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Nonlinear Elliptic Systems with Measure Data in Low Dimension

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Abstract. – *In this paper we prove existence of solutions to some elliptic systems with measure on the right hand side, in dimension two and three.*

1. – Introduction.

We consider the Dirichlet problem

$$-div(A(x, Du(x))) = \mu \qquad \text{in } \Omega \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{1.2}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, A is an elliptic operator and μ is a measure on $\mathbb{R}^n$ with values into $\mathbb{R}^N$; thus (1.1) is a system of N elliptic equations. When $N = 1$ (1.1) is one single equation and existence of distributional solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has been deeply studied starting from [3]; see also [5], [4] and the survey [2]; uniqueness is a delicate matter: see [17], [1], [10] and the introduction of [6]; nice regularity results are contained in [15] and [16], whose introductions and references contain additional information for the interested reader. Existence of solutions is obtained by a truncation argument; this shows why the vectorial case $N \geq 2$ is difficult and only few contributions are available; in [9] and [7] the authors deal with p-laplace operator $A(x, \xi) = |\xi|^{p-2}\xi$; more general systems are considered in [8]: they assume that

$$0 \leq \sum_{a=1}^{N} \sum_{i=1}^{n} A_i^a(x, \xi)((Id - a \times a)\xi)_i^a \tag{1.3}$$

for every $a \in \mathbb{R}^N$ with $|a| \leq 1$; in [18] the author assumes the componentwise sign condition

$$0 \leq \sum_{i=1}^{n} A_i^a(x, \xi)\xi_i^a \tag{1.4}$$

for every $a \in \{1, \dots, N\}$. When $N = 2$ then (1.3) implies (1.4): it is enough to take first $a = (1, 0)$, then $a = (0, 1)$. In our paper we consider the componentwise coercivity

$$\nu|\xi^a|^2 - M \leq \sum_{i=1}^{n} A_i^a(x, \xi)\xi_i^a \tag{1.5}$$

for every $a \in \{1, \dots, N\}$, for some constants $\nu \in (0, +\infty)$ and $M \in [0, +\infty)$. This condition is satisfied in the following example: take $n = N = 2$, $t_0 \in (0, +\infty)$ and $h(t) = \sqrt{1 + (t - t_0)^2}$; we set $f(\xi) = |\xi|^2 + h(\det(\xi))$ where ξ is any 2×2 matrix with real entries and $\det(\xi)$ its determinant; we consider

$$A_i^a(x, \xi) = \frac{\partial f}{\partial \xi_i^a}(\xi) = 2\xi_i^a + h'(\det(\xi)) Cof_i^a(\xi); \tag{1.6}$$

then (1.5) is verified with $\nu = 2$ and $M = t_0$; moreover, neither (1.4) nor (1.3) are satisfied. In this paper we prove existence of distributional solutions to (1.1), (1.2) under the componentwise coercivity (1.5); our proof needs to restrict ourselves to dimension two and three; moreover, our theorem can deal with measures concentrated on compact sets with zero Lebesgue measure; precise assumptions and result are in the next section; the proof appears in section 3.

2. – Assumptions and results.

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let $A : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ be a Caratheodory function, that is, $A(x, \xi)$ is measurable with respect to x and continuous with respect to ξ, where $N \geq 2$. For suitable constants $\nu \in (0, +\infty)$ and $M \in [0, +\infty)$, we assume componentwise coercivity: for every $\gamma \in \{1, \dots, N\}$ it results that

$$\nu|\xi^\gamma|^2 - M \leq \sum_{i=1}^{n} A_i^\gamma(x, \xi)\xi_i^\gamma \tag{2.1}$$

for almost every $x \in \Omega$, for any $\xi \in \mathbb{R}^{N \times n}$, where $\xi^1, \dots, \xi^N$ are the N rows of the $N \times n$ matrix ξ. We assume linear growth for A with respect to ξ: for a suitable constant $c_1 \in (0, +\infty)$ we have

$$|A(x, \xi)| \leq c_1[1 + |\xi|] \tag{2.2}$$

for almost every $x \in \Omega$, for any $\xi \in \mathbb{R}^{N \times n}$. We assume monotonicity for A with respect to ξ: for a suitable constant $c_2 \in (0, +\infty)$ we have

$$c_2|\xi - z|^2 \leq \langle A(x, \xi) - A(x, z); \xi - z \rangle \tag{2.3}$$

for almost every $x \in \Omega$, for any $\xi, z \in \mathbb{R}^{N\times n}$. Let μ be a finite Radon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$; moreover, we assume that

$$\operatorname{supp}|\mu| \subset \Omega \tag{2.4}$$

and

$$\mathcal{L}^n(\operatorname{supp}|\mu|) = 0 \tag{2.5}$$

where $\mathcal{L}^n(E)$ is the n dimensional Lebesgue measure of the set $E \subset \mathbb{R}^n$. Let us consider q such that

$$\frac{2n}{n+2} < q < \frac{n}{n-1}. \tag{2.6}$$

We remark that, for $n \geq 2$, we have $1 \leq \frac{2n}{n+2}$ and $\frac{n}{n-1} \leq 2$; thus $q \in (1,2)$; note that $\frac{2n}{n+2} < \frac{n}{n-1}$ for $n<4$; thus we are dealing only with low dimension: $n = 2$ or $n = 3$. In this paper we prove existence of weak solution to system (1.1) with zero Dirichlet boundary condition (1.2); more precisely, we prove the following

THEOREM 2.1. – *Under the previous assumptions* (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), *there exists a weak solution* $u = (u^1, \dots, u^N) \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ *of the system* (1.1), *that is,*

$$\begin{aligned}\int_\Omega \sum_{\gamma=1}^{N} \sum_{i=1}^{n} A_i^\gamma(x, Du(x)) D_i v^\gamma(x) d\mathcal{L}^n(x) \\ = \int_\Omega \sum_{\gamma=1}^{N} v^\gamma(x) d\mu(x) \qquad \forall v \in C_0^\infty(\Omega; \mathbb{R}^N).\end{aligned} \tag{2.7}$$

A model measure μ for the previous theorem can be obtained by means of $\mu = (\mu^1, \dots, \mu^N)$ with $\mu^\beta = \mathcal{H}^{s_\beta} \lfloor \mathcal{K}_\beta$ where $0 \leq s_\beta < n$, $\mathcal{H}^t$ is the t-dimensional Hausdorff measure in $\mathbb{R}^n$ and $\mathcal{K}_\beta \subset \Omega$ is a compact set with $\mathcal{H}^{s_\beta}(\mathcal{K}_\beta) < +\infty$.

REMARK 2.1. – *We take* $n = N = 2$, $t_0 \in (0, +\infty)$ *and*

$$h(t) = \sqrt{1 + (t - t_0)^2}; \tag{2.8}$$

we set

$$f(\xi) = |\xi|^2 + h(\det(\xi)) \tag{2.9}$$

where ξ *is any* 2×2 *matrix with real entries and* $\det(\xi)$ *its determinant:*

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix} \qquad Cof(\xi) = \begin{pmatrix} \xi_2^2 & -\xi_1^2 \\ -\xi_2^1 & \xi_1^1 \end{pmatrix} \tag{2.10}$$

$$\det \xi = \sum_{j=1}^{2} \xi_j^a Cof_j^a(\xi) \tag{2.11}$$

for every $a \in \{1, 2\}$*, so that*

$$\frac{\partial}{\partial \xi_i^a}(\det \xi) = Cof_i^a(\xi) \tag{2.12}$$

for every $i, a \in \{1, 2\}$*. Then*

$$\frac{\partial f}{\partial \xi_i^a}(\xi) = 2\xi_i^a + h'(\det(\xi))Cof_i^a(\xi); \tag{2.13}$$

we set

$$A_i^a(\xi) = 2\xi_i^a + h'(\det(\xi))Cof_i^a(\xi); \tag{2.14}$$

then (2.1) *is verified with* $\nu = 2$ *and* $M = t_0$, (2.2) *is satified with* $c_1 = 3$, (2.3) *is verified with* $c_2 = 1$*. We claim that neither* (1.4) *nor* (1.3) *are satisfied. Indeed, for* $\gamma = 1$*, we choose*

$$\tilde{\xi} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with } \varepsilon > 0 \tag{2.15}$$

and we get

$$\sum_{i=1}^{2} A_i^1(\tilde{\xi})\tilde{\xi}_i^1 = \varepsilon[2\varepsilon + h'(\varepsilon)]. \tag{2.16}$$

Since

$$\lim_{\varepsilon \to 0^+} [2\varepsilon + h'(\varepsilon)] = \frac{-t_0}{\sqrt{1 + t_0^2}} < 0 \tag{2.17}$$

it turns out that

$$\sum_{i=1}^{2} A_i^1(\tilde{\xi})\tilde{\xi}_i^1 < 0 \tag{2.18}$$

for suitable small $\varepsilon > 0$*; thus* (1.4) *does not hold true. When* $N = 2$, (1.3) *implies* (1.4)*: this shows that* (1.3) *does not hold true as well. The present example is obtained by slightly modifing the one given in* [12]; *see also examples* 2.4 *and* 2.5 *in* [13].

3. – Proof of Theorem 2.1.

Let $\{\varsigma_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers converging to zero. We mollify our measure μ and we obtain functions $h_k = \mu * \rho_{\varsigma_k} \in C^\infty(\mathbb{R}^n;\mathbb{R}^N)$ weakly* converging to μ, with

$$\text{(3.1)} \qquad supp\, h_k \subset (supp\,|\mu|)_{\varsigma_k} = \{x \in \Omega : dist(x, supp\,|\mu|) \le \varsigma_k\} \subset \Omega$$

and

$$\text{(3.2)} \qquad \|h_k\|_{L^1(\Omega)} \le |\mu|(\mathbb{R}^n) < +\infty.$$

We use Leray-Lions surjectivity result [14] in order to find $u_k \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ such that

$$\text{(3.3)} \qquad \begin{aligned} &\int_\Omega \sum_{a=1}^N \sum_{i=1}^n A_i^a(x, Du_k(x)) D_i v^a(x)\, d\mathcal{L}^n(x) \\ &= \int_\Omega \sum_{a=1}^N v^a(x) h_k^a(x)\, d\mathcal{L}^n(x) \qquad \forall v \in W_0^{1,2}(\Omega;\mathbb{R}^N). \end{aligned}$$

Now we want to get a priori estimates for u_k: we use a componentwise truncation argument, see [11] and [9], that allows us to use level sets as in [3]. For $j \in \{0, 1, 2, \dots\}$ we set

$$T(s) = \begin{cases} 0 & 0 \le s \le j \\ s - j & j < s < j+1 \\ 1 & s \ge j+1 \\ -T(-s) & s < 0. \end{cases}$$

Note that $|T(s)| \le 1$. We fix $\gamma \in \{1, \dots, N\}$ and we take $v = (v^1, \dots, v^N)$ with $v^a = 0$ for $a \ne \gamma$ and $v^\gamma = T(u_k^\gamma)$, where u_k^γ is the γ-th component of $u_k = (u_k^1, \dots, u_k^N)$. Then $v^\gamma \in W_0^{1,2}(\Omega)$ with $Dv^\gamma = 1_{B_{j,k}} Du^\gamma$ where $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$; moreover,

$$B_{j,k} = \left\{x \in \Omega : j \le \left|u_k^\gamma(x)\right| < j+1\right\}.$$

We use such a v as a test function in (3.3): on the left hand side we use the componentwise coercivity (2.1), on the right hand side we keep in mind the inequality $|T(s)| \le 1$ together with the L^1 bound (3.2) and we get

$$
\begin{aligned}
(3.4) \qquad & \int_{B_{j,k}} (\nu|Du_k^\gamma(x)|^2 - M) d\mathcal{L}^n(x) \\
& \leq \int_{B_{j,k}} \sum_{i=1}^n A_i^\gamma(x, Du_k(x)) D_i u_k^\gamma(x) d\mathcal{L}^n(x) \\
& = \int_\Omega \sum_{i=1}^n A_i^\gamma(x, Du_k(x)) D_i v^\gamma(x) d\mathcal{L}^n(x) \\
& = \int_\Omega \sum_{a=1}^N \sum_{i=1}^n A_i^a(x, Du_k(x)) D_i v^a(x) d\mathcal{L}^n(x) \\
& = \int_\Omega \sum_{a=1}^N v^a(x) h_k^a(x) d\mathcal{L}^n(x) \\
& = \int_\Omega T(u_k^\gamma) h_k^\gamma(x) d\mathcal{L}^n(x) \\
& \leq \|h_k^\gamma\|_{L^1(\Omega)} \leq |\mu|(\mathbb{R}^n)
\end{aligned}
$$

then

$$
(3.5) \qquad \int_{B_{j,k}} |Du_k^\gamma|^2 \leq \frac{M\mathcal{L}^n(\Omega) + |\mu|(\mathbb{R}^n)}{\nu}.
$$

Holder inequality, estimate (3.5), assumption $q < n/(n-1)$ in (2.6) and Sobolev embedding are used as in [3] in order to ensure the existence of constants $c_3, c_4 \in (0, +\infty)$, depending only on $n, q, \nu, M, \mathcal{L}^n(\Omega), |\mu|(\mathbb{R}^n)$ such that, for every $k \in \mathbb{N}$ and any $\gamma \in \{1, \ldots, N\}$, it results

$$
(3.6) \qquad \int_\Omega |u_k^\gamma|^{q^*} \leq c_3
$$

and

$$
(3.7) \qquad \int_\Omega |Du_k^\gamma|^q \leq c_4.
$$

Assumption (2.2) and estimate (3.7) guarantee that

$$
(3.8) \qquad \int_\Omega |A(x, Du_k(x))|^q dx \leq (c_1 2)^q (\mathcal{L}^n(\Omega) + N^{q/2} N c_4)
$$

for every $k \in \mathbb{N}$. Weak compactness allows us to get existence of $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$

and $\sigma \in L^q(\Omega; \mathbb{R}^{N\times n})$ such that, up to a subsequence,

$$u_k \rightharpoonup u \qquad \text{weakly in } W_0^{1,q}(\Omega;\mathbb{R}^N), \tag{3.9}$$

$$A(x, Du_k(x)) \rightharpoonup \sigma(x) \qquad \text{weakly in } L^q(\Omega;\mathbb{R}^N) \tag{3.10}$$

and

$$u_k \to u \ \text{ strongly in } L^t(\Omega, \mathbb{R}^N) \quad \forall\, t < q^*. \tag{3.11}$$

We want to prove pointwise convergence of Du_k following [9]. We fix $k_0 \in \mathbb{N}$ and a ball B_R with $B_R \subset \Omega \setminus supp\,|\mu|$ and $B_R \cap (supp\,|\mu|)_{\varsigma_k} = \emptyset$ for every $k \geq k_0$. We consider $\eta : \mathbb{R}^n \to \mathbb{R}$ such that $\eta \in C^\infty(\mathbb{R}^n)$ with $0 \leq \eta \leq 1$ in $\mathbb{R}^n$, $\eta = 0$ outside B_R, $\eta = 1$ on $B_{R/2}$; moreover, $|D\eta| \leq c_5/R$ in $\mathbb{R}^n$. In (3.3) we use the test function $v = \eta^2 u_k$; since $supp\, h_k \subset (supp|\mu|)_{\varsigma_k}$ then $B_R \cap supp\, h_k = \emptyset$ and we have

$$\int_\Omega \sum_{a=1}^N \sum_{i=1}^n A_i^a(x, Du_k) D_i(\eta^2 u_k^a) = \int_\Omega \sum_{a=1}^N h_k^a \eta^2 u_k^a = \int_{B_R} \sum_{a=1}^N h_k^a \eta^2 u_k^a = 0 \tag{3.12}$$

then

$$\int_{B_R} \sum_{a=1}^N \sum_{i=1}^n A_i^a(x, Du_k) \eta^2 D_i u_k^a = -\int_{B_R} \sum_{a=1}^N \sum_{i=1}^n A_i^a(x, Du_k) 2\eta (D_i\eta) u_k^a. \tag{3.13}$$

On the left hand side we use componentwise coercivity (2.1), on the right hand side we use linear growth (2.2); we recall properties of η and we get the following Caccioppoli estimate

$$\int_{B_{R/2}} |Du_k|^2 \leq \frac{2(MN+1)}{\nu} \mathcal{L}^n(B_R) + \left(\frac{c_1 c_5}{R} \left(1 + \frac{2}{\nu} \right) \right)^2 \int_{B_R} |u_k|^2. \tag{3.14}$$

We recall the assumption $2n/(n+2) < q$ in the left hand side of (2.6) and we get

$$2 < q^* \tag{3.15}$$

then we can use Holder inequality and estimate (3.6) in order to get

$$\begin{aligned} \int_{B_R} |u_k|^2 = \sum_{\gamma=1}^N \int_{B_R} |u_k^\gamma|^2 &\leq \sum_{\gamma=1}^N \left(\int_{B_R} |u_k^\gamma|^{q^*} \right)^{2/q^*} \left(\int_{B_R} 1 \right)^{1-(2/q^*)} \\ &\leq N (c_3)^{2/q^*} (\mathcal{L}^n(B_R))^{1-(2/q^*)} \end{aligned} \tag{3.16}$$

thus (3.14) gives us

$$\int_{B_{R/2}} |Du_k|^2 \leq c_6 \tag{3.17}$$

for a suitable constant $c_6 \in (0, +\infty)$ that does not depend on k. For $k' > k$, we take the corresponding solutions $u_{k'}$ and u_k of (3.3): we use the test function $v = \eta^2(u_k - u_{k'})$, where η is now a cut-off function between $B_{R/2}$ and $B_{R/4}$: $\eta \in C^\infty(\mathbb{R}^n)$, $0 \leq \eta \leq 1$ in $\mathbb{R}^n$, $\eta = 1$ on $B_{R/4}$, $\eta = 0$ outside $B_{R/2}$ and $|D\eta| \leq c_7/R$ in $\mathbb{R}^n$; we recall monotonicity assumption (2.3), linear growth (2.2) and we get

$$
\begin{aligned}
(3.18)\qquad & c_2 \int_{B_{R/4}} |Du_k - Du_{k'}|^2 \leq c_2 \int_{B_{R/2}} |Du_k - Du_{k'}|^2 \eta^2 \\
& \leq \int_{B_{R/2}} \langle A(x, Du_k) - A(x, Du_{k'}); Du_k - Du_{k'} \rangle \eta^2 \\
& = \int_{B_{R/2}} \langle A(x, Du_k) - A(x, Du_{k'}); D((u_k - u_{k'})\eta^2) \rangle \\
& - \int_{B_{R/2}} \langle A(x, Du_k) - A(x, Du_{k'}); (u_k - u_{k'}) 2\eta D\eta \rangle \\
& = \int_{B_{R/2}} \sum_{a=1}^{N} h_k^a (u_k^a - u_{k'}^a) \eta^2 - \int_{B_{R/2}} \sum_{a=1}^{N} h_{k'}^a (u_k^a - u_{k'}^a) \eta^2 \\
& - \int_{B_{R/2}} \langle A(x, Du_k) - A(x, Du_{k'}); (u_k - u_{k'}) 2\eta D\eta \rangle \\
& = - \int_{B_{R/2}} \langle A(x, Du_k) - A(x, Du_{k'}); (u_k - u_{k'}) 2\eta D\eta \rangle \\
& \leq \int_{B_{R/2}} 2c_1 (1 + |Du_k| + |Du_{k'}|) |u_k - u_{k'}| 2\eta |D\eta| \\
& \leq \left(\int_{B_{R/2}} (4c_1)^2 (1 + |Du_k| + |Du_{k'}|)^2 \right)^{1/2} \left(\int_{B_{R/2}} |u_k - u_{k'}|^2 |D\eta|^2 \right)^{1/2} \\
& \leq \frac{12 c_1 c_7}{R} \left(\int_{B_{R/2}} (1 + |Du_k|^2 + |Du_{k'}|^2) \right)^{1/2} \left(\int_{B_{R/2}} |u_k - u_{k'}|^2 \right)^{1/2} \\
& \leq \frac{12 c_1 c_7}{R} \left(\mathcal{L}^n(B_{R/2}) + 2c_6 \right)^{1/2} \left(\int_{B_{R/2}} |u_k - u_{k'}|^2 \right)^{1/2} .
\end{aligned}
$$

Since (3.15) holds true, we can use (3.11) with $t = 2$; the strong convergence of u_k in $L^2(\Omega)$ and the previous estimate guarantee the strong convergence of Du_k in

$L^2(B_{R/4})$ so that, up to a further subsequence,

$$Du_k(x) \to Du(x) \qquad \text{for almost every } x \in B_{R/4}. \tag{3.19}$$

We cover $\Omega \setminus supp\,|\mu|$ and the previous convergence holds true in $\Omega \setminus supp\,|\mu|$. Let us recall that

$$\mathcal{L}^n(supp\,|\mu|) = 0,$$

thus

$$Du_k \to Du \quad \text{almost everywhere in } \Omega. \tag{3.20}$$

We keep in mind that $z \to A(x, z)$ is continuous, thus

$$A(x, Du_k(x)) \to A(x, Du(x)) \quad \text{for almost every } x \in \Omega. \tag{3.21}$$

Weak convergence (3.10) and pointwise convergence (3.21) allow us to write

$$\sigma(x) = A(x, Du(x)). \tag{3.22}$$

Then we can pass to the limit, as $k \to +\infty$, into (3.3) and we get that u satisfies (2.7). This ends the proof. □

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