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A Note on VLO Functions

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Abstract – Inspired by a result from Leibov, we find that the supremum defining the BLO norm in $[0,1]$ is actually attained by a specific sub-interval of $[0,1]$ for $f \in VLO([0,1])$

Riassunto – Ispirati da un risultato di Leibov, proviamo che l'estremo superiore che definisce la norma BLO in $[0,1]$ è in realtà assunto da uno specifico sottointervallo di $[0,1]$ nel caso di una funzione $f \in VLO([0,1])$

1 - INTRODUCTION

Definition 1. A real valued locally integrable function $f(x) \in L^1_{loc}([0,1])$ is said to have *Bounded Mean Oscillation* ($f(x) \in BMO([0,1])$) if:

$$\sup_I \fint_I |f(x) - f_I| dx = \|f\|_{BMO} < \infty \tag{1}$$

where f_I denotes $\fint_I f(x)dx$ and I spans the set of all compact intervals contained in $[0,1]$.

One can prove that BMO is a vector space and, modulo the set of functions that are almost everywhere equal to a constant, $\|\cdot\|_{BMO}$ defines a norm on it. This space was introduced in (John and Nirenberg, 1961).

Definition 2. A BMO function $f(x)$ is said to have *Vanishing Mean Oscillation* ($f \in VMO([0,1])$) if it also satisfies:

$$\lim_{|I| \to 0} \sup \fint_I |f(x) - f_I| dx = 0. \tag{2}$$

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In particular, in (Leibov, 1990), Leibov was able to prove that

Lemma 1. If $f \in VMO([0,1])$ then there exists an interval $I^* \subseteq [0,1]$ such that

$$\|f\|_{BMO} = \fint_{I^*} |f - f_{I^*}| dt \tag{3}$$

The aim of the paper is to find an analogue of this result in the subclass of *BLO*-functions (see Section 2 for the definition).
To do so, it will be convenient to introduce some notation.
We will refer to an interval in terms of its center x and half-lenght h as in:

$$I_h^x := [x-h, x+h].$$

Let us fix $h \in \left(0, \frac{1}{2}\right]$ and define $S_h = [h, 1-h]$: we have that $I_h^x \subseteq [0,1]$ if and only if $x \in S_h$. Thus we can define the set

$$T = \left\{ (x,h) \in \mathbb{R}^2 : \ h \in \left(0, \frac{1}{2}\right], \ x \in S_h \right\} \tag{4}$$

such that $I_h^x \subseteq [0,1]$ if and only if $(x,h) \in T$.
Finally, for a generic function $f \in BMO([0,1])$, let us define the function $F : T \to \mathbb{R}$ given by

$$F(x,h) = \fint_{I_h^x} |f - f_{I_h^x}| dt, \tag{5}$$

which is always a continuous function since $f \in L^1([0,1])$. Thus the norm on $BMO([0,1])$ can be also defined as:

$$\|f\|_{BMO} = \sup_{(x,h)\in T} F(x,h). \tag{6}$$

We can also restate the VMO property in terms of F. We have that:

$$f \in VMO \iff \lim_{h\to 0} \sup_{x \in S_h} F(x,h) = 0 \tag{7}$$

that is to say that F converges to 0 as $h \to 0$ uniformly with respect to x.
The idea of the proof of the lemma by Leibov is to notice that F is continuous, and since f is in VMO, it can be extended by continuity to the closure of T, namely:

$$\tilde{T} = \left\{ (x,h) \in \mathbb{R}^2 : \ h \in \left[0, \frac{1}{2}\right], \ x \in S_h \right\} \tag{8}$$

by posing $F = 0$ on $[0,1] \times 0$.
A straightforward application of Weierstrass theorem concludes the proof. In particular three main ingredients emerge:

- The compactness of $\widetilde{T}$;

- The continuity of F on T;
- The fact that if $f \in VMO([0,1])$ then F can be extended with continuity of the whole $\widetilde{T}$.

In order to mimic this proof and obtain the same result for functions in $VLO([0,1])$ with respect to the norm in $BLO([0,1])$ we will need to write the norm of a function f in such space in terms of a suitable two variables function F and then assure this three hypotheses. We will see that even the second one is not necessarily satisfied by functions in $BLO([0,1]) \setminus VLO([0,1])$.

2 - THE SPACES $BLO([0,1])$ AND $VLO([0,1])$

The following is a definition by R.Coifman and R.Rochberg.

Definition 3. A real valued locally integrable function $f(x) \in L^1_{loc}(\mathbb{R})$ is said to have *Bounded Lower Oscillation* ($f(x) \in BLO(\mathbb{R})$) if

$$\sup_I \fint_I \left[f(x) - \inf_I f \right] dx = \sup_I \left[f_I - \inf_I f \right] = \|f\|_{BLO} < \infty. \tag{9}$$

Here and throughout the rest of this paper, we are using inf to denote the essential infimum.

Of course, as in the definition of BMO, we need to think of the class of $BLO([0,1])$ functions modulo the set of all functions which are almost everywhere equal to a constant.
In (Korey, 2001), Korey gave the following definition of a subclass in BLO.

Definition 4. A BLO function is said to have *Vanishing Lower Oscillation* ($f \in VLO([0,1])$) if it also satisfies:

$$\lim_{|I| \to 0} \sup \left[f_I - \inf_I f \right] = 0. \tag{10}$$

It will be useful to notice that $L^\infty([0,1]) \subset BLO([0,1])$ while $L^\infty([0,1]) \not\subset VLO([0,1])$. In particular an equivalent distance between BLO functions and L^∞ is given in (Angrisani, 2017). $BLO([0,1])$ is not a real vector space, since if $f \in BLO([0,1])$ then $-f \in BLO([0,1])$ if and only if $f \in L^\infty([0,1])$.
However, if $f, g \in BLO([0,1])$ then $f + g \in BLO([0,1])$ and if $f \in BLO([0,1])$ and $a \in [0, +\infty)$, then $af \in BLO([0,1])$.
We can define on $BLO([0,1])$ the following quantity:

$$\|f\|_{BLO} = \sup_{I \subseteq [0,1]} \fint_I f dt - \inf_I f \tag{11}$$

observing that:

- $\|f\|_{BLO} \geq 0$ and $\|f\|_{BLO} = 0$ if and only if f is constant;
- If $a > 0$ then $\|af\|_{BLO} = a\|f\|_{BLO}$;
- $\|f+g\|_{BLO} \leq \|f\|_{BLO} + \|g\|_{BLO}$.

Even if it lacks homogeneity for negative constants, following other authors (see (Korey, 2001) and (Coifman and Rochberg, 1980)) we will call this quantity a norm on $BLO([0,1])$ anyways.
By using the same notation in the previous section for I_h^x, S_h and T, for a generic function $f \in BLO([0,1])$ let us define the function $F : T \to \mathbb{R}$ to be

$$F(x,h) = \fint_{I_h^x} f dt - \inf_{I_h^x} f \tag{12}$$

Thus the norm on $BLO([0,1])$ can be also defined as

$$\|f\|_{BLO} = \sup_{(x,h)\in T} F(x,h). \tag{13}$$

In terms of F, we have that $f \in VLO([0,1])$ if and only if

$$\lim_{h\to 0} \sup_{x\in S_h} F(x,h) = 0 \tag{14}$$

that is to say that F converges to 0 as $h \to 0$ uniformly with respect to x.
However, this time it is not true that F is continuous in T for any $f \in BLO([0,1])$.
In particular, let us observe that for any $f \in BLO([0,1])$ we can define:

$$G(x,h) = \fint_{I_h^x} f dt \tag{15}$$

$$H(x,h) = \inf_{I_h^x} f \tag{16}$$

to obtain that

$$F(x,h) = G(x,h) + H(x,h) \tag{17}$$

thus, since G is continuous in T (since $f \in L^1([0,1])$), then F is continuous if and only if H is continuous.
Let us consider for instance the function $f(x) = \chi[1/2,1]$.
We have that $f \in BLO([0,1])$ since $f \in L^\infty$ but it is also easy to see that in this case H is not continuous. In particular, let us observe that since $f \notin VLO([0,1])$.
Moreover one can see that $\sup_{(x,h)\in T} F = 1$ but $F(x,h) < 1$ for any $(x,h) \in T$.

3 - THE NORM-ATTAINING PROPERTIES OF VLO FUNCTIONS IN BLO

Our aim is to show an analogue of Leibov's Lemma for VLO functions. To do so we need to prove that, at least for a function f in VLO, the corresponding F is continuous.

Proposition 2. Let $f \in VLO([0,1])$.Then

$$H(x,h) = \inf_{I_x^h} f$$

is continuous in T.

Proof. Let us prove this assertion by contradiction. Fix $(x,h) \in T$ and let us suppose there exists a $\overline{\varepsilon} > 0$ such that for any $\delta > 0$ there exists a point (x_δ, h_δ) such that

$$|x - x_\delta| + |h - h_\delta| \leq \delta \tag{18}$$

and

$$|H(x,h) - H(x_\delta, h_\delta)| \geq \overline{\varepsilon}. \tag{19}$$

To fix the ideas, let us consider $\delta_n = \frac{1}{n}$ and let us denote $x_n := x_{\delta_n}$, $h_n := h_{\delta_n}$, $I := I_h^x$ and $I_n := I_{h_n}^{x_n}$. Moreover, we can suppose n is big enough to have $I \cap I_n \neq \emptyset$. Posing $m_n := \inf_{I_n \cup I} f$, Equation (19) assures that

$$\inf_{I \cap I_n} f \neq m_n \tag{20}$$

thus

$$m_n = \inf_{I \Delta I_n} f \tag{21}$$

where $I \Delta I_n = (I \setminus I_n) \cup (I_n \setminus I)$. Let us suppose that $I_n \setminus I \neq \emptyset$ and $m_n = \inf_{I_n \setminus I} f$. Now let us observe that $I_n \setminus I$ has at most two connected components $I_1 = [x + h, x_n + h_n]$ and $I_2 = [x_n - h_n, x - h]$. Thus let us suppose that $m_n = \inf_{I_1} f$.
Let us suppose that n is big enough to have $x_n + h_n < x + 3h$ and then let us consider the interval

$$I_1^* = [2(x+h) - x_n - h_n, x_n + h_n] \tag{22}$$

on which we have that $\inf_{I_1^*} f = m_n$. Moreover, let us observe that by construction $|I_1^* \cap I| = |I_1|$ and $|I_1^*| = 2|I_1|$. Now, since we have supposed that $m_n = \inf_{I_n \setminus I} f$, then

$$m_n = H(x_n, h_n) \leq H(x,h) \tag{23}$$

and

$$H(x,h) \geq m_n + \overline{\varepsilon}. \tag{24}$$

By definition we have

$$\inf_{I_1^* \cap I} f \geq H(x,h) \geq m_n + \overline{\varepsilon}. \tag{25}$$

Now let us observe that

$$\int_{I_1^*} f dt = \int_{I_1^* \cap I} f dt + \int_{I_1} f dt \geq (2m_n + \overline{\varepsilon})|I_1| \tag{26}$$

and then

$$\fint_{I_1^*} f\,dt \geq m_n + \frac{\overline{\varepsilon}}{2} \tag{27}$$

so that

$$\fint_{I_1^*} f\,dt - \inf_{I_1^*} \geq \frac{\overline{\varepsilon}}{2}. \tag{28}$$

In such case, let us denote with $I_n^* := I_1^*$. If $m_n = \inf_{I_2} f$, we can construct in a similar way an interval I_2^* on which we have Eq. (28) and we can pose $I_n^* := I_2^*$. The construction of I_n^* can be done in the same way if $I \setminus I_n \neq \emptyset$ and $m_n = \inf_{I \setminus I_n} f$. Thus, for any n there exists an interval I_n^* with length $|I_n^*| = x_n + h_n - x - h$ such that

$$\fint_{I_n^*} f\,dt - \inf_{I_n^*} \geq \frac{\overline{\varepsilon}}{2} \tag{29}$$

that is a contradiction with the fact that $f \in VLO([0,1])$ since $|I_n^*| \to 0$. □

Now we can prove the main result

Proposition 3. If $f \in VLO([0,1])$ then there exists an interval $I^* \subseteq [0,1]$ such that

$$\|f\|_{BLO} = \fint_{I^*} f\,dt - \inf_{I^*} f \tag{30}$$

Proof. Since $f \in VLO([0,1])$, H (and then F) is continuous on T. Moreover, since F converges to 0 as $h \to 0$ uniformly with respect to x, we can extend F with continuity to

$$\widetilde{T} = \left\{ (x,h) \in \mathbb{R}^2 : \ h \in \left[0, \frac{1}{2}\right], \ x \in S_h \right\} \tag{31}$$

by setting $F(x,0) = 0$. Thus, since F is continuous on $\widetilde{T}$ that is compact, there exists a point $(x^*, h^*) \in \widetilde{T}$ such that

$$\sup_{(x,h)\in T} F(x,h) = \max_{(x,h)\in \widetilde{T}} F(x,h) = F(x^*, h^*) \tag{32}$$

and in particular we have $I^* = I_{h^*}^{x^*}$. □

Let us show that $f \in VLO([0,1])$ is a sufficient but not necessary condition. Consider

$$f(x) = \begin{cases} 4x - 1 & x \in \left[0, \frac{1}{2}\right] \\ 0 & x \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{33}$$

First let us observe that $f \in BLO([0,1])$ since $f \in L^\infty([0,1])$. Then let us consider the intervals $I_h = \left[\frac{1}{2} - h, \frac{1}{2} + h\right]$ for $h < \frac{1}{4}$. Then

$$\lim_{h\to 0} \fint_{I_h} f\,dt - \inf_{I_h} f = \lim_{h\to 0} h - \frac{1}{2} = -\frac{1}{2} \tag{34}$$

so $f \notin VLO([0,1])$. With some easy calculations, one can show that $\|f\|_{BLO} = 1$. However, posing $I^* = \left[0, \frac{1}{2}\right]$, we have

$$\fint_{I^*} f\,dt - \inf_{I^*} f = 0 + 1 = 1 = \|f\|_{BLO}, \tag{35}$$

so there is an interval I^* attaining $\|f\|_{BLO}$ even if $f \notin VLO([0,1])$.

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