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Geometria differenziale. — *Some commutation formulae arising from Lie differentiation in a Finsler Space.* Nota di R. B. MISRA e F. M. MEHER, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — La derivata di Lie di un campo di vettori è stata estesa da E. T. Davies e B. Laptew agli spazi di Finsler. Qui vengono date formole di commutazione per due successive derivazioni di Lie di un tensore, che definiscono operatori di Lie a due indici; da questi si passa poi ad operatori a tre e quattro indici.

The theory of Lie differentiation was extended to Finsler geometry by E. T. Davies [1] and B. Laptew [2]. Considering an r–parameter group of infinitesimal transformations generated by the vector-fields $\underset{\alpha}{v^i}$ (1) K. Yano [3] constructed the second order Lie derivative of the metric tensor with respect to infinitesimal transformations generated by the vector-fields $\underset{\alpha}{v^i}$ and $\underset{\beta}{v^j}$. Further it is shown [3] that the order of Lie differentiation for the above vector-fields is not commutative. Indeed, the difference $(\underset{\alpha}{\pounds}\underset{\beta}{\pounds} - \underset{\beta}{\pounds}\underset{\alpha}{\pounds}) g_{ij}$ of the corresponding Lie derivatives is seen to be a Lie derivative again with respect to a transformation generated by a vector-field $\underset{\alpha}{\pounds}\underset{\beta}{v^j}$ and it is denoted by $\underset{\alpha\beta}{\pounds} g_{ij}$.

The aim of the present paper is to verify whether the above property of Lie differentiation can be extended to the tensors of arbitrary rank too. Next it is to be examined whether the order of Lie differentiation can also be extended arbitrarily so that the property still holds good. Of the four sections of this paper the first one is introductory and includes the known formulae for their applications in the following sections. The second section deals with the commutation rules of the Lie operators $\underset{\alpha}{\pounds}$ and $\underset{\beta}{\pounds}$ for arbitrary vectors and tensors. The commutation rules of the Lie operator $\underset{\alpha}{\pounds}$ with the operator $\underset{\alpha\beta}{\pounds}$ are found in the third section giving rise to the operator $\underset{\alpha\beta\gamma}{\pounds}$. Some properties of this operator are also included in it. The commutation properties of this operator with the Lie operator $\underset{\alpha}{\pounds}$ are found in the last section. Introducing the operator $\underset{\alpha\beta\gamma\delta}{\pounds}$ some of its properties are also derived.

(*) Nella seduta del 9 gennaio 1971.

(1) The Latin indices run from 1 to n whereas the Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to r.

1. – INTRODUCTION

Let F_n be an n–dimensional Finsler space in which the fundamental function $F(x^i, \dot{x}^i)$ (2) satisfies the requisite conditions. The metric tensor $\boldsymbol{g}$ of F_n is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def}}{=} (1/2)\, \dot{\partial}_i \dot{\partial}_j F^2, \qquad \dot{\partial}_i \equiv \partial/\partial \dot{x}^i, \tag{1.1}$$

and

$$g^{ij}(x, \dot{x})\, g_{jk}(x, \dot{x}) = \delta^i_k . \tag{1.2}$$

The covariant derivative (in the sense of Cartan) of a vector-field $X^i(x, \dot{x})$ with respect to x^k is given by [4], [6]

$$\nabla_k X^i = \partial_k X^i - (\dot{\partial}_j X^i)\, \Gamma^{*j}_{kh} \dot{x}^h + \Gamma^{*i}_{kh} X^h, \tag{1.3}$$

where $\Gamma^{*i}_{kh}(x, \dot{x})$ are the connection parameters of Cartan. Also the Lie derivative of a vector with respect to an infinitesimal change

$$\bar{x}^i = x^i + \varepsilon \underset{\beta}{v^i}(x^j) \tag{1.4}$$

has been obtained in the form

$$\underset{\beta}{\pounds} X^i = \underset{\beta}{v^j} \nabla_j X^i - X^j \nabla_j \underset{\beta}{v^i} + (\dot{\partial}_j X^i) \nabla_l \underset{\beta}{v^j} \dot{x}^l . \tag{1.5}$$

It is known that the partial differential operator $\dot{\partial}_i$ commutes with the Lie operator $\underset{\alpha}{\pounds}$ whereas the covariant differential operator ∇_k commutes with $\underset{\alpha}{\pounds}$ according to ([3], equation (8.6.13))

$$(\underset{\alpha}{\pounds} \nabla_k - \nabla_k \underset{\alpha}{\pounds})\, X^i = (\underset{\alpha}{\pounds}\, \Gamma^{*i}_{jk})\, X^j - (\underset{\alpha}{\pounds}\, \Gamma^{*j}_{lk})\, (\dot{\partial}_j X^i)\, \dot{x}^l . \tag{1.6}$$

2. – COMMUTATION RULES FOR THE OPERATORS $\underset{\alpha}{\pounds}$ AND $\underset{\beta}{\pounds}$

In this section we consider the Lie derivatives of second order for various geometric entities and study the rules of commutation for the change of order of differentiation. First we consider a vector-field $X^i(x, \dot{x})$ and the corresponding commutation formula has been derived in the following theorem.

THEOREM 2.1. — *The Lie operators* $\underset{\alpha}{\pounds}$ *and* $\underset{\beta}{\pounds}$ *commute according to*

$$(\underset{\alpha}{\pounds}\underset{\beta}{\pounds} - \underset{\beta}{\pounds}\underset{\alpha}{\pounds})\, X^i = \underset{\alpha\beta}{\pounds} X^i, \tag{2.1}$$

where $\underset{\alpha\beta}{\pounds} X^i$ *is the Lie derivative of* X^i *with respect to the infinitesimal change for the vector* $\underset{\alpha}{\pounds}\underset{\beta}{v^i}$.

(2) In what follows the line-element $(x^i, \dot{x}^i)$ will be briefly denoted by $(x, \dot{x})$.

Proof. — The Lie derivative of X^i is given by (1.5). To consider the second order Lie derivative of X^i we operate (1.5) by $\underset{\alpha}{\pounds}$:

$$\underset{\alpha}{\pounds}\underset{\beta}{\pounds} X^i = (\underset{\alpha}{\pounds}\underset{\beta}{v^j}) \nabla_j X^i + \underset{\beta}{v^j}\underset{\alpha}{\pounds}\nabla_j X^i - (\underset{\alpha}{\pounds} X^j) \nabla_j \underset{\beta}{v^i}$$
$$- X^j \underset{\alpha}{\pounds}\nabla_j \underset{\beta}{v^i} + (\dot{\partial}_j \underset{\alpha}{\pounds} X^i) \nabla_l \underset{\beta}{v^j} \dot{x}^l + (\dot{\partial}_j X^i) \underset{\alpha}{\pounds}\nabla_l \underset{\beta}{v^j} \dot{x}^l .$$

Applying the commutation formula (1.6) this equation becomes

$$\underset{\alpha}{\pounds}\underset{\beta}{\pounds} X^i = (\underset{\alpha}{\pounds}\underset{\beta}{v^j}) \nabla_j X^i - X^j \nabla_j \underset{\alpha}{\pounds}\underset{\beta}{v^i} + (\dot{\partial}_j X^i) \nabla_l \underset{\alpha}{\pounds}\underset{\beta}{v^j} \dot{x}^l$$
$$+ \underset{\beta}{v^j} \nabla_j \underset{\alpha}{\pounds} X^i - (\underset{\alpha}{\pounds} X^j) \nabla_j \underset{\beta}{v^i} + (\dot{\partial}_j \underset{\alpha}{\pounds} X^i) \nabla_l \underset{\beta}{v^j} \dot{x}^l . \tag{2.2}$$

Knowing that $\underset{\alpha}{\pounds} X^i$ has vectorial characteristics if we apply (1.5) for its further Lie derivative we have

$$\underset{\beta}{\pounds}\underset{\alpha}{\pounds} X^i = \underset{\beta}{v^j} \nabla_j \underset{\alpha}{\pounds} X^i - (\underset{\alpha}{\pounds} X^j) \nabla_j \underset{\beta}{v^i} + (\dot{\partial}_j \underset{\alpha}{\pounds} X^i) \nabla_l \underset{\beta}{v^j} \dot{x}^l . \tag{2.3}$$

Thus, from (2.2) and (2.3), we find

$$(\underset{\alpha}{\pounds}\underset{\beta}{\pounds} - \underset{\beta}{\pounds}\underset{\alpha}{\pounds}) X^i = (\underset{\alpha}{\pounds}\underset{\beta}{v^j}) \nabla_j X^i - X^j \nabla_j \underset{\alpha}{\pounds}\underset{\beta}{v^i} + (\dot{\partial}_j X^i) \nabla_l \underset{\alpha}{\pounds}\underset{\beta}{v^j} \dot{x}^l . \tag{2.4}$$

Now if we write the Lie derivative of X^i with respect to the vector $\underset{\alpha}{\pounds}\underset{\beta}{v^i}$:

$$\underset{\alpha\beta}{\pounds} X^i = (\underset{\alpha}{\pounds}\underset{\beta}{v^j}) \nabla_j X^i - X^j \nabla_j \underset{\alpha}{\pounds}\underset{\beta}{v^i} + (\dot{\partial}_j X^i) \nabla_l \underset{\alpha}{\pounds}\underset{\beta}{v^j} \dot{x}^l, \tag{2.5}$$

the equation (2.4) immediately reduces to (2.1). This completes the proof of the above theorem.

Denoting the symmetric and skew-symmetric parts of any object, say Ω_{ij}, with respect to the indices i, j by $\Omega_{(ij)} \overset{\text{def}}{=} (1/2)(\Omega_{ij} + \Omega_{ji})$ and $\Omega_{[ij]} \overset{\text{def}}{=} (1/2)(\Omega_{ij} - \Omega_{ji})$ the formula (2.1) may be re-written as

$$2 \underset{[\alpha}{\pounds}\underset{\beta]}{\pounds} X^i = \underset{\alpha\beta}{\pounds} X^i . \tag{2.6}$$

Thus the operator $\underset{\alpha\beta}{\pounds}$ is skew-symmetric in α, β and therefore it satisfies

$$\underset{(\alpha\beta)}{\pounds} X^i = 0 . \tag{2.7}$$

The generalisation of the commutation formula (2.1) for a tensor $\mathbf{T}$ of type (p, q) may be obtained as

$$2 \underset{[\alpha}{\pounds}\underset{\beta]}{\pounds} \mathbf{T} = \underset{\alpha\beta}{\pounds} \mathbf{T} . \tag{2.8}$$

3. – COMMUTATION RULES FOR THE OPERATORS $\underset{\alpha}{\pounds}$ AND $\underset{\beta\gamma}{\pounds}$

The operator $\underset{\beta\gamma}{\pounds}$ has been defined by (2.5). Considering its further Lie derivative we obtain the Lie derivative of higher order. Rules for the interchange of the operators $\underset{\alpha}{\pounds}$ and $\underset{\beta\gamma}{\pounds}$ have been derived in the following theorem.

THEOREM 3.1. — *The operators* $\underset{\alpha}{\pounds}$ *and* $\underset{\beta\gamma}{\pounds}$ *commute according to*

$$(\underset{\alpha}{\pounds}\underset{\beta\gamma}{\pounds} - \underset{\beta\gamma}{\pounds}\underset{\alpha}{\pounds})\, X^i = \underset{\alpha\beta\gamma}{\pounds} X^i\,, \tag{3.1}$$

where $\underset{\alpha\beta\gamma}{\pounds}$ *is the Lie derivative of the vector-field* X^i *with respect to the infinitesimal transformation generated by the vector* $\underset{\alpha}{\pounds}\underset{\beta}{\pounds} \underset{\gamma}{v^i}$.

Proof. — The derivative $\underset{\beta\gamma}{\pounds} X^i$ given by (2.5) is also a vector quantity. Therefore its Lie derivative can be found by application of (1.5). Thus we have

$$\begin{aligned}\underset{\alpha}{\pounds}\underset{\beta\gamma}{\pounds} X^i = {} & (\underset{\alpha}{\pounds}\underset{\beta}{\pounds}\underset{\gamma}{v^j})\, \nabla_j X^i + (\underset{\beta}{\pounds}\underset{\gamma}{v^j})\, \underset{\alpha}{\pounds}\nabla_j X^i - (\underset{\alpha}{\pounds} X^j)\, \nabla_j \underset{\beta}{\pounds}\underset{\gamma}{v^i} \\ & - X^j \underset{\alpha}{\pounds}\nabla_j \underset{\beta}{\pounds}\underset{\gamma}{v^i} + (\dot{\partial}_j \underset{\alpha}{\pounds} X^i)\, \nabla_l \underset{\beta}{\pounds}\underset{\gamma}{v^j}\, \dot{x}^l + (\dot{\partial}_j X^i)\, \underset{\alpha}{\pounds}\nabla_l \underset{\beta}{\pounds}\underset{\gamma}{v^j}\, \dot{x}^l\,.\end{aligned} \tag{3.2}$$

Moreover, describing the Lie derivative of $\underset{\alpha}{\pounds} X^i$ for the vector $\underset{\beta}{\pounds}\underset{\gamma}{v^i}$ in the way analogous to (2.5) we have

$$\underset{\beta\gamma}{\pounds}\underset{\alpha}{\pounds} X^i = (\underset{\beta}{\pounds}\underset{\gamma}{v^j})\, \nabla_j \underset{\alpha}{\pounds} X^i - (\underset{\alpha}{\pounds} X^j)\, \nabla_j \underset{\beta}{\pounds}\underset{\gamma}{v^i} + (\dot{\partial}_j \underset{\alpha}{\pounds} X^i)\, \nabla_l \underset{\beta}{\pounds}\underset{\gamma}{v^j}\, \dot{x}^l\,. \tag{3.3}$$

From (3.2) and (3.3), when using (1.6), we find

$$(\underset{\alpha}{\pounds}\underset{\beta\gamma}{\pounds} - \underset{\beta\gamma}{\pounds}\underset{\alpha}{\pounds})\, X^i = (\underset{\alpha}{\pounds}\underset{\beta}{\pounds}\underset{\gamma}{v^j})\, \nabla_j X^i - X^j \nabla_j \underset{\alpha}{\pounds}\underset{\beta}{\pounds}\underset{\gamma}{v^i} + (\dot{\partial}_j X^i)\, \nabla_l \underset{\alpha}{\pounds}\underset{\beta}{\pounds}\underset{\gamma}{v^j}\, \dot{x}^l, \tag{3.4}$$

which is same as (3.1). Thus the theorem is established.

In view of (2.7) it may be easily seen that the operator $\underset{\alpha\beta\gamma}{\pounds}$ is skew-symmetric in the indices β, γ, i.e.

$$(\underset{\alpha\beta\gamma}{\pounds} + \underset{\alpha\gamma\beta}{\pounds})\, X^i = 0\,. \tag{3.5}$$

Observing this property of $\underset{\alpha\beta\gamma}{\pounds}$ the formula (3.1) may be also written as

$$(\underset{\beta\gamma}{\pounds}\underset{\alpha}{\pounds} - \underset{\alpha}{\pounds}\underset{\beta\gamma}{\pounds})\, X^i = \underset{\alpha\gamma\beta}{\pounds} X^i\,. \tag{3.6}$$

Another interesting property of the operator $\underset{\alpha\beta\gamma}{\pounds}$ is characterized by the following theorem.

THEOREM 3.2. — *The operator* $\underset{\alpha\beta\gamma}{\pounds}$ *satisfies*

$$(\underset{\alpha\beta\gamma}{\pounds} + \underset{\beta\gamma\alpha}{\pounds} + \underset{\gamma\alpha\beta}{\pounds})\, X^i = 0\,. \tag{3.7}$$

Proof. — Interchanging the indices α, β, γ cyclically in (3.1) and adding the resulting two identities in (3.1) we get

$$(\underset{\alpha\beta\gamma}{\pounds} + \underset{\beta\gamma\alpha}{\pounds} + \underset{\gamma\alpha\beta}{\pounds})\, X^i = (\underset{\alpha}{\pounds}\underset{\beta\gamma}{\pounds} - \underset{\beta\gamma}{\pounds}\underset{\alpha}{\pounds})\, X^i$$

$$+ (\underset{\beta}{\pounds}\underset{\gamma\alpha}{\pounds} - \underset{\gamma\alpha}{\pounds}\underset{\beta}{\pounds})\, X^i + (\underset{\gamma}{\pounds}\underset{\alpha\beta}{\pounds} - \underset{\alpha\beta}{\pounds}\underset{\gamma}{\pounds})\, X^i\,.$$

Applying (2.1) to the second member of this identity it may be easily seen to vanish. Thus the formula (3.7) is established.

Noting the skew-symmetry of $\underset{\alpha\beta\gamma}{\pounds}$ in β, γ we may also derive

$$\underset{[\alpha\beta\gamma]}{\pounds} X^i = 0\,, \tag{3.8}$$

and

$$\underset{(\alpha\beta\gamma)}{\pounds} X^i = 0 \tag{3.9}$$

from (3.7).

4. – COMMUTATION RULES FOR THE OPERATORS $\underset{\alpha}{\pounds}$ AND $\underset{\beta\gamma\delta}{\pounds}$

It has been proved in the second section that the difference $2\underset{[\alpha}{\pounds}\underset{\beta]}{\pounds} X^i$ is again a Lie derivative of X^i with respect to the vector $\underset{\alpha}{\pounds} \underset{\beta}{v^i}$. Similarly it is seen in the third section that the difference $(\underset{\alpha}{\pounds}\underset{\beta\gamma}{\pounds} - \underset{\beta\gamma}{\pounds}\underset{\alpha}{\pounds})\, X^i$ is also a Lie derivative of X^i for the vector $\underset{\alpha}{\pounds}\underset{\beta}{\pounds}\underset{\gamma}{v^i}$. In view of these observations it is expected that the difference $(\underset{\alpha}{\pounds}\underset{\beta\gamma\delta}{\pounds} - \underset{\beta\gamma\delta}{\pounds}\underset{\alpha}{\pounds})\, X^i$ should also be a Lie derivative of X^i with respect to the vector $\underset{\alpha}{\pounds}\underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^i}$. Thus to verify our stipulation we write the expression for the derivative $\underset{\beta\gamma\delta}{\pounds} X^i$ as in (3.4)

$$\underset{\beta\gamma\delta}{\pounds} X^i = (\underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^j})\, \nabla_j X^i - X^j\, \nabla_j \underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^i} + (\dot{\partial}_j X^i)\, \nabla_l \underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^j}\, \dot{x}^l. \tag{4.1}$$

Considering its further Lie derivative for the vector $\underset{\alpha}{v^i}$ we get

$$\underset{\alpha}{\pounds}\underset{\beta\gamma\delta}{\pounds} X^i = (\underset{\alpha}{\pounds}\underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^j})\, \nabla_j X^i - X^j\, \underset{\alpha}{\pounds} \nabla_j \underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^i} \tag{4.2}$$

$$+ (\dot{\partial}_j X^i)\, \underset{\alpha}{\pounds} \nabla_l \underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^j}\, \dot{x}^l + (\underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^j})\, \underset{\alpha}{\pounds} \nabla_j X^i$$

$$- (\underset{\alpha}{\pounds} X^j)\, \nabla_j \underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^i} + (\underset{\alpha}{\pounds}\dot{\partial}_j X^i)\, \nabla_l \underset{\beta}{\pounds}\underset{\gamma}{\pounds}\underset{\delta}{v^j}\, \dot{x}^l\,.$$

Also from (4.1) we may obtain

$$\text{(4.3)}\qquad \underset{\beta\gamma\delta}{\pounds}\,\underset{\alpha}{\pounds}\,X^i = (\underset{\beta}{\pounds}\,\underset{\gamma}{\pounds}\,\underset{\delta}{v^j})\,\nabla_j \underset{\alpha}{\pounds}\,X^i - (\underset{\alpha}{\pounds}\,X^j)\,\nabla_j \underset{\beta}{\pounds}\,\underset{\gamma}{\pounds}\,\underset{\delta}{v^i} + (\dot{\partial}_j \underset{\alpha}{\pounds}\,X^i)\,\nabla_l \underset{\beta}{\pounds}\,\underset{\gamma}{\pounds}\,\underset{\delta}{v^j}\,\dot{x}^l.$$

Subtracting (4.3) from (4.2) and simplifying by means of (1.6) we get, after a little simplification, the equation

$$\text{(4.4)}\qquad (\underset{\alpha}{\pounds}\,\underset{\beta\gamma\delta}{\pounds} - \underset{\beta\gamma\delta}{\pounds}\,\underset{\alpha}{\pounds})\,X^i = \underset{\alpha\beta\gamma\delta}{\pounds}\,X^i,$$

where

$$\text{(4.5)}\qquad \underset{\alpha\beta\gamma\delta}{\pounds}\,X^i = (\nabla_j X^i)\,\underset{\alpha}{\pounds}\,\underset{\beta}{\pounds}\,\underset{\gamma}{\pounds}\,\underset{\delta}{v^j} - X^i\,\nabla_j \underset{\alpha}{\pounds}\,\underset{\beta}{\pounds}\,\underset{\gamma}{\pounds}\,\underset{\delta}{v^i} + (\dot{\partial}_j X^i)\,\nabla_l \underset{\alpha}{\pounds}\,\underset{\beta}{\pounds}\,\underset{\gamma}{\pounds}\,\underset{\delta}{v^j}\,\dot{x}^l.$$

Comparison of (4.5) with (1.5) gives rise to the following theorem.

THEOREM 4.1. — *The quantity* $\underset{\alpha\beta\gamma\delta}{\pounds}\,X^i$ *is the Lie derivative of the vector-field* X^i *with respect to an infinitesimal transformation generated by a vector* $\underset{\alpha}{\pounds}\,\underset{\beta}{\pounds}\,\underset{\gamma}{\pounds}\,\underset{\delta}{v^i}$.

Analogous to the operator $\underset{\beta\gamma\delta}{\pounds}$ the operator $\underset{\alpha\beta\gamma\delta}{\pounds}$ also satisfies the identities

$$\text{(4.6)}\qquad (\underset{\alpha\beta\gamma\delta}{\pounds} + \underset{\alpha\beta\delta\gamma}{\pounds})\,X^i = 0,$$

and

$$\text{(4.7)}\qquad (\underset{\alpha\beta\gamma\delta}{\pounds} + \underset{\beta\gamma\delta\alpha}{\pounds} + \underset{\gamma\delta\alpha\beta}{\pounds} + \underset{\delta\alpha\beta\gamma}{\pounds})\,X^i = 0.$$

The formulae (2.1), (3.1) and (4.4) lead us to stipulate the truth of the following general rule of commutation for the operators $\underset{1}{\pounds}$ and $\underset{23\cdots r}{\pounds}$.

$$\text{(4.8)}\qquad (\underset{1}{\pounds}\,\underset{23\cdots r}{\pounds} - \underset{23\cdots r}{\pounds}\,\underset{1}{\pounds})\,X^i = \underset{123\cdots r}{\pounds}\,X^i,$$

where $\underset{123\cdots r}{\pounds}\,X^i$ is the Lie derivative of X^i for the vector $\underset{123\cdots r-1}{\pounds}\,\underset{r}{v^i}$.

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