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Squarefree Lexsegment Ideals with Linear Resolution

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Sunto. – *In questo articolo determiniamo tutti gli ideali completamente lexsegmento squarefree con risoluzione lineare. Sia M_d l'insieme di tutti i monomi squarefree di grado d in un anello di polinomi $k[x_1, \ldots, x_n]$ in n variabili su un campo k. Ordiniamo i monomi lessicograficamente in modo che $x_1 > x_2 > \ldots > x_n$, così un lexsegmento (di grado d) è un sottoinsieme di M_d del tipo $L(u, v) = \{w \in M_d : u \geq w \geq v\}$ per qualche u, $v \in M_d$ con $u \geq v$. Un ideale generato da un lexsegmento è chiamato ideale lexsegmento. Descriviamo la procedura per determinare quando un tale ideale ha risoluzione lineare.*

Abstract. – *In this paper we determine all squarefree completely lexsegment ideals which have a linear resolution. Let M_d denote the set of all squarefree monomials of degree d in a polynomial ring $k[x_1, \ldots, x_n]$ in n variables over a field k. We order the monomials lexicographically such that $x_1 > x_2 > \ldots > x_n$, thus a lexsegment (of degree d) is a subset of M_d of the form $L(u, v) = \{w \in M_d : u \geq w \geq v\}$ for some u, $v \in M_d$ with $u \geq v$. An ideal generated by a lexsegment is called a lexsegment ideal. We describe the procedure to determine when such an ideal has a linear resolution.*

Introduction.

Let M_d denote the set of all squarefree monomials of degree d in a polynomial ring $R = k[x_1, ..., x_n]$ in n variables over a field k. We order the monomial lexicographically such that $x_1 > x_2 > \ldots > x_n$. A squarefree lexsegment of degree d is a subset of M_d of the form $L = L(u, v) = \{w \in M_d : u \geq w \geq v\}$ for some u, $v \in M_d$ with $u \geq v$. An ideal generated by a lexsegment is called lexsegment ideal. Lexsegment ideals in this generality were first introduced by Hulett and Martin [7]. In extremal combinatorics one usually considers initial lexsegment ideals. These are ideals which are spanned by an initial lexsegment $L^i(v) = \{w \in M_d : w \geq v\}$. A final lexsegment is a set of the form $L^f(u) = \{w \in M_d : w \leq u\}$. The squarefree shadow of a set S of squarefree monomials in M_d is the set of monomials $\mathrm{Shad}(S) = \{sx_i : \forall s \in S, \forall i \notin \mathrm{supp}(s)\}$, where $\mathrm{supp}(s) = \{i : x_i \text{ divides } s\}$. We define the i-th shadow recursively by $\mathrm{Shad}^i(S) = \mathrm{Shad}(\mathrm{Shad}^{i-1}(S))$. A squarefree lexsegment ideal is called completely lexsegment if all the iterated shadows of L are again squarefree lexsegments.

Lexsegment ideals have been considered in the non-squarefree case in [5] and in [2]. In [2], Aramova, De Negri and Herzog determined all lexsegment ideals which have a linear resolution. In [3], Aramova, Herzog and Hibi introduced the concept of squarefree initial lexsegment ideal and showed that squarefree initial lexsegment ideals are stable. In addition they computed the explicit minimal free resolution of squarefree stable ideals, and showed that these resolutions are linear. In the present paper we want to give a description for arbitrary squarefree lexsegment ideals with a linear resolution. In order to do this, we will often use the fact that initial squarefree lexsegment ideals are stable and have linear resolutions [3], and that final lexsegment ideals are stable with respect to the lexicographic order with $x_n > x_{n-1} > \ldots > x_1$, and thus by [3], they have a linear resolution.

In the first section we translate some results for completely lexsegment ideals in the exterior algebra to the corresponding statements in the polynomial ring. In Section 2 we give a full characterization of squarefree completely lexsegment ideals with a linear resolution. Our approach is similar to that used in [2]. Here the following isomorphism $\mathrm{Tor}_I^R(R/I, k) \cong H_i(x_1, \ldots, x_n; R/I)$, where $H_i(x_1, \ldots, x_n; R/I)$ is the Koszul homology, plays a crucial role. As an application we consider some special classes of completely lexsegment ideals with linear resolution. Finally, in Section 3 we consider arbitrary lexsegment ideals and we describe the procedure to determine when such an ideal has a linear resolution.

This paper may be considered as a contribution to the more general problem of describing all monomial ideals with linear resolution.

1. – Completely squarefree lexsegment ideals.

In [4], the first author characterized completely lexsegment ideals in the exterior algebra and gave some sufficient conditions for lexsegment ideals to have a linear resolution. The following theorem shows that the problem of studying minimal resolutions of lexsegment ideals in the exterior algebra can be reduced to studying minimal resolutions of squarefree lexsegment ideals.

REMARK 1.1. – Let V be a vector space with basis $e_1, \ldots, e_n$ over a field k and let E be the exterior algebra over V. Let $J \subseteq E$ be an ideal in the exterior algebra and let I be the corresponding squarefree monomial ideal in the polynomial ring. Then:

1) I is completely lexsegment in R if, and only if, J is completely lexsegment in E.

2) I has a linear free resolution over R if, and only if, the corresponding ideal J has a linear free resolution over E .

PROOF. – The assertion (1) follows from the definition of completely lexsegment ideals in the exterior algebra [4]. The assertion (2) is taken from Aramova, Avramov, Herzog (see Corollary 2.2 in [1]). □

Now we introduce some notation which we will use throughout this paper. For $w \in M_d$ we set

$$\begin{array}{ll} m(w) = \min\{i : i \in \operatorname{supp} w\}, & M(w) = \max\{i : i \in \operatorname{supp} w\} \\ w'' = w/x_{m(w)}, & w' = w/x_{M(w)}. \end{array}$$

Using 1.1 one can easily translate the results of Bonanzinga (Theorem 2.7, Theorem 3.10 in [4]), on squarefree lexsegment ideals in the exterior algebra, to the corresponding statements in the polynomial ring.

THEOREM 1.1. – *Let $u = x_{i_1} \cdots x_{i_d} \geq v = x_{j_1} \cdots x_{j_d}$ be two monomials in M_d and $I = (L(u, v))$. Let k be the smallest integer such that $(i_k, j_k) \neq (k, k)$. Then I is completely lexsegment if and only if one of the following conditions holds:*

(a) *$u = x_1 x_2 \cdots x_{k-1} x_{i_k} \cdots x_{i_d}$ with $x_{i_k} \cdots x_{i_d} \geq x_{k+2} x_{k+3} \cdots x_{d+2}$ and $v = x_1 x_2 \cdots x_{k-1} x_{n-d+k} \cdots x_n$;*

(b) *$i_k = k$ and for every $w < v$ there exists $i > k$ such that x_i divides w and $\dfrac{x_k w}{x_i} \leq u$.*

THEOREM 1.2. – *Let u, $v \in M_d$ be squarefree monomials with $u \geq v$ and $I = (L(u, v))$ be a squarefree completely lexsegment ideal. Let k be the smallest integer such that $(i_k, j_k) \neq (k, k)$. Then I has a linear resolution in the following cases:*

(a) *$u = x_1 \cdots x_{k-1} x_{i_k} \cdots x_{i_d}$ with $x_{i_k} \cdots x_{i_d} \geq x_{k+2} \cdots x_{d+2}$ and $v = x_1 \cdots x_{k-1} x_{n-d+k} x_{n-d+k+1} \cdots x_n$,*

(b) *$1 = i_1 \neq j_1$ and $x_1 w' \leq u$ for every $w < v$.*

2. – A characterization of squarefree completely lexsegment ideals with a linear resolution.

In this section we give a full description of squarefree completely lexsegment ideals with a linear resolution. In order to prove the main theorem we need the following Proposition which generalizes 1.2(b).

PROPOSITION 2.1.– *Let $u = x_{i_1} \cdots x_{i_d} \geq v = x_{j_1} \cdots x_{j_d}$ be two monomials in M_d and $I = (L(u, v))$. Let k be the smallest integer such that $(i_k, j_k) \neq (k, k)$. Suppose that $k = i_k \neq j_k$ and $x_k w' \leq u$ for the largest $w < v$. Then I has a linear resolution.*

PROOF. – We can suppose $k = 1$. Since I is completely lexsegment with $k \neq 1$ it follows that u and v are of the form $u = x_1 \cdots x_{k-1} x_{i_k} \cdots x_{i_d}$, $v = x_1 \cdots x_{k-1} x_{j_k} \cdots x_{j_d}$. In this case I is isomorphic to the ideal $\tilde{I}$ generated by $L(x_{i_k-(k-1)} \cdots x_{i_d-(k-1)}, x_{j_k-(k-1)} \cdots x_{j_d-(k-1)})$ in $k[x_{i_k-(k-1)}, \dots, x_{n-(k-1)}]$. Obviously I has a linear resolution if, and only if, $\tilde{I}$ has a linear resolution. Moreover I is completely lexsegment if, and only if, $\tilde{I}$ is completely lexsegment. Then the assertion follows from Theorem 1.2. □

THEOREM 2.1. – *Let u, $v \in M_d$ be squarefree monomials with $u \geq v$, and $I = (L(u, v))$ be a squarefree completely lexsegment ideal. Let k be the smallest integer such that $(i_k, j_k) \neq (k, k)$ and $B = \{w \in M_d : w < v, x_k w' > u\}$. Then I has a linear resolution if, and only if, one of the following conditions holds:*

(a) *$u = x_1 \cdots x_{k-1} x_{i_k} \cdots x_{i_d}$ with $x_{i_k} \cdots x_{i_d} \geq x_{k+2} \cdots x_{d+2}$ and $v = x_1 \cdots x_{k-1} x_{n-d+k} x_{n-d+k+1} \cdots x_n$,*

(b) *$i_k = k$ and the following condition holds: for every $(w_1, w_2) \in B \times B$ with $w_1 \neq w_2$ there exists an index l, $m(w_1) \leq l < M(w_2)$ such that $x_k w_1'' \leq u$, $\dfrac{x_k w_2}{x_l} \leq u$ and $w_1'' \neq \dfrac{w_2}{x_l}$.*

In order to prove Theorem 2.1 we need some preliminary results. We set: $J = (L^i(v)), K = (L^f(u)), L = J + K$. Then one has $I \subset J \cap K$ but in general this is a proper inclusion.

EXAMPLE 2.1. – *Let $u = x_1 x_3 x_6$, $v = x_2 x_3 x_4$ in $k[x_1, \dots, x_6]$ and $I = (L(u, v))$. Then $L(u, v)$ consists of the monomials $L(u, v) = \{u_1 = u = x_1 x_3 x_6, u_2 = x_1 x_4 x_5, u_3 = x_1 x_4 x_6, u_4 = x_1 x_5 x_6, u_5 = x_2 x_3 x_4 = v\}$. The inclusion $I \subset J \cap K$ is proper, because $x_1 x_2 x_3 x_5 \notin I$, but $x_1 x_2 x_3 x_5 \in J \cap K$. Indeed, I is not completely lexsegment, since for $w = x_2 x_3 x_5 < v = x_2 x_3 x_4$ condition (b) of Theorem 1.1 does not hold.*

Since L is the ideal spanned by all squarefree monomials of degree d, it is stable. We note that K is stable in the sense of Eliahou and Kervaire with respect to the lexicographic order with $x_n > x_{n-1} > \ldots > x_1$. Since initial squarefree lexsegment ideals are stable, it follows that J is stable, too. In particular, it follows from the Aramova-Herzog-Hibi resolution [3] that the ideals J, K and L have linear resolutions. In other words, we have

$$\operatorname{Tor}_i^R(k, R/J)_j = 0, \text{ for } i > 0 \text{ and } \quad j \neq i + d - 1$$

and similarly for K and L. We note that for an arbitrary graded ideal D the group $\operatorname{Tor}_i^R(k, R/D)$ is isomorphic to the Koszul homology group $H_i(x, R/D)$ where x is the sequence $x_1, ..., x_n$. We denote this homology group simply by $T_i(D)$. In [3], Aramova, Herzog and Hibi determined cycles whose homology classes form a basis of $T_i(D)$ where D is a squarefree stable ideal. We consider the Koszul algebra as the exterior algebra over the $k-$vector space with basis $e_1, \ldots, e_n$. Then, the $i-$th component of the Koszul complex has a basis $e_\sigma = e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_i}$ where $\sigma = \{j_1, \ldots, j_i\} \subset \{1, \ldots, n\}$, with $j_1 < \ldots < j_i$. Now for $J = (L^i(v))$ the cycles whose homology classes form a basis of $T_i(J)$ are

$$w' e_\sigma \wedge e_{M(w)}, \;\; w \in L^i(v), \;\; |\, \sigma \,| = i - 1, \quad \max(\sigma) < M(w), \quad \sigma \bigcap \mathrm{supp}(w) = \emptyset.$$

The homology classes of the same cycles, but with $w \in M_d$ arbitrary, form a basis of $T_i(L)$. We call this kind of cycles classical cycles.

Analogously, the cycles whose homology classes form a basis of $T_i(K)$ are the following:

$$w'' e_{m(w)} \wedge e_\sigma, \quad w \in L^f(u), \quad |\, \sigma \,| = i - 1, \quad m(w) < min(\sigma), \quad \sigma \cap \mathrm{supp}(w) = \emptyset$$

which we call non-classical cycles. The exact sequence of $R-$modules

$$0 \longrightarrow R/J \cap K \longrightarrow R/J \oplus R/K \longrightarrow R/L \longrightarrow 0$$

yields the long exact sequence

$$(1) \qquad \ldots \longrightarrow T_{i+1}(L) \longrightarrow T_i(J \cap K) \longrightarrow T_i(J) \oplus T_i(K) \xrightarrow{\varphi} T_i(L) \longrightarrow \ldots.$$

Provided $I = J \cap K$, this sequence implies (since J, K and L have linear resolutions) that I has a linear resolution if and only if the maps φ are surjective for all $i > 1$.

Note that φ maps $[w' e_\sigma \wedge e_{M(w)}]$ onto itself in $T_i(L)$. Therefore we have to describe the induced map

$$\varphi' : T_i(K) \longrightarrow T_i(L)/T_i(J)$$

where $ker(\varphi) = ker(\varphi')$, and where a basis of $T_i(L)/T_i(J)$ is given by $W = \{[w' e_\sigma \wedge e_{M(w)}], w \in M_d, w < v\}$. On the other hand a basis of $T_i(K)$ consists of the homology classes of the non-classical cycles. Therefore to understand the image of $T_i(K)$ in $T_i(L)$ we write a non-classical cycle as a linear combination of classical cycles in $T_i(L)$.The next result can be easily proved by following the arguments of [2, Lemma 1.2].

LEMMA 2.1. – *Let* $[z'' e_m(z) \wedge e_\tau]$ *be a basis element of* $T_i(K)$. *If* $\max(\tau) > M(z)$ *one has*

$$\varphi([z'' e_{m(z)} \wedge e_\tau]) = [(x_{\max(\tau)} z'')' e_{m(z)} \wedge e_{\tau \setminus \max(\tau)} \wedge e_{M(x_{\max(\tau)} z'')}]$$

and if $\max(\tau) < M(z)$, *then*

$$\varphi([z'' e_{m(z)} \wedge e_\tau]) = (-1)^{i-1} [z' e_\tau \wedge e_{M(z)}] + \sum_{t \in \tau} \pm [(x_t z'')' e_{m(z)} \wedge e_{\tau \setminus t} \wedge e_{M(x_t z'')}].$$

After these preparations we can prove Theorem 2.1.

PROOF. – [Proof of 2.2] In case (a) it follows from Theorem 1.2 that I has a linear resolution. In case (b) we may assume $k = 1$. If $|B| = 0$ it follows from Proposition 2.1 that I has a linear resolution. If $|B| = 1$, in order to prove that I has a linear resolution, we show that the maps

$$\varphi : T_i(J) \oplus T_i(K) \to T_i(L)$$

are surjective for every $i \geq 3$. We have already noted that the homology classes of the cycles $w'e_\sigma \wedge e_{M(w)}$ with $w \geq v$ are in the image of φ. One can easily show that the homology classes of the cycles $w'e_\sigma \wedge e_{M(w)}$, with $w < v$ and $m(\sigma) = 1$ are in Imφ, arguing as in the proof of Theorem 1.3 in [2]. Let U be the $k-$ subvector space of $T_i(L)$ generated by the homology classes of the following classical cycles: $w'e_\sigma \wedge e_{M(w)}$ for every $w \in M^s_d$ where $min(\sigma) \geq 2$, and $w'e_1 \wedge e_\rho \wedge e_{M(w)}$ where $w \geq v$. Then $U \subset \mathrm{Im}\,\varphi$, and φ is surjective if, and only if, the induced map

$$\overline{\varphi} : T_i(K) \to T_i(L)/U$$

is surjective. Moreover, the homology classes of $w'e_1 \wedge e_\rho \wedge e_{M(w)}$, $w < v$ form a basis of $T_i(L)/U$. For every $w < v$ with $x_1w' \leq u$ we have that $w'e_1 \wedge e_\rho \wedge e_{M(w)} = z''e_{m(z)} \wedge e_\tau$ is a non-classical cycle, with $m(z) = 1$, $\max(\tau) = \max(w)$ and $z = x_1w'$. So it remains to show that for the only $w < v$ with $x_1w' > u$ the cycles $w'e_1 \wedge e_\rho \wedge e_{M(w)}$ are in $Im\overline{\varphi}$. Since I is completely lexsegment then, from Theorem 1.1, there exists an index $l \in \mathrm{supp}(w')$ such that $\dfrac{x_1w}{x_l} \leq u$. Let $z = \dfrac{x_1w}{x_l}$. In this case $M(z) = M(w)$. We can consider the cycle $[z''e_1 \wedge e_\tau]$ with $\tau = \rho \cup l$, $\max(\tau) < M(z)$. For a non-classical cycle $[z''e_{m(z)} \wedge e_\tau]$ set $T = \{t \in \tau : x_tz'' < v\}$. From Lemma 2.1 we obtain

$$\overline{\varphi}([z''e_1 \wedge e_\tau]) = \sum_{t\in T} \pm [(x_tz'')'e_1 \wedge e_{\tau\setminus t} \wedge e_{M(x_tz'')}]$$

since $z'e_\tau \wedge e_{M(z)} \in U$. If $T = \{t_1, \dots, t_r\}$ we can write:

$$\overline{\varphi}([z''e_1 \wedge e_\tau]) = \sum_{i=1}^{r} \pm [w_i'e_1 \wedge e_{\tau\setminus t_i} \wedge e_{M(w_i)}] \tag{2}$$

with $x_{t_i}z'' = w_i < v$, $i = 1, \dots, r$. Since $|B| = 1$ there exists an index i such that $w_i = w$, $t_i = l$ and for all $w_j < v$ $j \neq i$, $x_1w_j' \leq u$. Since $z_j = x_1w_j' \leq u$ then

$$w_j'e_1 \wedge e_{\tau\setminus t_j} \wedge e_{M(w_j)} = z_j''e_1 \wedge e_{\tau_j}, \quad \forall j \neq i. \tag{3}$$

So $\forall w_j < v$, and $j \neq i$, the cycles $[w'e_1 \wedge e_\rho \wedge e_{M(w_j)}]$ are in $\mathrm{Im}\,\overline{\varphi}$. It remains to prove that the cycles $[w_i'e_1 \wedge e_\rho \wedge e_{M(w)}]$ are in Im$\overline{\varphi}$. From (2) and (3) we obtain

$$[w_i'e_1 \wedge e_{\tau\setminus t_i} \wedge e_{M(w_i')}] = \overline{\varphi}([z''e_1 \wedge e_\tau]) - \sum_j \pm [z_j''e_1 \wedge e_{\tau_j}].$$

So the cycles $[w_i'e_1 \wedge e_\rho \wedge e_{M(w_i)}]$ can be expressed as a linear combination of non-classical cycles, thus they are in $\operatorname{Im}\overline{\varphi}$. Now we prove that in case $|B|>1$ I has a linear resolution by induction on $|B|$.

If $|B|=2$ then there exist $w_1<v\,,w_2<v$ and an index l such that $m(w_1)\le l<M(w_2)$, with $x_1w_1'>u$, $x_1w_2'>u$, $x_1w_1''\le u$, $\dfrac{x_1w_2}{x_l}\le u$ and $w_1''\neq\dfrac{w_2}{x_l}$, and for all the remaining $w_j<v$, $x_1w_j'\le u$. Assume $z_1=x_1w_1''$, $z_2=x_1\dfrac{w_2}{x_l}$. Then, $M(z_1)=M(w_1)$ and $M(z_2)=M(w_2)$. We can consider the cycles

$$[z_1''e_1\wedge e_{\tau_1}],\quad \text{with } \tau_1=\rho_1\cup m(w_1),\ \max(\tau_1)<M(z_1)$$

$$[z_2''e_1\wedge e_{\tau_2}],\quad \text{with } \tau_2=\rho_2\cup l,\ \max(\tau_2)<M(z_2).$$

Set $T_k=\{t\in\tau_k : x_tz_k''<v\}$, with $k=1,2$. As in case $|B|\le 1$ we have

$$\overline{\varphi}([z_k''e_1\wedge e_{\tau_k}])=\sum_j \pm[w_j'e_1\wedge e_{\tau_k\setminus t_j}\wedge e_{M(w_j)}]\ \text{ with } w_j=x_{t_j}z_k''<v, t_j\in T_k.$$

We distinguish four cases.

Case a) $x_tz_1''\neq x_lz_2''=w_2$, $\forall t\in T_1$. In this case we have:

$$\overline{\varphi}([z_1''e_1\wedge e_{\tau_1}])=\sum_{j\neq 1,2}\pm[w_j'e_1\wedge e_{\tau_1\setminus t_j}\wedge e_{M(w_j)}]\pm[w_1'e_1\wedge e_{\tau_1\setminus m(w_1)}\wedge e_{M(w_1)}]. \tag{4}$$

Therefore, $[w_1'e_1\wedge e_{\tau_1\setminus m(w_1)}\wedge e_{M(w_1)}]$ is a linear combination of non-classical cycles.

Case b) $x_tz_1''=x_lz_2''=w_2$, for some $t\in T_1$. In this case $M(z_1)=M(w_1)=M(w_2)$ and we have:

$$\overline{\varphi}([z_1''e_1\wedge e_{\tau_1}])=\sum_{j\neq 1,2}\pm[w_j'e_1\wedge e_{\tau_1\setminus t_j}\wedge e_{M(w_j)}]\pm[w_1'e_1\wedge e_{\tau_1\setminus m(w_1)}\wedge e_{M(w_1)}]\pm[w_2'e_1\wedge e_{\tau_1\setminus t}\wedge e_{M(w_1)}]. \tag{5}$$

Case a$'$) $x_sz_2''\neq x_{m(w_1)}z_1''=w_1$, $\forall s\in T_2$. In this case we have:

$$\overline{\varphi}([z_2''e_1\wedge e_{\tau_2}])=\sum_{j\neq 1,2}\pm[w_j'e_1\wedge e_{\tau_2\setminus s_j}\wedge e_{M(w_j)}]\pm[w_2'e_1\wedge e_{\tau_2\setminus l}\wedge e_{M(w_2)}]. \tag{6}$$

Therefore, $[w_2'e_1\wedge e_{\tau_2\setminus l}\wedge e_{M(w_2)}]$ is a linear combination of non-classical cycles.

Case b$'$) $x_sz_2''=x_{m(w_1)}z_1''=w_1$, for some $s\in T_2$. In this case $M(z_2)=M(z_1)=M(w_1)$ and we have:

$$\overline{\varphi}([z_2''e_1\wedge e_{\tau_2}])=\sum_{j\neq 1,2}\pm[w_j'e_1\wedge e_{\tau_2\setminus s_j}\wedge e_{M(w_j)}]\pm[w_2'e_1\wedge e_{\tau_2\setminus l}\wedge e_{M(w_2)}]\pm[w_1'e_1\wedge e_{\tau_2\setminus s}\wedge e_{M(w_2)}]. \tag{7}$$

Note that conditions b) and b') are not compatible. In fact suppose that

$$x_s z_2'' = x_{m(w_1)} z_1'', x_t z_1'' = x_l z_2'', \tag{8}$$

then, by substitution we obtain $\frac{x_t x_s}{x_l} = x_{m(w_1)}$. It follows that $t = m(w_1)$ and $s = l$ or $t = l$ and $s = m(w_1)$. If $t = m(w_1)$ and $s = l$, it follows from (8) that $w_1 = w_2$. But this is a contradiction. If $t = l$ and $s = m(w_1)$ it follows from (8) that $z_1'' = z_2''$. But by hypothesis we have $z_1'' \neq z_2''$. We want to show that the cycles $[w_1' e_1 \wedge e_{\rho_1} \wedge e_{M(w_1)}]$, $[w_2' e_1 \wedge e_{\rho_2} \wedge e_{M(w_2)}]$ are in $\operatorname{Im} \overline{\varphi}$. We have already noted that in case (a) the cycles $[w_1' e_1 \wedge e_{\rho_1} \wedge e_{M(w_1)}]$ are in $\operatorname{Im} \overline{\varphi}$. If condition b) is satisfied then also condition a') is satisfied, because b) and b') are not compatible. So, from equation (6) we obtain

$$\pm [w_2' e_1 \wedge e_{\tau_2 \setminus l} \wedge e_{M(w_2)}] = \overline{\varphi}([z_2'' e_1 \wedge e_{\tau_2}]) - \sum_{j \neq 1,2} \pm [w_j' e_1 \wedge e_{\tau_2 \setminus s_j} \wedge e_{M(w_j)}]. \tag{9}$$

By substituting (9) in (5) we obtain that $[w_1' e_1 \wedge e_{\rho_1} \wedge e_{M(w_1)}]$ is a linear combination of non-classical cycles; thus it is in $\operatorname{Im} \overline{\varphi}$. With similar arguments we obtain that the cycles $[w_2' e_1 \wedge e_{\rho_2} \wedge e_{M(w_2)}]$ are in $\operatorname{Im} \overline{\varphi}$. Then $\overline{\varphi}$ is surjective.

Now, let $B = \{w_1, \dots, w_n\}$, and for all $w_i, w_j \in B$, $w_i \neq w_j$, $x_1 w_i'' \leq u$, $\frac{x_1 w_j}{x_{l_j}} \leq u$, $w_i'' \neq \frac{w_j}{x_{l_j}}$. From induction hypothesis, the cycles $[w_i' e_1 \wedge e_{\rho_i} \wedge e_{M(w_i)}]$ are in $\operatorname{Im} \overline{\varphi}$, for every $i = 1, \dots, n-1$. Then, $[w_i' e_1 \wedge e_{\rho_i} \wedge e_{M(w_i)}] = \overline{\varphi}[z_i'' e_1 \wedge e_{\tau_i}]$, $i = 1, \dots, n-1$. We want to prove that the cycles $[w_n' e_1 \wedge e_{\rho_n} \wedge e_{M(w_n)}]$ are in $\operatorname{Im} \overline{\varphi}$. Let $z_n = \frac{x_1 w_n}{x_{l_n}} \leq u$. From hypothesis $z_n \neq z_j = x_1 w_j''$, $\forall j \neq n$. Let us consider the cycle $[z_n'' e_1 \wedge e_{\tau_n}]$, with $\max(\tau_n) < M(z_n)$. Set $T_n = \{t \in \tau_n : x_t z_n'' < v\}$. Then

$$\begin{aligned}\overline{\varphi}[z_n'' e_1 \wedge e_{\tau_n}] &= \sum_{t \in T_n} \pm (x_t z_n'')' e_1 \wedge e_{\tau_n \setminus t} \wedge e_{M(x_t z_n'')} \\ &= \sum \pm \overline{\varphi}([z_i'' e_1 \wedge e_{\tau_i}]) \pm [w_n' e_1 \wedge e_{\tau_n \setminus l_n} \wedge e_{M(w_n)}].\end{aligned}$$

Then $[w_n' e_1 \wedge e_{\tau_n \setminus l_n} \wedge e_{M(w_n)}]$ is a linear combination of non-classical cycles. Then it is in $\operatorname{Im} \overline{\varphi}$.

To conclude it remains to prove that if neither of conditions (a) and (b) holds, then I does not have a linear resolution. Let $i_1 = 1$ and assume that there exists $(w_1, w_2) \in B \times B$, $w_1 \neq w_2$ such that for all index $l \in \operatorname{supp}(w_2')$, $m(w_1) \leq l < M(w_2)$ with $x_k w_1'' \leq u$, $\frac{x_k w_2}{x_l} \leq u$ the condition $w_1'' = \frac{w_2}{x_l}$ holds. Then $l \neq m(w_1)$. In fact if $l = m(w_1)$ then $w_1 = w_2$. Set $\rho_1 = \{t \in c(w_1) : 1 < t < M(w_1)\}$, $\rho_2 = \{s \in c(w_2) : 1 < s < M(w_2)\}$. So $l \in \rho_1$ and $m(w_1) \in \rho_2$. The cycles $[w_1' e_1 \wedge e_{\rho_1} \wedge e_{M(w_1)}]$ and $[w_2' e_1 \wedge e_{\rho_2} \wedge e_{M(w_2)}]$ are two distinct basis elements in

$T_i(L)/U$ for some $i \geq 3$. For a non-classical cycle $[z'' e_{m(z)} \wedge e_\tau]$, set

$$B(z,\tau) = \begin{cases} \max(\tau), & \text{if } \max(\tau) > M(z)\ ; \\ \{t \in \tau : x_t z'' < v\}, & \text{if } \max(\tau) < M(z). \end{cases}$$

Then from 2.1

$$\overline{\varphi}([z'' e_{m(z)} \wedge e_\tau]) = \sum_{t \in B(z,\tau)} \pm [(x_t z'')' e_{m(z)} \wedge e_{\tau \setminus t} \wedge e_{M(z)}]$$

because $[z' e_\tau \wedge e_{M(z)}] \in U$. So, if $\overline{\varphi}([z'' e_m(z) \wedge e_\tau]) \neq 0$ then $m(z) = 1$. Since $\overline{\varphi}$ is surjective then $[w_1' e_1 \wedge e_{\rho_1} \wedge e_{M(w_1)}]$ and $[w_2' e_1 \wedge e_{\rho_2} \wedge e_{M(w_2)}]$ are in $\operatorname{Im} \overline{\varphi}$. Then there exists a non-classical cycle $[z_1'' e_1 \wedge e_{\tau_1}]$ such that $z_1 = x_1 w_1''$ and

$$(x_t z_1'')' e_1 \wedge e_{\tau_1 \setminus t} \wedge e_{M(x_t z_1'')} = w_1' e_1 \wedge e_{\rho_1} \wedge e_{M(w_1)}, \tag{10}$$

for some $t \in \tau_1$. Moreover, there exists a non-classical cycle $[z_2'' e_1 \wedge e_{\tau_2}]$ such that $z_2 = \dfrac{x_1 w_2}{x_l}$ and

$$(x_s z_2'')' e_1 \wedge e_{\tau_2 \setminus s} \wedge e_{M(x_s z_2'')} = w_2' e_1 \wedge e_{\rho_2} \wedge e_{M(w_2)}, \tag{11}$$

for some $t \in \tau_2$. From (10) we obtain $t = m(w_1)$, $\rho_1 = \tau_1 \setminus m(w_1)$, $M(x_{m(w_1)} z_1'') = M(w_1)$. From (11) we obtain $s = l$, $\rho_2 = \tau_2 \setminus l$, $M(x_l z_2'') = M(w_2)$. Since $\rho_1 \setminus l = \rho_2 \setminus m(w_1)$ then $\tau_1 = \tau_2$. So, $[z_1'' e_1 \wedge e_{\tau_1}] = [z_2'' e_1 \wedge e_{\tau_2}]$ and the cycles $[w_1' e_1 \wedge e_{\rho_1} \wedge e_{M(w_1)}]$, $[w_2' e_1 \wedge e_{\rho_2} \wedge e_{M(w_2)}]$ are the image of the same cycle, but this is a contradiction. $\square$

In what follows we consider some special classes of completely lexsegment ideals with linear resolution.

COROLLARY 2.1. – *Let $u = x_{i_1} \cdots x_{i_d} \geq v = x_{j_1} \cdots x_{j_d}$ be monomials in M_d, $I = (L(u,v))$. Let k be the smallest integer such that $(i_k, j_k) \neq (k,k)$. If $i_{k+1} = k+1$ then I has a linear resolution.*

PROOF. – According to the previous considerations we may assume $i_2 = 2$. Then we prove that if $u = x_1 x_2 x_{i_3} \ldots x_{i_d}$, $v = x_{j_1} \ldots x_{j_d}$, with $j_1 > 1$, I has a linear resolution. Let B be the set defined in Theorem 2.1. Let $w < v$, then $m(w) \geq m(v) \geq 2$. If $m(w) > 2$ then $x_1 w' < u$ and this implies $m \notin B$. Take two monomials $w_1, w_2 \in B$, $w_1 \neq w_2$, then $m(w_1) = m(w_2) = 2$. Since $w_1 < v, w_2 < v$ and I is completely lexsegment, it follows from 1.1 that there exists an index $l_1 > 1$ and an index $l_2 > 2$ such that l_1 divides w_1 and l_2 divides w_2, and $\dfrac{x_1 w_1}{x_{l_1}} \leq u$, $\dfrac{x_1 w_2}{x_{l_2}} \leq u$. We have: $x_1 w_1'' \leq \dfrac{x_1 w_1}{x_{l_1}} \leq u$, $x_1 w_2'' \leq \dfrac{x_1 w_2}{x_{l_2}} \leq u$ and $w_1'' \neq \dfrac{w_2}{m(w_2)}$. In fact, since $m(w_1) = m(w_2)$, if $w_1'' = \dfrac{w_2}{m(w_2)}$ then $w_1 = w_2$, which is a contradiction. Then the assertion follows from Theorem 2.1. $\square$

We also can give a direct proof of Corollary 2.1 not using Theorem 2.1. In order to do this we need the following technical lemmas. Note that I is not necessarily completely lexsegment.

LEMMA 2.2. – *Let* $u = x_1 \cdots x_{k-1} x_k x_{i_{k+1}} \cdots x_{i_d}$, $v = x_1 \cdots x_{k-1} x_{j_k} \cdots x_{j_d}$, $u \geq v$ *be monomials in* M_d. *Let* $f = x_1 \cdots x_{k-1}$, $w = u/f$, $z = v/f$, *with* $w, z \in M_{d-k+1}$ *and* $\tilde{J} = x_k(L^f(w''))$, $\tilde{K} = (L(x_{k+1} \cdots x_{d+1}, z))$. *Then, the ideal* $I = (L(u, v))$ *has a linear resolution if, and only if, the ideal generated by* $\mathrm{Shad}(L^f(w'')) \cap L^i(z)$ *in* $k[x_{k+1}, \ldots, x_n]$ *has a linear resolution and* $\tilde{J} \cap \tilde{K}$ *is generated in degree* $d - k + 2$.

PROOF. – The lemma follows by arguments similar to those of Lemma 2.2 in [2]. $\square$

LEMMA 2.3. – *Let* $u = x_1 \cdots x_{k-1} x_k x_{i_{k+1}} \cdots x_{i_d}$ *and* $v = x_1 \cdots x_{k-1} x_l x_{n-d+k+1} \cdots x_n$ *with* $i_{k+1} > l$ *and* $k + 1 \leq l < n - d + k$. *Then* I *has a linear resolution.*

PROOF. – The lemma follows by arguments similar to those of Lemma 2.3 in [2].

PROOF. – [Proof of 2.1] Let us consider the ideals $\tilde{J}$ and $\tilde{K}$ as in Lemma 2.2. We prove that if $i_2 = 2$ then

$$\tilde{J} \cap \tilde{K} = (L^f(x_1 v)). \tag{12}$$

Take a monomial $w \in \tilde{J} \cap \tilde{K}$. From $w \in \tilde{J} = x_1(L^f(u''))$ it follows that $w = x_1 pq$ with p monomial, $q \leq u''$. From $w \in \tilde{K} = (L(x_2 \cdots x_{d+1}, v))$ it follows that $w = rs$, with $x_2 \cdots x_{d+1} \geq s \geq v$, r monomial. Since $1 \notin \mathrm{supp}(s)$ then $1 \in \mathrm{supp}(r)$. It follows that $r = x_1 m$, with m monomial. So $w = x_1 ms$ with $x_1 s \geq x_1 v$. Then $w \in (L^i(x_1 v))$. So $\tilde{J} \cap \tilde{K} \subset x_1(L^f(v))$. Take a monomial $w \in (L^i(x_1 v))$, then $w = x_1 pq$, with p monomial, $q \geq v$. Since $1 \notin \mathrm{supp}(q)$ then $q \leq x_2 \cdots x_{d+1}$, then $w \in \tilde{K}$. Since $x_2 \ldots x_{d+1} \geq q \geq v$, then $\min(q) \geq 2$. Then $q'' \leq u''$. It follows that $q'' \in L^f(u'')$ and $w = x_1 p x_{m(q)} q'' \in x_1(L^f(u'')) = \tilde{J}$. So we obtain $\tilde{J} \cap \tilde{K} \supset (L^i(x_1 v))$. The ideal I has a linear resolution if and only if $\tilde{J} \cap \tilde{K}$ has a linear resolution and $(\tilde{J} \cap \tilde{K})_{d+1} \neq 0$. From equality (12) we obtain that $(\tilde{J} \cap \tilde{K})_{d+1} \neq 0$. Moreover $\tilde{J} \cap \tilde{K}$ has a linear resolution because the initial lexsegment ideal $(L^i(x_1 v))$ has a linear resolution. Hence we obtain the assertion. $\square$

As another special case of Theorem 2.1 we get

COROLLARY 2.2. – *Let* $u = x_{i_1}\cdots x_{i_d} \geq v = x_{j_1}\cdots x_{j_d}$ *be monomials in* M_d, $I = (L(u,v))$ *a completely lexsegment ideal. Let* k *be the smallest integer such that* $(i_k,j_k) \neq (k,k)$. *If* $i_{k+1} > k+1$ *and* $u = x_1\cdots x_{k-1}x_kx_{k+2}\ldots x_{d+1}$ *then* I *has a linear resolution.*

PROOF. – We may assume $k=1$ and $i_1 = 1$. We prove that if $u = x_1x_3x_4\ldots x_{d+1}$ and $v = x_{j_1}\ldots x_{j_d}$, $j_1 > 1$ then I has a linear resolution. Let $w < v$, then $m(w) \geq m(v) > 1$. Let B be the set defined in 2.1. If $m(w) \geq 3$ then $x_1w' \leq u$ and this implies that $w \notin B$. So, if $w < v$ and $w \in B$ then $m(w) = 2$. Let $w_1,w_2 \in B$, then from Theorem 1.1 $x_1w_1'' \leq u$ and $x_1w_2'' \leq u$. Since $w_1 \neq w_2$ and $m(w_1) = w_2$ then $w_1'' \neq w_2''$. Then the assertion follows from 2.1. □

We also can give a direct proof of Corollary 2.2 not using Theorem 2.1.

PROOF. – We may assume $k = 1$. Let us consider the ideals $\tilde{J}$ and $\tilde{K}$ as in Lemma 2.2. We prove that if $u = x_1x_3\cdots x_{d+1}$ then

$$\tilde{J} \cap \tilde{K} = (L^i(x_1v)).$$

Let $p \in \tilde{J}\cap\tilde{K}$ a monomial. Since $p \in \tilde{J}$, then $p = qr$, with q monomial, $r = x_1s$, $s \leq u''$ squarefree monomial, $1 \notin \mathrm{supp}(s)$, then $p = x_1qs$. Since $p \in \tilde{K}$, then $p = mt$ with m monomial, t squarefree monomial, $x_2\cdots x_{d+1} \geq t \geq v$. So we obtain $p = x_1qs = mt$. Since $1 \notin \mathrm{supp}(t)$, then $1 \in \mathrm{supp}(m)$. Therefore, $p = \frac{m}{x_1}x_1t$ with $x_1t \geq x_1v$. It follows that $p \in (L^i(x_1v))$. So $\tilde{J}\cap\tilde{K} \subset (L^i(x_1v))$. Conversely, let $p \in (L^i(x_1v))$, then $p = qr$ with q monomial, r squarefree monomial such that $r \geq x_1v$. It follows that $1 \in \mathrm{supp}(r)$ and $x_2\cdots x_{d+1} \geq \dfrac{r}{x_1} \geq v$. Then, $p \in \tilde{K}$. Moreover, since $1 \in \mathrm{supp}(r)$ it follows that $r = x_1t$ with t squarefree monomial, $t \geq v$. Therefore, $p = qx_1t$ with $1 \notin \mathrm{supp}(t)$. Since $1 \notin \mathrm{supp}(t)$ then $m(t) \geq 2$ and $m(t'') \geq 3$. Moreover, $u'' = x_3\cdots x_{d+1}$ and $p = x_1qx_{m(t)}t''$, $t'' \leq u''$. It follows that $p \in \tilde{J}$. So we obtain $(L^i(x_1v)) \subset \tilde{J}\cap\tilde{K}$. The equality $\tilde{J}\cap\tilde{K} = (L^i(x_1v))$ shows that $\tilde{J}\cap\tilde{K}$ is generated in degree $d+1$ and has a linear resolution and the assertion follows. □

In the following we consider another special class of completely lexsegment ideals with linear resolution for which we give a direct, simple proof.

PROPOSITION 2.2.– *Let* $u = x_{i_1}\cdots x_{i_d} \geq v = x_{j_1}\cdots x_{j_d}$ *be monomials in* M_d, $I = (L(u,v))$ *a completely lexsegment. Let* k *be the smallest integer such that* $(i_k,j_k) \neq (k,k)$. *If* $i_{k+1} > k+1$ *and* $v = x_1\cdots x_{k-1}x_lx_{n-d+k+1}\cdots x_n$ *with* $k+1 \leq l \leq n-d+k$ *then* I *has a linear resolution.*

PROOF. – We can suppose $k=1$. We prove that if $u = x_lx_{n-d+k+1}\cdots x_n$ then I has a linear resolution. We distinguish two cases:

Case 1. For every $w < v$ $x_1 w' \le u$. In this case, it follows from Proposition 2.1 that I has a linear resolution.

Case 2. There exists $w < v$ such that $x_1 w' > u$. It follows from $w < v$ that $m(w) = m > l$. Since $x_1 w' > u$ then $m \le i_{k+1}$. So we obtain $l < m \le i_{k+1}$ and the assertion follows from Lemma 2.3. $\square$

3. – Squarefree lexsegment ideals with linear resolution.

In the following we describe the procedure to determine whether or not a squarefree lexsegment ideal has a linear resolution. Let $I = (L(u, v))$ be a lexsegment ideal with $u = x_{i_1} \dots x_{i_d}$, $v = x_{j_1} \dots x_{i_d}$.

- If $u = v$ then I has a linear resolution. In the next steps we may therefore assume that $u > v$.
- If I is completely lexsegment, see Theorem 2.1.
- If I is not completely lexsegment, we let $f = x_1 \cdots x_{k-1}$, and let $\tilde{I}$ be the ideal generated by $L(u/f, v/f)$ in $k[x_{i_k}, \dots, x_n]$. Obviously I has a linear resolution in $k[x_1, \dots, x_n]$ if, and only if, $\tilde{I}$ has a linear resolution in $k[x_{i_k}, \dots, x_n]$.
- If $\tilde{I}$ is completely lexsegment, see Theorem 2.1, and if $\tilde{I}$ is not completely lexsegment, see Theorem 3.1 below.

It remains to prove 3.1; it characterizes all ideals with the property that $i_k = k$, which have a linear resolution and which are not completely lexsegment.

THEOREM 3.1. – *Let $u = x_{i_1} \cdots x_{i_d}$, $v = x_{j_1} \dots x_{j_d}$ be squarefree monomials in M_d with $i_k = k$. Suppose that the ideal I generated by $L(u, v)$ is not completely lexsegment, then I has a linear resolution if, and only if, u and v are of the form*

$$u = x_1 \cdots x_{k-1} x_k x_{i_{k+1}} \cdots x_{i_d}, v = x_1 \cdots x_{k-1} x_l x_{n-d+k+1} \cdots x_n$$

for some l, $k + 1 \le l < n - d + k$.

For the proof we need the following results:

LEMMA 3.1. – *Let $u, v \in M_d$, $u \ge v$ with $i_k = k$. If $i_{k+1} = k + 1$ then $I = (L(u, v))$ is completely lexsegment.*

PROOF. – We prove that if I is not completely lexsegment then $i_{k+1} > k + 1$. It follows from 1.1 that there exists a squarefree monomial $w < v$ such that, for all $i > k$, $i \in \operatorname{supp}(w)$, $\dfrac{w x_k}{x_i} > u$. Suppose that

$w = x_{r_1} \cdots x_{r_d}$ with $1 \leq r_1 < r_2 < \ldots < r_d \leq n$. The inequality $\frac{w}{x_i} > \frac{u}{x_k}$ implies $x_{r_1} \cdots x_{r_k} > x_1 \cdots x_{k-1} x_{i_{k+1}}$ and we obtain $r_1 = 1, r_2 = 2, \ldots, r_{k-1} = k-1$ and $r_k < i_{k+1}$. It follows from $w < v$ that $r_k \geq k+1$. Then $k+1 \leq r_k < i_{k+1}$ and the assertion follows. □

LEMMA 3.2. – *Let*

$$u = x_1 \cdots x_{k-1} x_k x_{i_{k+1}} \cdots x_{i_d}, v = x_1 \cdots x_{k-1} x_l x_{n-d+k+1} \cdots x_n$$

be monomials in M_d *with* $k+1 \leq l < n-d+k$. *Suppose that* $I = (L(u,v))$ *is not completely lexsegment. Then* $i_{k+1} > l$.

PROOF. – It follows from 1.1 that there exists a squarefree monomial $w < v$ such that, for all $i > k$, $i \in \mathrm{supp}(w)$, $\frac{w x_k}{x_i} > u$. Suppose that $w = x_{r_1} \cdots x_{r_d}$ with $1 \leq r_1 < r_2 < \ldots < r_d \leq n$. The inequality $\frac{w}{x_i} > \frac{u}{x_k}$ implies $x_{r_1} \cdots x_{r_k} > x_1 \cdots x_{k-1} x_{i_{k+1}}$ and we obtain $r_1 = 1, r_2 = 2, \ldots, r_{k-1} = k-1$ and $r_k < i_{k+1}$. It follows from $w < v$ that $r_k \geq l$. Then $l \leq r_k < i_{k+1}$ and the assertion follows. □

PROOF. – [Proof of 3.1] Suppose that u, v are of the form

$$u = x_1 \cdots x_{k-1} x_k x_{i_{k+1}} \cdots x_{i_d}, v = x_1 \cdots x_{k-1} x_l x_{n-d+k+1} \cdots x_n, \tag{13}$$

for some $k+1 \leq l < n-d+k$. It follows from Lemma 3.2 that $i_{k+1} = m(w'') > l$. Then from 2.3 it follows that I has a linear resolution. Conversely, we prove that if $i_k = k$ and I has a linear resolution (I is not completely lexsegment) then u and v are of the form (13). Since $i_k = k$, we have $u = x_1 \cdots x_{k-1} x_k x_{i_{k+1}} \cdots x_{i_d}$. Since I is not completely lexsegment v is of the form $v = x_1 \cdots x_{k-1} x_{j_k} x_{j_{k+1}} \cdots x_{j_d}$, with $j_k = m(z) = l < n-d+k$. To conclude, it is enough to prove that $z = x_l x_{n-d+k+1} \cdots x_n$. Obviously $l = j_k \geq k+1$. We distinguish two cases $l = k+1$ and $l > k+1$.

Case 1. $j_k = k+1$. In this case we will prove that $z = x_{k+1} x_{n-d+k+1} \cdots x_n$. We first show that in $k[x_{k+1}, \ldots, x_n]$ one has

$$\mathrm{Shad}(L^f(w'')) \cap L^i(z) = L(x_{k+1} w, z). \tag{14}$$

If $q \in L(x_{k+1} w'', z)$ then $q \geq z$. It follows that $q \in L^i(z)$. Moreover, since $z \leq q \leq x_{k+1} w''$ then $q = x_{k+1} r$, with $r \leq w''$. Then $q \in \mathrm{Shad}(L^f(w''))$. It follows that $q \in \mathrm{Shad}(L^f(w'')) \cap L^i(z)$. Conversely, if $q \in \mathrm{Shad}(L^f(w'')) \cap L^i_s(z)$, then $z \leq q \leq x_{k+1} x_{k+2} \cdots x_{d+1}$ and there exists $j \in \mathrm{supp}(q), j \notin \mathrm{supp}(w'')$, $k+1 \leq j \leq n$ such that $\frac{q}{x_j} \leq w''$. Then $z \leq q \leq x_j w'' \leq x_{k+1} w''$. It follows that $q \in L(x_{k+1} w'', z)$.

We proved in 2.2 that $(\tilde{J} \cap \tilde{K})_{d-k+2}$ is spanned by $x_k(\mathrm{Shad}(L^f(w'')) \cap L^i(z)) = x_k L(x_{k+1}w'', z)$. Then, since I has a linear resolution it follows that $L(x_{k+1}w'', z) \neq \emptyset$, that is $z \leq x_{k+1}w''$. Now we will show that if $z \neq x_{k+1}x_{n-d+k+1}\cdots x_n$ then $x_k x_{k+1} x_{k+2} x_{n-d+k+1}\cdots x_n$ is a generator of degree $d-k+3$ of $\tilde{J} \cap \tilde{K}$ and from 2.2 this is a contradiction. Since $z \neq x_{k+1}x_{n-d+k+1}\cdots x_n$ then $z > x_{k+1}x_{n-d+k+1}\cdots x_n$. From $x_{n-d+k+1}\cdots x_n < z'' \leq w''$ it follows that $x_{n-d+k+1}\cdots x_n \in L^f(w'')$. Then, $x_{k+1}x_{k+2}x_{n-d+k+1}\cdots x_n \in (L^f(w''))$ and we obtain $x_k x_{k+1} x_{k+2} x_{n-d+k+1}\cdots x_n \in \tilde{J} = x_k(L^f(w''))$. Since $z \leq x_{k+1}w''$ one has $m(z'') \geq m(w'') \geq k+2$ (Lemma 3.1). In order to obtain $x_k x_{k+1} x_{k+2} x_{n-d+k+1}\cdots x_n \in \tilde{K}$ we prove that from $z \leq x_{k+1}w''$, and I not completely lexsegment, it follows that $m(z'') \geq m(w'') > k+2$. Let us suppose that $m(z'') = k+2$, then $m(w'') = k+2$. But I not completely lexsegment implies that there exists a monomial $b < v$ such that $\dfrac{b}{x_i} > \dfrac{u}{x_k}$ for all $i > k$. Suppose $b = x_1\cdots x_{k-1}x_{r_k}\cdots x_{r_d}$, with $r_k < r_{k+1} < \ldots < r_d$. Then $\dfrac{b}{x_i} = \dfrac{x_1\cdots x_{k-1}x_{r_k}\cdots x_{r_d}}{x_i} > x_1\cdots x_{k-1}x_{k+2}\cdots x_{i_d}$ and we obtain $m\left(\dfrac{b}{f}\right) < k+2$. Moreover $b < v$ implies $\dfrac{b}{f} < \dfrac{v}{f}$ and then $m\left(\dfrac{b}{f}\right) \geq m\left(\dfrac{v}{f}\right) = k+1$. It follows that $k+1 \leq m\left(\dfrac{b}{f}\right) < k+2$. Then $m\left(\dfrac{b}{f}\right) = k+1$. Since $\dfrac{b}{f}\cdot\dfrac{1}{x_{k+1}} \leq \dfrac{u}{f}\cdot\dfrac{1}{x_k}$, then $\dfrac{b}{x_{k+1}} \leq \dfrac{u}{x_k}$. So there exists an index $i = k+1 > k$, $i \in \mathrm{supp}(b)$ such that $\dfrac{x_k b}{x_i} \leq u$. Then from condition (b) of 1.1 we obtain that I is completely lexsegment and this is a contradiction. Then $m(z'') \geq m(w'') > k+2$. It follows that $x_{k+1}\cdots x_{d+1} \geq x_k x_{k+1} x_{k+2} x_{n-d+k1}\cdots x_n > z$. Then $x_k x_{k+1} x_{k+2} x_{n-d+k+1}\cdots x_n \in \tilde{K}$. So we obtain $x_k x_{k+1} x_{k+2} x_{n-d+k+1}\cdots x_n \in (\tilde{J} \cap \tilde{K})_{d-k+3}$. Now we prove that $x_k x_{k+1} x_{k+2} x_{n-d+k+1}\cdots x_n$ is a generator of degree $d-k+3$. Suppose that $x_{k+1}x_{k+2}x_{n-d+k+1}\cdots x_n \in \mathrm{Shad}(L(x_{k+1}w'', z))$, then $x_{k+1}x_{k+2}x_{n-d+k+1}\cdots x_n = x_t y$, with $y \in L(x_{k+1}w'', z)$, $t \notin \mathrm{supp}(y)$, $k+2 \leq t \leq n$. If $t = k+2$ we obtain $y = x_{k+1}x_{n-d+k+1}\ldots x_n < z$ (because from hypothesis $z \neq x_{k+1}x_{n-d+k+1}\cdots x_n$). If $t > k+2$ then $y = \dfrac{x_{k+1}x_{k+2}x_{n-d+k+1}\cdots x_n}{x_t} > x_{k+1}w''$ (it follows from $m(z'') \geq m(w'') > k+2$). So we obtain a contradiction. Hence $x_k x_{k+1} x_{k+2} x_{n-d+k+1}\cdots x_n$ is a generator of $\tilde{J} \cap \tilde{K}$ of degree $d-k+3$.

Case 2. $l = j_k = m(z) > k+1$, $m = m(w'')$. It follows from I not completely lexsegment (Lemma 3.1) that $m(w'') > m(z) \geq k+2$. Then $m > l \geq k+2$. Let Q be the ideal generated by $L(w,z)$ in $k[x_k, x_l, x_{l+1}, \ldots, x_n]$. Then, from I not completely lexsegment it follows that Q is not completely lexsegment. In fact, suppose that I is not completely lexsegment, then there exists a monomial $b < v$ such that $\dfrac{b}{x_i} > \dfrac{u}{x_k}$, for all $i > k$. It follows that $b = x_1\cdots x_{k-1}x_{r_k}\cdots x_{r_d}$. Take $q = \dfrac{b}{f}$, we obtain $q < z$ and $\dfrac{q}{x_i} > \dfrac{w}{x_k}$, for all $i > k$. Since $\mathrm{supp}(q) \subset \{l, \ldots, n\}$, then

$\operatorname{supp}(q) \subset \{l, \dots, n\}$, for all $q < z$. Hence there exists a monomial $q \in k[x_k, x_l, \dots, x_n]$ such that $\frac{q}{x_i} > \frac{w}{x_k}$, for all $i > k$. Then Q is not completely lexsegment. Now, arguing as in Case 1 we obtain that if Q has a linear resolution then u and v are of the form (13). We prove that $z = x_l x_{n-d+k+1} \cdots x_n$. We first show that in $k[x_l, x_{l+1}, \dots, x_n]$ one has

$$\operatorname{Shad}(L^f(w'')) \cap L^i(z) = L(x_l w, z). \tag{15}$$

If $q \in L(x_l w'', z)$ then $q \geq z$. It follows that $q \in L^i_s(z)$. Moreover, since $z \leq q \leq x_l w''$ then $q = x_l r$, with $r \leq w''$, and $q \in \operatorname{Shad}(L^f(w''))$. It follows that $q \in \operatorname{Shad}(L^f(w'')) \cap L^i(z)$. Conversely, if $q \in \operatorname{Shad}(L^f(w'')) \cap L^i(z)$, then $z \leq q \leq x_l x_{l+1} \cdots x_{d+1}$ and there exists $j \in \operatorname{supp}(q)$, $j \notin \operatorname{supp}(w'')$, $l \leq j \leq n$ such that $\frac{q}{x_j} \leq w''$. Then $z \leq q \leq x_j w'' \leq x_l w''$. It follows that $q \in L(x_l w'', z)$. We proved in 2.2 that $(\tilde{J} \cap \tilde{K})_{d-k+2}$ is spanned by $x_k(\operatorname{Shad}(L^f(w'')) \cap L^i(z)) = x_k L(x_l w'', z)$. Then, since I has a linear resolution it follows that $L(x_l w'', z) \neq \emptyset$, that is $z \leq x_l w''$. Now we will show that if $z \neq x_l x_{n-d+k+1} \dots x_n$ then $x_k x_l x_{l+1} x_{n-d+k+1} \cdots x_n$ is a generator of degree $d-k+3$ of $\tilde{J} \cap \tilde{K}$ and from 2.2 this is a contradiction. Since $z \neq x_l x_{n-d+k+1} \cdots x_n$ then $z > x_l x_{n-d+k+1} \cdots x_n$. From $x_{n-d+k+1} \cdots x_n < z'' \leq w''$ it follows that $x_{n-d+k+1} \cdots x_n \in L^f(w'')$. Then $x_l x_{l+1} x_{n-d+k+1} \dots x_n \in (L^f(w''))$ and we obtain $x_k x_l x_{l+1} x_{n-d+k+1} \cdots x_n \in \tilde{J} = x_k(L^f(w''))$. Since $z \leq x_l w''$ one has $m(z'') \geq m(w'') \geq l+1$ (Lemma 3.1).

In order to obtain $x_k x_l x_{l+1} x_{n-d+k+1} \cdots x_n \in \tilde{K}$ we prove that from $z \leq x_l w''$ and I not completely lexsegment it follows that $m(z'') \geq m(w'') > l+1$. In fact, suppose that $m(z'') = l+1$, then $m(w'') = l+1$. But I not completely lexsegment implies that there exists a monomial $b < v$ such that $\frac{b}{x_i} > \frac{u}{x_k}$ for all $i > k$. Suppose $b = x_1 \cdots x_{k-1} x_{r_k} \cdots x_{r_d}$, with $r_k < r_{k+1} < \ldots < r_d$. Then $\frac{b}{x_i} = \frac{x_1 \cdots x_{k-1} x_{r_k} \cdots x_{r_d}}{x_i} > x_1 \cdots x_{k-1} x_{l+1} x_{i_{k+2}} \cdots x_{i_d}$ and we obtain $m\left(\frac{b}{f}\right) < l+1$. Moreover, $b < v$ implies $\frac{b}{f} < \frac{v}{f}$ and then $m\left(\frac{b}{f}\right) \geq m\left(\frac{v}{f}\right) = l$. It follows that $l \leq m\left(\frac{b}{f}\right) < l+1$. Then $m\left(\frac{b}{f}\right) = l$. Since $\frac{b}{f} \cdot \frac{1}{x_l} \leq \frac{u}{f} \cdot \frac{1}{x_k}$, then $\frac{b}{x_l} \leq \frac{u}{x_k}$. So, we obtain that there exists an index $i = l > k$, $i \in \operatorname{supp}(b)$ such that $\frac{x_k b}{x_i} \leq u$. Then, from condition (b) of 1.1 we obtain that I is completely lexsegment and this is a contradiction. Therefore, $m(z'') \geq m(w'') > l+1$. It follows that $x_l x_{l+1} \cdots x_{l+d-k} \geq x_k x_{k+1} x_{k+2} x_{n-d+k+1} \cdots x_n > z$. Then $x_k x_l x_{l+1} x_{n-d+k+1} \cdots x_n \in \tilde{K}$. So we obtain $x_k x_l x_{l+1} x_{n-d+k+1} \cdots x_n \in (\tilde{J} \cap \tilde{K})_{d-k+3}$. Now we prove that $x_k x_l x_{l+1} x_{n-d+k+1} \cdots x_n$ is a generator of degree $d-k+3$. Suppose that $x_l x_{l+1} x_{n-d+k+1} \cdots x_n \in \operatorname{Shad}(L(x_l w'', z))$. Then $x_l x_{l+1} x_{n-d+k+1} \cdots x_n = x_t y$, with

$y \in L(x_l w'', z)$, $t \notin \mathrm{supp}(y)$, $l+1 \leq t \leq n$. If $t = l+1$ we obtain $y = x_l x_{n-d+k+1} \cdots x_n < z$ (because, by hypothesis, $z \neq x_l x_{n-d+k+1} \cdots x_n$). If $t > l+1$ then $y = \dfrac{x_l x_{l+1} x_{n-d+k+1} \cdots x_n}{x_t} > x_l w''$ (it follows from $m(z'') \geq m(w'') > l+1$). So we obtain a contradiction. Hence $x_k x_l x_{l+1} x_{n-d+k+1} \cdots x_n$ is a generator of $\tilde{J} \cap \tilde{K}$ of degree $d-k+3$. Now we claim that if I has a linear resolution, then Q has a linear resolution, and this concludes the proof. In order to prove the claim we introduce the following lexsegments and ideals. We let $\tilde{Q}$ be the extension ideal of Q in $k[x_k, \ldots, x_n]$, P be the ideal generated by $L(x_{k+1} \ldots x_{d+1}, x_{l-1} x_{n-d+k+1} \cdots x_n)$ in $k[x_{k+1}, \ldots, x_n]$ and $\tilde{P}$ the extension ideal of P in $k[x_k, \ldots, x_n]$. Obviously it suffices to show that if I has linear resolution, then $\tilde{Q}$ has a linear resolution. We notice that $\tilde{I} = \tilde{P} + \tilde{Q}$. This yields a long exact sequence of $k-$vector spaces, for all $a \in Z^n$

$$\cdots \rightarrow T_i(\tilde{P} \cap \tilde{Q})_a \rightarrow T_i(\tilde{P})_a \oplus T_i(\tilde{Q})_a \rightarrow T_i(\tilde{I})_a \rightarrow \cdots$$

where the lower index a denotes the $a-$th graded component of the corresponding vector space. Suppose now that $\tilde{Q}$ does not have a linear resolution. Then there exists an integer i and an element $a \in Z^n$, $a = (a_1, \ldots, a_n)$, such that $T_i(\tilde{Q})_c = 0$ and $\sum_{j=1}^{n} a_j > d-k-i+2$. Since the generators of $\tilde{Q}$ are in $k[x_k, x_l, x_{l+1}, \ldots, x_n]$ one has that $T_i(\tilde{Q})_c = 0$ if $c_j \neq 0$ for some j, $k+1 \leq j \leq l-1$. It follows that $a_j = 0$ for $j = k+1, \ldots, l-1$. In particular, the $a-$th graded component of these T_i vanishes. This implies that $T_i(\tilde{I})_a \cong T_i(\tilde{Q})_a \neq 0$, contradicting the fact that $\tilde{I}$ has a linear resolution. □

In the proof of the previous theorems the characteristic of the base field is arbitrary. Thus we can give the following

COROLLARY 3.1. – *The linearity of the resolution of a squarefree lexsegment ideal does not depend on the characteristic of the base field, k.*

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