$ weighing, which proves the assumption in the present case. Hence we can assume $h \geq 2$ and proceed by induction with respect to h, distinguishing two alternatives according as to whether in (7):

$$(9) \qquad \alpha_1 = 1, \qquad\qquad (10) \qquad \text{or} \quad \alpha_1 = 2.$$

If (7) and (9) hold, we have

$$(11) \qquad n = 2n' + n''$$

where it is assumed that

$$n' = \sum_{i=1}^{h-1} \alpha_i \, 3^{h-i-1}, \qquad n'' = n' + \alpha_h;$$

and we may observe that, taking account of (9) and the non-validity of (8), these values of n', n'' satisfy

$$3^{h-1} > n' \geq 3^{h-2} \quad , \quad 3^{h-1} > n'' > 3^{h-2}.$$

Let the n objects of N now be distributed in three subsets A , B , C, disjoint in twos, comprising n', n' and n'' objects respectively. By weighing A against B in the two scale pans of the balance, we determine, as above, which of the three subsets A , B , C contains the object x of excess weight; since, through the assumed induction, the choice of x in this set can be obtained with not more than $h - 1$ weighings, theor. II follows in the present case.

If (7), (10) hold, then (11) still holds where it is now assumed that

$$n' = 3^{h-1} \quad , \quad n'' = \sum_{i=2}^{h} \alpha_i \, 3^{h-i}.$$

Proceeding in a similar way to that indicated above and observing that $n'' < 3^{h-1}$, theor. II is established since it is now possible to identify x in one of the three sets A , B , C with not more than $h - 1$ weighings: as regards A or B this is true by virtue of the case already considered relative to (6), whereas for C this at once follows from the assumed induction.

THE CASE OF ONE SINGLE OBJECT OF ANOMALOUS WEIGHT

6. From now on let us refer to a set N of n (≥ 3) objects, among which there is known to be a single one x (unknown) of *anomalous* weight, i.e. different from the weight of the other $n - 1$ objects, which are known to be

all of equal weight. As in no. 5, by means of no more than $n - 1$ weighings it is immediately possible to identify that object x and also to establish the fact that it is of greater or of lesser weight than the others, in which cases the object will be indicated by x^+ or x^- respectively. We have to reach these goals with a limited number of weighings. In this respect, first of all we have

THEOREM III.—*If in a set* N *of* $n = 3^k$ *objects there is one, and only one, of anomalous weight, by means of no more than* $k + 1$ *weighings it is possible to determine this object* x *and establish whether it is of greater or lesser weight than the other objects.*

The result being obvious for $k = 1$, since in that case we have $k + 1 = 2 = n - 1$, we can assume $k \geq 2$ and proceed by induction with respect to k. Dividing up N arbitrarily into the sum of three disjoint subsets A , B , C, each consisting of 3^{k-1} objects, let us carry out a first weighing by placing A and B in the two scale pans of the balance.

If $\sigma(A) = \sigma(B)$ it follows that $x \in C$, so that—for the assumed induction—the problem is solved by performing no more than $(k - 1) + 1 = k$ further weighings, hence the statement is proved.

If $\sigma(A) \neq \sigma(B)$, let for instance be $\sigma(A) < \sigma(B)$. It follows that necessarily

$$x^- \in A \qquad \text{or} \quad x^+ \in B.$$

Putting, for the sake of brevity

$$n' = 3^{k-2} \quad , \quad n'' = 2 \cdot 3^{k-2},$$

let us arbitrarily divide up each of the three subsets A , B , C into the sum of two disjoint subsets

$$A = A' + A'' \quad , \quad B = B' + B'' \quad , \quad C = C' + C'',$$

the first consisting of n' and the second of n'' objects. Let us then perform a second weighing, placing $A' + C''$ and $B' + A''$ in the scale pans, and let us distinguish the three possibilities that it may present.

(i) If $\sigma(A' + C'') = \sigma(B' + A'')$, the object x cannot be either in $A = A' + A''$ or in B'. Since it satisfies (12), then necessarily

$$x^+ \in B''.$$

We can therefore apply theor. II (no. 5), in which B'', n'', $k - 1$ are substituted for N , n , h respectively, and conclude that the object x can be determined (with the relative + sign) by means of no more than $k - 1$ further weighings, hence the statement is proved.

(ii) If $\sigma(A' + C'') > \sigma(B' + A'')$, on the basis of (12) the object x cannot belong either to A' or to B', so that we have

$$x^- \in A''.$$

Also in this case, similarly to what has been said in (i), it may be concluded that the object x (with the relative — sign) can be determined by means of no more than $k-1$ further weighings, hence the statement is proved.

(iii) If $\sigma(A'+C'')<\sigma(B'+A'')$, then necessarily

$$x^{-}\in A' \qquad \text{or} \quad x^{+}\in B'. \tag{13}$$

If $k=2$, each of the sets A', B' consists of $n'=1$ elements; the choice between the two alternatives (13) is then made by a third weighing alone: e.g., the single element of B' against any element of C, the weight of which can only be less than or equal to that of B'; in these two cases either the second or the first of (13) hold, respectively, hence the statement is proved.

We can therefore assume $k>2$ and establish inductively with respect to k that, in any case, *x can be determined, satisfying* (13), *by means of no more than $k-1$ further weighings.*

For this purpose it is sufficient to proceed, in relation to (13), by a method similar to that previously followed for (12) (which naturally involves the substitution of $k-1$ for k). A suitable third weighing will then be performed, which will give alternatives similar to those mentioned above [(i), (ii), (iii)]. In the first two cases the assumption is immediately proved from what has just been seen in (i), (ii), while in the third case the proof is exactly provided by the assumed induction.

Thus theor. III is completely proved.

7. Without advancing any further hypothesis regarding n, let us put for brevity

$$k=\mathrm{ilog}_3 n \tag{14}$$

and prove

THEOREM IV.—*If in a set* N *of $n\,(\geq 3)$ objects it is known that there is one object—and only one—(unidentified) of anomalous weight, by means of no more than $k+2$ weighings—where k is expressed by* (14)—*it is possible to identify this object x with its proper sign, i.e. establishing whether it weighs more or less than the other objects of* N.

The theorem is proved immediately and directly if $3\leq n\leq 8$. Let us therefore assume $n\geq 9$, such that, by virtue of (14), we have $k\geq 2$. Putting, for brevity,

$$n=3\,n'+\varepsilon \qquad \text{with} \quad \varepsilon=0\,,1\,,2\,,$$
$$k'=\mathrm{ilog}_3 n'=k-1\geq 1\,,$$

let us subdivide N into the sum of four disjoint subsets N_1', N_2', N_3', E comprising n', n', n', ε objects respectively (so that E will be empty if $\varepsilon=0$).

Let us compare one of the three N' with the other two, which is done by means of two weighings. If those three N' are found to be of equal weight, then necessarily $x \in \mathrm{E}$, which requires that ε be 1 or 2. It is now sufficient to compare the single elements of E with any element chosen from N_1' in order to establish the required object x with its proper sign. For this purpose the total number of weighings carried out amounts to

$$2 + \varepsilon \leq 4 \leq k + 2 ,$$

and hence the assumption is proved.

If, on the other hand, the two first weighings show that one of those three N' is of greater or lesser weight than the other two, then the required object x must be an anomalous element of this N' and of $+$ or $-$ sign respectively. In accordance with theor. II (no. 5), x is identified by means of no more than $k' + 1 = k$ further weighings, and hence the statement is proved.

Theor. IV is thus completely established.

8. Defining k once more by means of (14) and denoting by $\tau(n)$ the minimum number of weighings sufficient for identifying the object x with its proper sign, as in theor. IV, this theorem shows that in every case

$$\tau(n) \leq k + 2 . \tag{15}$$

However, (15) can be improved when n is a power of 3, in this case theor. III giving

$$\tau(n) \leq k + 1 . \tag{15'}$$

It will then be found from theor. V (no. 10) that (15′) holds also for other values of n; furthermore, it appears plausible that $\tau(n)$ may be a *non-decreasing function of n*.

In fact a direct and straightforward analysis shows that we have

$$\begin{array}{lll} \text{for } n = 3 : & k = 1 , & \tau(n) = 2 = k + 1 ; \\ \text{for } n = 4, 5, 6, 7, 8 : & k = 1 , & \tau(n) = 3 = k + 2 ; \\ \text{for } n = 9, 10, 11, 12 : & k = 2 , & \tau(n) = 3 = k + 1 \end{array}$$

(relative to this last series of values of n, cf. theor. V already cited) (1).

We can therefore *conjecture* that for special values of n we have

$$\tau(n) = k + 1 ,$$

while for the others we have

$$\tau(n) = k + 2 ;$$

but the precise statement and proof of such a result (for which the above referred theor. V will have to be taken into account) certainly represents a fairly hard problem.

(1) Added in proof. — I have been told by Professor Ferenc Kárteszi that a few special cases are already considered in JAGLOM, *Az információelmélet matematikai alapjai* (1959).

Also for this reason it is therefore interesting to determine some values of n (apart from the powers of 3) for which (15) can be improved by adopting (15′). This is what we now propose to do, leaving open the question of seeing whether or not in this way we obtain all the values of n of such a type.

9. We first state the following

LEMMA.—*Supposing* $n \geq 3$ *of the form*

$$n = \sum_{i=0}^{k} \alpha_i \, 3^{k-i} \qquad \textit{with} \quad \alpha_i = 0 , 1 , \tag{16}$$

and if we add to the hypotheses of theor. IV the possibility of employing 3^k *regular supplementary objects—i.e., different from those of* N *and having the same weight as the non-anomalous objects among the latter—in order to attain the purpose of theor. IV it is sufficient to perform* $k + 1$ *weighings.*

If $\alpha_0 = 0$ the lemma already follows theor. IV. It is therefore not restrictive to assume $\alpha_0 = 1$, or

$$n = 3^k + n' \tag{17}$$

with $n' = \sum_{i=1}^{k} \alpha_i \, 3^{k-i}$ $(\alpha_i = 0 , 1)$ of the same type (16) as n, apart from the substitution of $k - 1$ in place of k.

First of all let us assume $k = 1$. If $\alpha_1 = 0$, or $n = 3$, the lemma holds by virtue of theor. III (no. 6). Thus there remains only the case in which $k_1 = \alpha_1 = 1$, or $n = 4$; in this case let a_1 , a_2 , a_3 , a_4 be the objects of N and b_1 , b_2 , b_3 the supplementary ones. Comparing a_1 , a_2 , a_3 with b_1 , b_2 , b_3 by means of a first weighing, we distinguish the three possibilities that can be offered.

If $\sigma(a_1 , a_2 , a_3) = \sigma(b_1 , b_2 , b_3)$, then necessarily $x = a_4$ and its sign is established by means of one further weighing.

If $\sigma(a_1 , a_2 , a_3) > \sigma(b_1 , b_2 , b_3)$, we have

$$x^+ \in (a_1 , a_2 , a_3) ,$$

and only one further weighing is sufficient to identify this x (theor. II).

An analogous conclusion is reached in the case where $\sigma(a_1 , a_2 , a_3) < < \sigma(b_1 , b_2 , b_3)$, and hence the lemma is completely established for $k = 1$.

It is therefore permissible to assume $k \geq 2$ and establish the lemma by arguing by induction with respect to k. Having regard to (17), we can subdivide N into two disjoint sets, one (N_1) consisting of 3^k objects, the other (N') of n' objects. Let us perform a first weighing, comparing the set N_1 with that of the 3^k supplementary objects. If the two sets are of unequal weight, this implies that the unknown object x must be in N_1 and have a determined sign. In accordance with theor. II (no. 5),

k further weighings are sufficient to identify this x, and hence the lemma is proved. If, on the other hand, those two sets are of equal weight, this necessarily means that $x \in \mathrm{N}'$ and hence, by the assumed induction, the object x may be identified, with its proper sign, by means of not more than k weighings, so that also in this case the lemma is proved.

10. Finally we can establish

THEOREM V.—*Let $n\ (\geq 3)$ be an integer of the form*

$$n = 3^k + m_1 \qquad (k \text{ being a whole number} \geq 1), \tag{18}$$

where m_1 satisfies

$$0 \leq m_1 \leq 3\,(3^{k-1} - 1)/2\,. \tag{19}$$

Then if, in a set N *of n objects, it is known that there is one and only one (not identified) of anomalous weight, by means of not more than $k+1$ weighings it is possible to identify this object x with its proper sign, i.e. establishing whether it is of greater or lesser weight than the other objects of* N.

If in (18) it is assumed that $m_1 = 0$, theor. V holds by virtue of theor. III (no. 6); we can therefore confine ourselves to the case in which $m_1 > 0$, such that, on account of (19), necessarily $k \geq 2$. For brevity, let us introduce the positive integers n_1, n^* by putting

$$m_1 = 3\,n_1 + \varepsilon\,, \qquad \text{with} \quad \varepsilon = 0\,,1\,,2\,. \tag{20}$$

$$n^* = (3^{k-1} - 1)/2 = \sum_{i=0}^{k-2} 3^i, \tag{21}$$

and let us then define n', n'' by assuming

$$\begin{cases} \text{if} \quad \varepsilon = 0: & n' = n'' = n_1\,; \\ \text{if} \quad \varepsilon = 1: & n' = n_1 \quad , \quad n'' = n_1 + 1\,; \\ \text{if} \quad \varepsilon = 2: & n' = n_1 + 1 \quad , \quad n'' = n_1\,. \end{cases} \tag{22}$$

By virtue of (18)–(22), it is at once seen that in every case

$$m_1 = 2\,n' + n'', \tag{23}$$

$$n' \leq n^* \quad , \quad n'' \leq n^*. \tag{24}$$

This being stated, we observe that—having regard to (18), (23)—it is permissible to split N up into the sum of three disjoint subsets A, B, C, containing

$$3^{k-1} + n' \quad , \quad 3^{k-1} + n' \quad , \quad 3^{k-1} + n''$$

objects respectively. With a f i r s t w e i g h i n g we compare the sets A and B, and we pass on to distinguish two cases according as to whether they are of equal or different weight.

In the former case, the objects contained in A and B are of regular weight, so that necessarily $x \in C$. Adding $n^* - n'$ (regular) elements to C, taking them arbitrarily from A (which is possible since $0 \leq n^* - n' < n^* < 3^{k-1} + n'$), we obtain a set C^* that consists, by virtue of (21), of

$$3^{k-1} + n^* = \sum_{i=0}^{k-1} 3^i$$

objects, among which there is the unknown object x. Since we can use 3^{k-1} regular supplementary objects taken, for example, from B, on the basis of the Lemma of no. 9 this object x, with its proper sign, can be identified by means of no more than $(k-1)+1=k$ further weighings. This proves theor. V in the present hypotheses.

In the second case, let us assume for instance that

$$\sigma(A) < \sigma(B);$$

this implies that every element of C is of regular weight and that

$$x^- \in A \quad \text{or} \quad x^+ \in B. \tag{25}$$

In any case, let us begin by dividing up C into the sum $C' + C''$ of two disjoint sets, consisting of 3^{k-1} and n'' elements respectively.

We then observe that, having regard to (22), (24), (21), an integer ν can be determined which satisfies

$$0 \leq 2\nu \leq n''$$

and such that the positive integer $m^* = n' + \nu$ can be written, to the base 3, in the form

$$m^* = \sum_{i=0}^{k-2} \alpha_i \, 3^{k-i-2} \quad \text{with} \quad \alpha_i = 0, 1. \tag{26}$$

We can then extract from C'' two disjoint sets C_1'', C_2'', each containing ν elements, and form the disjoint sets

$$\bar{A} = A + C_1'' \quad , \quad \bar{B} = B + C_2'',$$

each of which consists of $3^{k-1} + m^*$ elements. It is then clear that (25) are equivalent to

$$x^- \in \bar{A} \quad \text{or} \quad x^+ \in \bar{B}. \tag{27}$$

This being said, let us split up $\bar{A}$ into the sum $A' + A^*$ of two disjoint sets A', A^*, comprising 3^{k-1} and m^* objects respectively; let us also do the same for $\bar{B} = B' + B^*$. Let us then perform a second weighing, placing $A' + B^*$ and $C' + A^*$ in the two pans of the balance, and let us distinguish the three possibilities that can be offered.

(i) If $\sigma(A' + B^*) = \sigma(C' + A^*)$, the object x can be neither in $\bar{A} = A' + A^*$ nor in B^*. Since it satisfies (27), then necessarily

$$x^+ \in B'.$$

We can then apply theor. II (no. 5), in which B', $k-1$ are substituted for N, k respectively, and conclude that the object x (having a $+$ sign) is identified by means of no more than $k-1$ further weighings, which proves theor. V in the present circumstances.

(ii) If $\sigma(A' + B^*) < \sigma(C' + A^*)$, according to (27) the object x cannot be either in A^* or in B^*; so we have that

$$x^- \in A',$$

and the conclusion is the same as that stated in (i).

(iii) If $\sigma(A' + B^*) > \sigma(C' + A^*)$, (27) imply that necessarily

(28) $$x^- \in A^* \quad \text{or} \quad x^+ \in B^*;$$

and it is a question of proving that also at present *the object x (with its sign) can be determined by means of no more than $k-1$ weighings*. This is clear if $k = 2$, in which case—by virtue of (26)—each of the sets A^*, B^* consists of one object only. We can therefore assume $k \geq 3$ and establish what has just been asserted by proceeding by induction with respect to k. Then, if it should be that $\alpha_0 = 0$, the assumed induction would at once give the required answer.

It can therefore be assumed that $\alpha_0 = 1$ and we can proceed with respect to (28) in a precisely similar way to that followed with respect to (27), apart from the substitution of $k-1$ for k. We then perform a suitable third weighing, after which the object x, with its proper sign, is identified by means of no more than $k-2$ further weighings: this is immediately clear if the occurring case is similar to (i) or (ii); while the result follows on the basis of the assumed induction, in the remaining possibility of a case of type (iii).

Theor. V is thus completely proved.