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A Note on Lax Projective Embeddings of Grassmann Spaces

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Abstract – In the paper (Ferrara Dentice *et al.*, 2004) a complete exposition of the state of the art for lax embeddings of polar spaces of finite rank ≥ 3 is presented. As a consequence, we have that if a Grassmann space $\mathscr{G}$ of dimension 3 and index 1 has a lax embedding in a projective space over a skew–field K, then $\mathscr{G}$ is the Klein quadric defined over a subfield of K. In this paper, I examine Grassmann spaces of arbitrary dimension $d \geq 3$ and index $h \geq 1$ having a lax embedding in a projective space.

Riassunto – Il lavoro (Ferrara Dentice *et al.*, 2004)) contiene una trattazione completa delle immersioni deboli di spazi polari che abbiano rango finito almeno tre. Come conseguenza dei risultati in esso contenuti, si ha che se uno spazio di Grassmann $\mathscr{G}$ di dimensione 3 ed indice 1 ha un'immersione debole in uno spazio proiettivo su un corpo K, allora $\mathscr{G}$ è la quadrica di Klein di uno spazio proiettivo di dimensione 5 coordinato su un sottocampo di K. Nel presente lavoro si esaminano spazi di Grassmann di dimensione $d \geq 3$ e indice $h \geq 1$ dotati di immersioni deboli in spazi proiettivi su corpi.

1 - INTRODUCTION

Given a skew–field H and a H–vector space V, let $PG(V)$ be the projective space of all linear subspaces of V. Following the custom, when V has finite dimension $d+1$, we denote $PG(V)$ by the symbol $PG(d,H)$.

In this paper, a *point–line geometry* is a pair $\Gamma = (P_\Gamma, \mathscr{L}_\Gamma)$ where P_Γ is a non–empty set, whose elements are called *points*, $\mathscr{L}_\Gamma$ is a collection of subsets of points, called *lines*, any two distinct points belong to at most one common line, every line contains at least two points and every point belongs to at least one line. Furthermore, an *isomorphism* between two point–line geometries $\Gamma = (P_\Gamma, \mathscr{L}_\Gamma)$ and $\Gamma' = (P'_\Gamma, \mathscr{L}'_\Gamma)$ is a bijection $\vartheta : P_\Gamma \longrightarrow P'_\Gamma$ such that ϑ and ϑ^{-1} transform lines onto lines.

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The *Grassmann space of index h* of a projective space $PG(V)$ is the point–line geometry $Gr(h, PG(V))$ whose points are all h–dimensional subspaces of $PG(V)$ and whose lines are the *pencils* of h–subspaces, a pencil being the set of all h–subspaces containing a fixed $(h-1)$–subspace and contained in a fixed $(h+1)$–subspace. When $PG(V) = PG(d,H)$, the Grassmann space $Gr(h, PG(V))$ is simply denoted by $Gr(d,h,H)$. We explicitely note that $Gr(d,0,H)$ is the projective space $PG(d,H)$ and $Gr(d,d-1,H)$ is the dual projective space $PG^*(d,H)$, thus we can suppose $1 \leq h \leq d-2$. It follows that $d \geq 3$.

A *projective embedding* of a connected point–line geometry Γ is an injective mapping e from the point–set P_Γ of Γ to the point–set of a desarguesian projective space Σ such that

(E1) the image $e(P_\Gamma)$ of P_Γ spans Σ;

(E2) for every line L of Γ, $e(L)$ spans a line of Σ;

(E3) no two distinct lines of Γ are mapped by e into the same line of Σ.

If moreover $e(L)$ is a line of Σ for every line L of Γ, then e is said to be *full*. Clearly, e is an isomorphism between the point–line geometries Γ and $e(\Gamma) := (e(P_\Gamma), e(\mathscr{L}_\Gamma))$, where $e(\mathscr{L}_\Gamma) = \{e(L), L \in \mathscr{L}_\Gamma\}$.
According to (Ferrara Dentice *et al.*, 2004), if e is non–full, or we do not know if it is full, or we do not care of that, then we say that e is *lax*.

If H is a (commutative) field, then the Grassmann space $Gr(d,h,H)$ admits full embeddings. More precisely, $Gr(d,h,H)$ is isomorphic to the Grassmann variety $\mathscr{G}(d,h,H)$ of $PG(D,H)$, where $D = \begin{pmatrix} d+1 \\ h+1 \end{pmatrix} - 1$, via the the *Plücker embedding*, sending every h–subspace $\mathscr{X}$ of $PG(d,H)$ into the point of $PG(D,H)$ whose coordinates are the Plücker coordinates of $\mathscr{X}$. Furthermore, all full embeddings of Grassmann varieties are known (Wells A.L. jr., 1983).

In the paper (Ferrara Dentice *et al.*, 2004) a complete exposition of the state of the art for lax embeddings of polar spaces of finite rank ≥ 3 is presented. Moreover, some improvements solving open cases and questions are given. As a consequence of Theorem 1.6 of (Ferrara Dentice *et al.*, 2004), we have the following result.

Theorem 1. *Let $\Gamma = Gr(3,1,H)$, H a skew–field. If Γ admits a lax embedding $e : \Gamma \longrightarrow \Sigma$ in a projective space Σ defined over a skew–field K, then H is a (commutative) sub–field of K and $Gr(3,1,H)$ is isomorphic to the Klein quadric $Q^+(5,H)$ of $PG(5,H)$.*

So far, Theorem 1 is the unique result concerning lax embeddings of Grassmann spaces. The next theorem extends Theorem 1 to Grassmann spaces of arbitrary index $h \geq 1$. We shall prove it in section 2.

Theorem 2. *Let (h,d) be a pair of integers such that $1 \leq h \leq d-2$ and let $\Gamma = Gr(d,h,H)$ be the Grassmann space of index h of a projective space $PG(d,H)$,*

H a skew–field. If Γ admits a lax embedding $e : \Gamma \longrightarrow \Sigma$ in a projective space Σ defined over a skew–field K, then H is a (commutative) sub–field of K and $Gr(d,h,H)$ is isomorphic to the Grassmann variety $\mathscr{G}(d,h,H)$ of $PG(D,H)$, $D = \begin{pmatrix} d+1 \\ h+1 \end{pmatrix} - 1$.

2 - PROOF OF THEOREM 2.

In this section, Γ is a Grassmann space satisfying the hypotheses of Theorem 2 of the previous section.

Lemma 1. *Up to isomorphisms, Γ contains a Grassmann space $\Gamma' = Gr(3,1,H)$.*

Proof. Let X and Y be projective subspaces of $PG(d,H)$ of dimensions $h+2$ and $h-2$, respectively, such that $Y \subset X$. Let P be the set of all h–dimensional subspaces of $PG(d,H)$ containing Y and contained in X and let $\mathscr{L}$ be the subset $\{L \cap P \mid L \in \mathscr{L}_\Gamma, |L \cap P| \geq 2\}$ of $\mathscr{L}_\Gamma$. Clearly, $\mathscr{L} = \{L \in \mathscr{L}_\Gamma \mid L \subset P\}$. Finally, let Z be a 3–dimensional subspace of $PG(d,H)$ such that $Z \subset X$ and $Z \cap Y = \emptyset$. Now, it is easy to see that $\Gamma' := (P, \mathscr{L})$ is a point–line geometry contained in Γ and isomorphic to $Gr(1,Z)$.

$\square$

Lemma 2. *The Grassmann space Γ' contained in Γ admits a lax embedding into a projective subspace Σ' of Σ.*

Proof. Let $\Sigma' := [e(P)]$ be the projective subspace of Σ generated by $e(P)$, and let us define e' as the restriction $e_{|P}$ of e from P to the point–set of Σ'. It is easy to see that $e' : \Gamma' \longrightarrow \Sigma'$ is a lax embedding.

$\square$

End of the Proof. From Lemma 2 and Theorem 1, H is a (commutative) sub–field of K. It follows that Γ is isomorphic to the Grassmann variety $\mathscr{G}(d,h,H)$ of $PG(D,H)$, $D = \begin{pmatrix} d+1 \\ h+1 \end{pmatrix} - 1$.

3 - REFERENCES

Ferrara Dentice E., Marino G. and Pasini A. (2004), *Lax Projective Embeddings of Polar Spaces*, Milan Journal of Mathematics, **72**, 335–377.

Wells A.L. jr. (1983), *Universal Projective Embeddings of the Grassmann, Half Spinor and Dual Orthogonal Geometries*, Quarterly Journal of Mathematics, Oxford ser. (2), **34**, 375–386.