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Sunto. – *I morfismi su una curva possono essere visti come classi di omologia nel prodotto della curva con se stessa. In questo lavoro descriviamo queste classi come elementi di una intersezione: il luogo dei punti interi di un insieme algebrico nello spazio di omologia complesso, e il luogo delle classi di divisori effettivi. Scriviamo equazioni esplicite per l'insieme algebrico, e nel caso di genere tre calcoliamo alcune soluzioni esplicite sui razionali.*

Summary. – *Morphisms on a curve may be seen as homology classes in the self product. We describe these classes as belonging to an intersection: the locus of integral points of an algebraic set in the complex homology group, and the locus of effective divisor classes. We write down explicit equations for the algebraic set, and in the case of genus three we compute a few explicit solutions over the rationals.*

1. – Introduction.

Let X be a curve of genus $g \geq 2$. We mean a nonsingular complex projective curve. Define $\mathcal{F}(X)$ to be the collection of equivalence classes of morphisms $f : X \to Y$ onto curves Y of genus $g' \geq 2$, up to isomorphisms $Y \to Y'$. This is a finite set, according to the theorem of De Franchis-Severi.

Associated to a morphism $f : X \to Y$ is an effective divisor Z_f in $X \times X$, the (locally principal) divisor defined by $f(x) = f(y)$. It is a reduced divisor. It only depends on the equivalence class $[f]$ in $\mathcal{F}(X)$, and the correspondence $[f] \mapsto Z_f$ is injective.

The main result about this is the rigidity theorem of Kani [1] saying that, taking the homology class of the divisor, the map

$$\mathcal{F}(X) \longrightarrow H_2(X \times X, \mathbb{Z}) \qquad [f] \mapsto [Z_f]$$

is still injective. The aim here is to describe the image of this map.

Define $\mathcal{F}_n(X)$ to be the collection of equivalence classes of morphisms of degree n. We describe its image as a subset of an intersection: the locus of integral points of an algebraic set V_n in $H_2(X \times X, \mathbb{C})$, and the locus of effective homology classes in the product. Here $V_n(\mathbb{Z})$ is a finite set. We write down explicit algebraic equations for V_n, in a remarkably simple form. In the case of genus $g = 3$, we present some first computations which lead to a few explicit solutions in $V_n(\mathbb{Q})$.

We mention that different but related approaches to the subject have been developed, by Martens [2], in terms of endomorphisms of the Jacobian $J(X)$ arising from correspondences, and recently by Tanabe [5], in terms of sublattices of $H^1(X, \mathbb{Z})$ arising via pullbacks, and by Naranjo and Pirola [3], for higher dimensions.

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2. – Basic facts: morphisms as divisors.

The basic properties of the associated divisor express the concept of an algebraic equivalence relation. One has $Z_f \geq \Delta$, in other words $Z_f = \Delta + S_f$ with S_f effective. Moreover Z_f is symmetric, under the involution $(x, y) \mapsto (y, x)$, and satisfies the fundamental equation in terms of the composition of correspondences:

$$Z \circ Z = nZ$$

if $Z = Z_f$ and $n = \deg f$.

REMARK 2.1. – Conversely, if Z is a symmetric effective divisor and $Z \circ Z = nZ$ then $Z_{red} = Z_f$ for some f with $n \geq \deg f$. Examples: $n\Delta$, multiple of the diagonal, and more generally pZ_f if $n = p \deg f$. However, if Z is reduced then $Z = Z_f$.

QUESTION: is there any other non-reduced solution?

The intersection numbers of the associated divisor encode the global invariants of the morphism. Assume that f has degree n and ramification r, and that Y has genus g'. Consider in $X \times X$ the two fibres $X \times p$ and $p \times X$, and the diagonal Δ. We will denote the associated homology classes by ξ_1 and ξ_2, and by Δ again. Then

$$\begin{aligned} S \cdot \xi_1 &= S \cdot \xi_2 = n - 1 & Z \cdot \xi_1 &= Z \cdot \xi_2 = n \\ S \cdot \Delta &= r & Z \cdot \Delta &= 2 - 2g + r = n(2 - 2g') \\ S \cdot S &= (n - 1)(2 - 2g) + (n - 2)r & Z \cdot Z &= n(2 - 2g + r) \end{aligned}$$

if Z and S are associated to f as defined above. A reference for the present section is Samuel [4].

3. – Morphisms as homology classes.

The homology classes of divisors in the product surface are described as $NS(X \times X) = H_{1,1}(X \times X) \cap H_2(X \times X, \mathbb{Z})$, where $H_{1,1}$ is Poincaré dual of the Hodge space $H^{1,1}$.

DEFINITION 3.1. – *Let $n \geq 1$ and $g' \geq 2$ be integers such that $n(g'-1) \leq (g-1)$. Define $\mathrm{V}_{n.g'} \subset H_{1,1}(X \times X)$ to be the subset of classes z which are symmetric and satisfy: $z \circ z = nz$, of degrees $z \cdot \xi_1 = z \cdot \xi_2 = n$, and with $z \cdot \Delta = n(2-2g')$. The integer g' may be called the virtual genus for these classes. This $\mathrm{V}_{n.g'}$ will be described as an algebraic set in the affine space $H_2(X \times X, \mathbb{C})$. Define moreover V_n to be the union of $\mathrm{V}_{n.g'}$ for all g'. Then $\mathrm{V}_n(\mathbb{Z}) := \mathrm{V}_n \cap H_2(X \times X, \mathbb{Z})$, the subset of algebraic classes, will be the locus of integral points of the algebraic set.*

The basic facts quoted in the previous section say that

$$\mathcal{F}_n(X) \subset \mathrm{V}_n(\mathbb{Z}),$$

and there is a classic argument implying

PROPOSITION 3.2. – $\mathrm{V}_n(\mathbb{Z})$ *is a finite set.*

PROOF. – It is easy to see that $\mathrm{V}_n(\mathbb{Z})$ maps injectively into $NS(X \times X)/(\xi_1, \xi_2)$. The key tool is that on the quotient the quadratic form $(z \cdot \xi_1)^2 + (z \cdot \xi_2)^2 - (z \cdot z)$ is positive definite, cf. Weil [6]. Also recall the intersection formula $(z \cdot z) = ((z \circ z) \cdot \Delta)$. It follows that if $z \in \mathrm{V}_n(\mathbb{Z})$ then $\|z\|^2 = 2n^2 - n(z \cdot \Delta) = 2ng'$, so $\|z\|$ is bounded; and there is at most a finite number of points of bounded norm in a discrete subspace. □

Consider moreover the locus $\mathrm{E} \subset NS(X \times X)$ of effective divisor classes, usually denoted by $\mathrm{E}(X \times X)$. More precisely we have

$$\mathcal{F}_n(X) \subset \mathrm{V}_n(\mathbb{Z}) \cap (\Delta + \mathrm{E}).$$

QUESTION: if z belongs to the right hand side is it possible to write $z = [Z]$ for some Z effective symmetric such that $Z \circ Z = nZ$?

REMARK 3.3. – *This is true for $n = 2$. Let $Z = \Delta + S$ with S effective, of degrees $S \cdot \xi_1 = S \cdot \xi_2 = 1$. Then $S = \Gamma_g$ is the graph of an automorphism $g \in \mathrm{Aut}(X)$.*

Moreover $[Z]$ *symmetric implies* $\Gamma_g \sim \Gamma_{g^{-1}}$ *and* $\Gamma_{g^2} \sim \Delta$. *Then* $\Gamma_{g^2} \cdot \Delta < 0$ *implies* $\Gamma_{g^2} = \Delta$ *and* $g^2 = 1$, *so* g *is an involution. If* $f : X \to X/g$ *is the quotient map, then* $Z_f = Z$.

4. – Homological equivalence relations.

We introduce coordinates in homology using the Künneth decomposition

$$H_2(X\times X,\mathbb{Z})\cong H_2(X,\mathbb{Z})\otimes H_0(X,\mathbb{Z})\oplus H_0(X,\mathbb{Z})\otimes H_2(X,\mathbb{Z})\oplus H_1(X,\mathbb{Z})\otimes H_1(X,\mathbb{Z}).$$

Let ξ be the fundamental class in $H_2(X,\mathbb{Z})$, let p be the point in $H_0(X,\mathbb{Z})$, and let $\gamma_1,\dots,\gamma_{2g}$ be a standard basis in $H_1(X,\mathbb{Z})$, for which the nonzero intersection numbers are $(\gamma_i\cdot\gamma_{g+i})=1$ for $i=1,\dots,g$, and their opposites. Then in $H_2(X\times X,\mathbb{Z})$ one has the basis

$$\xi_1 := \xi\times p,\quad \xi_2 := p\times\xi,\quad \gamma_i\times\gamma_j\quad i,j\in\{1,\dots,2g\}.$$

We will use the rough notation $\gamma_i\gamma_j$.

LEMMA 4.1. – *Table of composition of correspondences:*

$$\begin{array}{llll} \xi_1\circ\xi_1=\xi_1 & \xi_2\circ\xi_2=\xi_2 & \xi_1\circ\xi_2=0 & \xi_2\circ\xi_1=0\\ \xi_1\circ\gamma_i\gamma_j=0 & \gamma_i\gamma_j\circ\xi_1=0 & \xi_2\circ\gamma_i\gamma_j=0 & \gamma_i\gamma_j\circ\xi_2=0\\ \gamma_i\gamma_j\circ\gamma_h\gamma_k=\delta_{jh}\,\gamma_i\gamma_k & \text{where } \delta_{jh}:=(\gamma_j\cdot\gamma_h). & & \end{array}$$

PROOF. – Compute with the definition: $z\circ z = p((z\times\xi)\cdot(\xi\times z))$, where p is the projection $X\times X\times X\longrightarrow X\times X$ to the first and third factors, using the general intersection formula: $(u\times v)\cdot(u'\times v') = (-1)^{p'q}(u\cdot u')\times(v\cdot v')$, where $p'=\operatorname{codim}(u')$ and $q=\operatorname{codim}(v)$. $\square$

A class in $H_2(X\times X,\mathbb{Z})$ may be written as $z=a_1\xi_1+a_2\xi_2+\sum a_{ij}\gamma_i\gamma_j$. The condition $z\cdot\xi_1=z\cdot\xi_2=n$ says that $a_1=a_2=n$. Therefore

$$z = n(\xi_1+\xi_2)+\sum a_{ij}\gamma_i\gamma_j. \tag{1}$$

Here z is symmetric if and only if

$$A := (a_{ij})$$

is antisymmetric (the involution $(x,y)\mapsto(y,x)$ implies $\gamma_i\times\gamma_j\mapsto -\gamma_j\times\gamma_i$ in homology). For instance the diagonal class is

$$\Delta = (\xi_1+\xi_2)+\sum_1^g(-\gamma_i\gamma_{g+i}+\gamma_{g+i}\gamma_i). \tag{2}$$

It follows that $z \circ z = n^2(\xi_1 + \xi_2) + \sum a_{ik}\gamma_i\gamma_k$ with coefficients $a_{ik} = \sum_{jh} a_{ij}\delta_{jh}a_{hk}$. Therefore $z \circ z = nz$ requires that $a_{ik} = na_{ik}$ for every i, k, in matrix form

$$ADA = nA \tag{3}$$

where $D = (\delta_{jh})$ is the standard matrix for the intersection product on $H_1(X, \mathbb{Z})$. Writing as a block matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

the equation becomes

$$\begin{matrix} -A_{12}A_{11} + A_{11}A_{21} = nA_{11} & -A_{12}A_{12} + A_{11}A_{22} = nA_{12} \\ -A_{22}A_{11} + A_{21}A_{21} = nA_{21} & -A_{22}A_{12} + A_{21}A_{22} = nA_{22} \end{matrix}.$$

Finally it is not difficult to compute the intersection number

$$z \cdot \Delta = 2 \operatorname{tr} A_{12} + 2n,$$

so the intersection formula in section 2 implies

$$\operatorname{tr} A_{12} = -ng'. \tag{4}$$

Using formula (1) for z, and formula (2) written as $\Delta = (\xi_1 + \xi_2) + \sum \delta_{ji}\gamma_i\gamma_j$, one finds that $z \cdot \Delta = 2n + \sum a_{ij}\delta_{kh}(\gamma_i\gamma_j \cdot \gamma_h\gamma_k) = 2n - \sum a_{ij}\delta_{kh}(\delta_{ih}\delta_{jk}) = 2n + 2\sum a_{i,g+i}$.

5. – Homology classes of type (1, 1).

We introduce coordinates in De Rahm cohomology using the Künneth decomposition

$$H^2(X \times X, \mathbb{C}) \cong H^2(X, \mathbb{C}) \otimes H^0(X, \mathbb{C}) \oplus H^0(X, \mathbb{C}) \otimes H^2(X, \mathbb{C}) \oplus H^1(X, \mathbb{C}) \otimes H^1(X, \mathbb{C}).$$

Let θ be the dual fundamental class in $H^2(X, \mathbb{Z})$, and let $\varphi_1, \ldots, \varphi_{2g}$ in $H^1(X, \mathbb{Z})$ be the dual basis of a standard basis $\gamma_1, \ldots, \gamma_{2g}$ in $H_1(X, \mathbb{Z})$, so that $\iint_X \theta = 1$ and $\int_{\gamma_i} \varphi_j = \delta_{ij}$, in the notation of the previous section. Then in $H^2(X \times X, \mathbb{Z})$ one has the De Rahm dual basis

$$\theta_1 := p_1^*(\theta), \quad \theta_2 := p_2^*(\theta), \quad p_1^*(\varphi_i) \wedge p_2^*(\varphi_j) \quad i, j \in \{1, \ldots, 2g\}.$$

We use the rough notation $\varphi_i\varphi_j$.

A reminder of Poincaré duality. The duality $H_1(X, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ is given by $\gamma_i \mapsto \varphi_{g+i}$, $\gamma_{g+i} \mapsto -\varphi_i$. It follows that the duality $P : H_2(X \times X, \mathbb{Z}) \to$

$H^2(X \times X, \mathbb{Z})$ is given by $\xi_1 \mapsto \theta_2,\ \xi_2 \mapsto \theta_1$, and

$$\gamma_i\gamma_j \mapsto -\varphi_{g+i}\varphi_{g+j}, \quad \gamma_i\gamma_{g+j} \mapsto \varphi_{g+i}\varphi_j, \quad \gamma_{g+i}\gamma_j \mapsto \varphi_i\varphi_{g+j}, \quad \gamma_{g+i}\gamma_{g+j} \mapsto -\varphi_i\varphi_j.$$

If z is given by the coordinate matrix A as in (1), then $P(z)$ is given by

$$P(A) := \begin{pmatrix} -A_{22} & A_{21} \\ A_{12} & -A_{11} \end{pmatrix}.$$

For instance $P(\Delta) = (\theta_1 + \theta_2) + \sum_1^g (\varphi_i\varphi_{g+i} - \varphi_{g+i}\varphi_i)$.

Then we introduce coordinates in the Hodge group by means of the Künneth decomposition $H^{1,1}(X \times X) \cong H^{1,1}(X) \otimes H^{0,0}(X) \oplus H^{0,0}(X) \otimes H^{1,1}(X) \oplus H^{1,0}(X) \otimes H^{0,1}(X) \oplus H^{0,1}(X) \otimes H^{1,0}(X)$.

If $\omega_1, \ldots, \omega_g$ is a basis in $H^{1,0}(X)$, then in $H^{1,1}(X \times X)$ there is the basis

$$\theta_1, \ \theta_2, \quad p_1^*(\omega_i) \wedge p_2^*(\bar{\omega}_j), \quad p_1^*(\bar{\omega}_i) \wedge p_2^*(\omega_j), \quad i, j \in \{1, \ldots, g\},$$

for which we also use the symbols $\omega_i\bar{\omega}_j$ and $\bar{\omega}_i\omega_j$.

The relation of coordinates in the De Rahm and the Hodge groups is as follows. The inclusion $H^{1,0}(X) \subset H^1(X, \mathbb{C})$ is described by

$$\omega_i = \sum_j q_{ij}\varphi_j.$$

Then $\bar{\omega}_i = \sum_j \bar{q}_{ij}\varphi_j$, where overline means complex conjugate. The complex matrix

$$\Pi := (q_{ij})$$

of type $(g, 2g)$ is a period matrix for X. The Riemann relations say that the basis $\omega_1, \ldots, \omega_g$ may be chosen so that $\Pi = (Q, 1)$ where Q is symmetric and $\mathrm{im}(Q)$ is positive definite.

Let z belong to $H_{1,1}(X \times X)$. The dual will be of the form $P(z) = n(\theta_1 + \theta_2) + \sum b_{ij}\omega_i\bar{\omega}_j + \sum b'_{ij}\bar{\omega}_i\omega_j$. This is symmetric if and only if $b'_{ij} = -b_{ji}$ (the involution $(x, y) \mapsto (y, x)$ implies $\omega_i\bar{\omega}_j \mapsto -\bar{\omega}_j\omega_i$ in cohomology). Therefore

$$P(z) = n(\theta_1 + \theta_2) + \sum b_{ij}(\omega_i\bar{\omega}_j - \bar{\omega}_j\omega_i)$$

and $B := (b_{ij})$ is a coordinate matrix for $P(z)$. Then write $\omega_i\bar{\omega}_j - \bar{\omega}_j\omega_i = \sum_{hk} (q_{ih}\bar{q}_{jk} - \bar{q}_{jh}q_{ik})\, \varphi_h\varphi_k$. An easy computation shows that

$$P(z) = n(\theta_1 + \theta_2) + \sum (\beta_{hk} - \beta_{kh})\varphi_h\varphi_k$$

where $\beta_{hk} := \sum_{ij} q_{ih}b_{ij}\bar{q}_{jk}$ is coefficient (h, k) in the product ${}^t\Pi B\bar{\Pi}$. Therefore

$$P(A) = \mathrm{asym}({}^t\Pi B\,\bar{\Pi}).$$

LEMMA 5.1. – *The condition for an antisymmetric matrix A to be Poincaré dual of some $asym({}^t\Pi B\bar{\Pi})$ is*

$$QA_{12} + A_{21}Q + QA_{11}Q + A_{22} = 0,$$

which in compact form is written as

$$ {}^t\Pi A\,\Pi = 0. \tag{5}$$

PROOF. – We have a proof by crude calculation, at present. Given A antisymmetric, let us search for B such that ${}^t\Pi B\bar{\Pi} = A + S$ for some S symmetric. Writing in block matrices:

$$\begin{pmatrix} A_{11} + S_{11} & A_{12} + S_{12} \\ A_{21} + S_{21} & A_{22} + S_{22} \end{pmatrix} = \begin{pmatrix} QB\bar{Q} & QB \\ B\bar{Q} & B \end{pmatrix}.$$

First we look at the equation $A_{22} + S_{22} = B$ with S_{22} symmetric, so we try to solve in S_{22}. Next consider the equations:

$$A_{12} + S_{12} = QB = Q(A_{22} + S_{22}),\quad A_{21} + S_{21} = B\bar{Q} = (A_{22} + S_{22})\bar{Q},$$

with the condition $S_{21} = {}^tS_{12}$.

This requires that $(A_{22} + S_{22})\bar{Q} - A_{21} = (-A_{22} + S_{22})Q + A_{21}$, hence $S_{22}(Q - \bar{Q}) - A_{22}(Q + \bar{Q}) + 2A_{21} = 0$. Write

$$Q = R + iI$$

in terms of the real and imaginary parts. The equation becomes $iS_{22}I - A_{22}R + A_{21} = 0$, and this says that

$$S_{22} = -i(A_{22}R - A_{21})I^{-1}$$

is purely imaginary. Moreover the condition S_{22} symmetric requires

$$I(A_{22}R - A_{21}) = (-RA_{22} + A_{12})I. \tag{a}$$

Finally the last equation $A_{11} + S_{11} = QB\bar{Q} = Q(A_{22} + S_{22})\bar{Q}$ with S_{11} symmetric requires $Q(A_{22} + S_{22})\bar{Q} - A_{11} = \bar{Q}(-A_{22} + S_{22})Q + A_{11}$, equivalent to $S_{11} = -\bar{S}_{11}$, i.e. that S_{11} is purely imaginary.

Write $S_{22} = iS'_{22}$ with $S'_{22} = -(A_{22}R - A_{21})I^{-1}$ as found before. Then

$$\begin{aligned} A_{11} + S_{11} &= (R + iI)(iS'_{22} + A_{22})(R - iI) \\ &= (R + iI)\{(A_{22}R + S'_{22}I) + i(S'_{22}R - A_{22}I)\} \\ &= R(A_{22}R + S'_{22}I) - I(S'_{22}R - A_{22}I) + i(\ldots) \end{aligned}$$

implies that $R(A_{22}R + S'_{22}I) - I(S'_{22}R - A_{22}I) = A_{11}$, hence

$$RA_{21} - RA_{22}R + A_{12}R + IA_{22}I = A_{11}. \tag{b}$$

Summing up equations (a) and (b) we end with

$$QA_{21} + A_{12}Q - QA_{22}Q = A_{11}.$$

Finally replace A with $P(A)$ in this formula.

6. – The variety of special homology classes.

We recollect the preceding computations. Write the antisymmetric A as a block matrix

$$A = \begin{pmatrix} U & W \\ -{}^tW & V \end{pmatrix}$$

with U and V antisymmetric. Then the equation $ADA = nA$ is written as

$$(3') \qquad \begin{matrix} -WU - U\,{}^tW = nU & -WW + UV = nW \\ \cdots & -VW - {}^tWV = nV \end{matrix}$$

(we omit block (2,1) by antisymmetry), in which we have the condition

$$(4') \qquad \mathrm{tr}\; W = -ng'$$

with $g' \geq 2$ and $n(g' - 1) \leq (g - 1)$, and the equation ${}^t\Pi A\,\Pi = 0$ is written as

$$(5') \qquad QW - {}^tWQ + QUQ + V = 0.$$

PROPOSITION 6.1. – *The set* V_n*, viewed in the space of triplets* U, V, W*, is defined by the equations above.*

Some first properties are seen from the equations. In V_n two isolated points are $-nD$ (the only point in the region A invertible), and 0 (the tangent space to $ADA = nA$ at the origin reduces to $A = 0$). So the algebraic set is not irreducible. It may be studied using the natural stratification according to $\mathrm{rk}(A)$. One finds for instance that the maximum dimension of strata is $\left[\dfrac{g^2}{2}\right]$ (the integral part). So the algebraic set is not small in dimension.

A naive hope in the beginning was that V_n might be something of general type, accounting in this way for the finite number of morphisms. It seems instead that every stratum is a unirational variety.

QUESTION: find a good parametrization for counting the integral points.

The following observation is useful for working with the equations above.

REMARK 6.2. – There is an action of nondegenerate matrices P on the datum of U, V, W and Q given by

$$W' = P^{-1}WP, \quad U' = P^{-1}U\,{}^tP^{-1}, \quad V' = {}^tPVP \quad \text{and} \quad Q' = {}^tPQP,$$

for which are invariant all the equations above, the trace function, and the Riemann relations for the period matrix.

The action has the following interpretation. Define

$$\tilde{P} := \begin{pmatrix} {}^tP & 0 \\ 0 & P^{-1} \end{pmatrix}.$$

This belongs to the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Q})$, and is viewed as a change of symplectic basis in $H_1(X, \mathbb{Q})$.

The action of $\tilde{P}$ on Q is congruence (under P), and is a special case of the action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on the Siegel space, usually written as $(\alpha Q + \beta)(\gamma Q + \delta)^{-1}$. The action of $\tilde{P}$ on the datum U, V, W is congruence:

$$A' = {}^t\tilde{P}A\tilde{P}$$

and the induced action on W is conjugation: this allows to simplify the form of W, at least to some extent.

7. – Some computation for $g = 3$.

We make some experimental computation for genus 3. Write

$$U = \begin{pmatrix} 0 & u_1 & u_2 \\ -u_1 & 0 & u_3 \\ -u_2 & -u_3 & 0 \end{pmatrix}, \; V = \begin{pmatrix} 0 & v_1 & v_2 \\ -v_1 & 0 & v_3 \\ -v_2 & -v_3 & 0 \end{pmatrix}, \; W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}.$$

In equations (3′) we observe the action of W on antisymmetric matrices given by $U \mapsto WU + U^tW$. For an antisymmetric matrix U as above define the coordinate vector $U' = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Then express the action in terms of the coordinate vector $(WU + U^tW)'$. It is given by

$$\begin{pmatrix} w_{11}u_1 - w_{13}u_3 + w_{22}u_1 + w_{23}u_2 \\ w_{11}u_2 + w_{12}u_3 + w_{32}u_1 + w_{33}u_2 \\ w_{21}u_2 + w_{22}u_3 - w_{31}u_1 + w_{33}u_3 \end{pmatrix} = \begin{pmatrix} w_{11} + w_{22} & w_{23} & -w_{13} \\ w_{32} & w_{11} + w_{33} & w_{12} \\ -w_{31} & w_{21} & w_{22} + w_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Define

$$t := \mathrm{tr}(W) \quad \text{and} \quad \widehat{W} := \begin{pmatrix} w_{33} & -w_{23} & w_{13} \\ -w_{32} & w_{22} & -w_{12} \\ w_{31} & -w_{21} & w_{11} \end{pmatrix}.$$

Then

$$(WU + U\,{}^tW)' = (tI - \widehat{W})\,U'.$$

Let us confine to the search of solutions with $t = -2n$, among which are the morphisms onto curves of genus $g' = 2$. Equations (3') become:

$$(\widehat{W} + nI)U' = 0, \quad ({}^t\widehat{W} + nI)\,V' = 0, \quad W(W + nI) = UV, \tag{6}$$

and may be read in terms of eigenvectors.

Now observe that W and $\widehat{W}$ and their transposed matrices have the same characteristic polynomial (if W is the matrix of an endomorphism relative to e_1, e_2, e_3, then $\widehat{W}$ is the matrix of the same endomorphism relative to $e_3, -e_2, e_1$).

It follows that $-n$ is an eigenvalue of W, and the other eigenvalues are p and $-n-p$ for some complex number p of degree at most 2 over the rationals. Let us confine to the case p rational.

Because of the remark at the end of the preceding section, we may assume that W is in Jordan form over the rationals. The possibilities are:

$$\begin{pmatrix} -n & 1 & \\ 0 & -n & \\ & & 0 \end{pmatrix}, \begin{pmatrix} -n & & \\ & -n & \\ & & 0 \end{pmatrix}, \begin{pmatrix} -n & & \\ & \frac{-n}{2} & 1 \\ & 0 & \frac{-n}{2} \end{pmatrix}, \begin{pmatrix} -n & & \\ & p & \\ & & -n-p \end{pmatrix}.$$
$$p \neq 0, -n$$

We call these matrices $W_1, W_2, W_3, W_4(p)$.

It is not difficult now to compute the solutions of (6):

(a) $W_1,\ U' = (0, u, 0),\ V' = (0, 0, v),$ with $uv = n$;
(b) $W_2,\ U' = 0$ or $V' = 0$;
(c) $W_3,\ U' = (0, 0, u),\ V' = (0, 0, v),$ with $uv = \dfrac{n^2}{4}$;
(d) $W_4(p),\ U' = (0, 0, u),\ V' = (0, 0, v),$ with $-uv = p(p + n)$.

Next consider equation (5'): $QW - {}^tWQ + QUQ + V = 0$. Let $\bar{q}_{ij}$ denote the complementary subdeterminant of the element q_{ij} (without the algebraic sign). Here are some formulas useful for computations:

$$(QW - {}^tWQ)' = \begin{pmatrix} q_{12}(w_{22} - w_{11}) \\ q_{13}(w_{33} - w_{11}) \\ q_{23}(w_{33} - w_{22}) \end{pmatrix} + w_{12} \begin{pmatrix} q_{11} \\ 0 \\ -q_{13} \end{pmatrix} + w_{23} \begin{pmatrix} 0 \\ q_{12} \\ q_{22} \end{pmatrix} + \cdots,$$

$$(QUQ)' = u_{23}\begin{pmatrix}\bar{q}_{31}\\ \bar{q}_{21}\\ \bar{q}_{11}\end{pmatrix} + u_{13}\begin{pmatrix}\bar{q}_{32}\\ \bar{q}_{22}\\ \bar{q}_{12}\end{pmatrix} + u_{12}\begin{pmatrix}\bar{q}_{33}\\ \bar{q}_{23}\\ \bar{q}_{13}\end{pmatrix}$$

It is now easy to check that (5′) becomes, in the various cases:

(a) $q_{11} + u\bar{q}_{32} = 0$, $nq_{13} + u\bar{q}_{22} = 0$, $nq_{23} - q_{13} + u\bar{q}_{12} + v = 0$, with $uv = n$;
(b) we have $U = 0$ or $V = 0$, and $(0, nq_{13}, nq_{23}) + (QUQ)' + V' = 0$;
(c) $q_{12}\frac{n}{2} + u\bar{q}_{31} = 0$, $q_{13}\frac{n}{2} + q_{12} + u\bar{q}_{21} = 0$, $q_{22} + u\bar{q}_{11} + v = 0$, with $uv = \frac{n^2}{4}$;
(d) $q_{12}(p+n) + u\bar{q}_{31} = 0$, $q_{13}(-p) + u\bar{q}_{21} = 0$, $q_{23}(-n-2p) + u\bar{q}_{11} + v = 0$, with $-uv = p(p+n)$.

We get the following easy:

COROLLARY 7.1. – *In* $\mathrm{V}_n(\mathbb{Q})$ *there is at most* 1 *solution of Jordan type* W_1, *at most* 1 *solution of type* W_2, *and there are at most* 2 *solutions of type* W_3.

So it is case (d) that seems to require some more detailed analysis.

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