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A Constructive Boolean Central Limit Theorem.

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Sunto. – *Si fornisce una costruzione dei processi di creazione, distruzione e numero sullo spazio di Fock Booleano a mezzo di un teorema di limite centrale quantistico partendo da processi di creazione, distruzione e numero con tempo discreto.*

Summary. – *We give a construction of the creation, annihilation and number processes on the Boolean Fock space by means of a quantum central limit theorem starting from creation, annihilation and number processes with discrete time.*

1. – Introduction.

In quantum probability the notion of independence is not unique. During the past years emerged various inequivalent types of non-commutative independence, studied in a unified abstract picture in many papers (see [15], [5], [13]). Furthermore one knows that generally any notion of non-commutative independence generates a central limit theorem. Also for this case a unified abstract approach has been made by Accardi, Hashimoto and Obata in [4] and by Speicher and von Waldenfels in [16], where non-commutative general central limits are proved. More recently the interacting Fock space structure has suggested an idea to obtain constructive central limit theorems and this led to prove in [3] that any mean zero measure on the real line with finite moments of any order is the central limit in the sense of moments of a sequence of processes, satisfying rather weak conditions, on 1-mode type interacting Fock space.

Furthermore Accardi and Bach showed in [1] that creation, annihilation and number operators in Boson Fock space without *test function,* can be obtained as central limit of operators in toy Fock spaces. This is possible also for fermionic, free and monotone Fock space (see [8], [10], [11]).

One wish naturally to find a similar central limit theorem also for the boolean independence and Boolean Fock space and to this aim is devoted the present paper, where, after pointing out that Boolean Fock space can be seen as a 1-mode type interacting Fock space, we use some useful techniques of such a space in order to reach this goal.

The notion of Boolean independence was introduced in [18] by W. von Waldenfels (namely "*interval partition*") and by Bozejko in [7] in the theory of the free groups and successively Speicher and Woroudi in [17] defined the Boolean convolution. This kind of non-commutative independence has been used also by Lenczewski in [9] and by Skeide in [14] for the theory of quantum stochastic calculus in the framework of Hilbert modules. More recently a quantum stochastic calculus on Boolean Fock space has been developed by Ben Ghorbal and Schürmann in [6].

The paper is organized as follows: in Section 2 we define the Boolean Fock space, the simplest quantum probability model - quantum Bernoulli process - and proper constructive quantum stochastic processes (namely creation - annihilation - number processes) on the Boolean Fock space and with discrete time. In Section 3 we give results on the moments of such operator processes with respect to the "vacuum" state. In Section 4 we present our main result: a quantum central limit argument is used to get the creation - annihilation - number processes on the *Boolean Fock* space starting from discrete-time processes. It is worth to mention that the family of processes used in our central limit theorem satisfies the singleton condition with respect to the vacuum state and the boundedness of the mixed moments (see [4] for more details). Then, as a consequence of [4], Lemma 2.4, only the pair partitions survive in the limit. Our constructive approach shows that discrete-time processes give exactly a particular pair partition, i.e. the *interval partition*.

Finally we want to point out the differences between [16] and our paper:

- we give a concrete construction rather than an abstract and general approach of existence;
- in [16] the *-algebra and the state are the same either for the convergent sequence of the quantum random variables or for their resulting limit. Our sequence is made of operators on $\left(\mathbb{R}^2\right)^{\otimes N}$, but in the limit one finds operators on the Boolean Fock space;
- in [16] the GNS representation for the limit state has a Fock like structure on which the elements act as creation and annihilation operators, instead in our case the constructive approach allows to find also the number operator either in the convergent sequence or in the limit.

2. – Notations and definitions.

We firstly introduce some definitions and notations on Boolean Fock space. To this goal we consider an Hilbert space $\mathfrak{H}$. The Hilbert space

$$\Gamma(\mathfrak{H}) := \mathbb{C} \oplus \mathfrak{H}$$

is called the Boolean Fock space over $\mathfrak{H}$. We denote by Φ the vector

$$\Phi := 1 \oplus 0$$

which is called the *vacuum vector* in $\Gamma(\mathfrak{H})$ and define the *vacuum expectation* $\langle\cdot\rangle$ as the state

$$\langle\cdot\rangle := \langle \Phi, \cdot\Phi\rangle$$

In the following definitions the three basic operators of creation, annihilation and preservation on $\Gamma(\mathfrak{H})$ are presented in the usual way.

DEFINITION 2.1. – *For any $f \in \mathfrak{H}$ and for any $a \in \mathbb{C}$, $g \in \mathfrak{H}$ the creation operator*

$$A^+(f)(a \oplus g) := 0 \oplus af$$

is defined as a linear operator $A^+(f) : \Gamma(\mathfrak{H}) \to \Gamma(\mathfrak{H})$ and has an adjoint $A(f) : \Gamma(\mathfrak{H}) \to \Gamma(\mathfrak{H})$ called the annihilation operator.

REMARK 2.1. – For any $f \in \mathfrak{H}$ and for any $a \in \mathbb{C}$, $g \in \mathfrak{H}$

$$A(f)(a \oplus g) := \langle f, g\rangle \oplus 0$$

REMARK 2.2. – For any $f, g \in \mathfrak{H}$

$$\langle A(f)A^+(g)\rangle = \langle f, g\rangle$$

DEFINITION 2.2. – *For any $B \in \boldsymbol{B}(\mathfrak{H})$ and for any $a \in \mathbb{C}$, $g \in \mathfrak{H}$ we define the preservation (number) operator $\Lambda(B) : \Gamma(\mathfrak{H}) \to \Gamma(\mathfrak{H})$ such that*

$$\Lambda(B)(a \oplus g) := 0 \oplus B(g)$$

For any $\varepsilon \in \{-1, 0, 1\}$, for any $B \in \boldsymbol{B}(\mathfrak{H})$ and $f \in \mathfrak{H}$ we will denote

$$A^\varepsilon(f, B) := \begin{cases} A^+(f) & \text{if } \varepsilon = 1 \\ \Lambda(B) & \text{if } \varepsilon = 0 \\ A(f) & \text{if } \varepsilon = -1 \end{cases}$$

In order to state our main result we need some further definitions and notations starting from the quantum Bernoulli process as in [8, 12]. Let us denote $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Clearly $\{e_1, e_2\}$ gives the usual base of $\mathbb{R}^2$. For any $N \geq 1$, $0 \leq n \leq N$ and $1 \leq k_1 < k_2 < \ldots < k_n \leq N$, we define

$$\delta^{(N)}_{(k_1,\ldots,k_n)} := \begin{cases} \delta^{(N)}_{\emptyset} := e_1^{\otimes N} & \text{if } n = 0 \\ e_1^{\otimes(k_1-1)} \otimes e_2 \otimes e_1^{\otimes(k_2-k_1-1)} \otimes e_2 \otimes \ldots \otimes e_2 \otimes e_1^{\otimes(N-k_n)} & \text{if } 1 \leq n \leq N \end{cases} \tag{2.1}$$

where hereinafter, when $n = 0$, $(k_1, \ldots, k_n)$ is understood as $\emptyset$. We notice that the family

$$\Big\{\delta^{(N)}_{(k_1,\dots,k_n)} \otimes \delta^{(N)}_{(h_1,\dots,h_m)} : 0 \le n, m \le N,\ 1 \le k_1 < \dots < k_n \le N,$$
$$1 \le h_1 < \dots < h_m \le N\Big\}$$

is a base of the space $\mathbb{M}_2^{\otimes N}$, where $\mathbb{M}_2 := \mathbb{R}^2 \otimes \mathbb{R}^2$ is the space of 2×2-matrices.

The definition (2.1) can be generalized by inserting suitable "test functions". Hence we consider the space of all complex Riemann integrable functions $\mathcal{L}([0,1])$ and for $1 \le n \le N$, $1 \le k_1 < \dots < k_n \le N$, $f_1, \dots, f_n \in \boldsymbol{L}^2([0,1], dx)$, define

$$\delta^{(N)}_{(k_1,\dots,k_n)}(f_1, \dots, f_n) := f_n\left(\frac{k_n}{N}\right) \prod_{h=1}^{n-1} f_h\left(\frac{k_h}{k_{h+1}}\right) \cdot \delta^{(N)}_{(k_1,\dots,k_n)}$$

For any $k = 1, \dots, N$, for any $f \in \mathcal{L}([0,1])$, for any $0 \le n \le N$ and $1 \le k_1 < \dots < k_n \le N$ define a linear operator $\boldsymbol{T}^+_N(f,k)$ from $\left(\mathbb{R}^2\right)^{\otimes N}$ into itself as follows

$$\boldsymbol{T}^+_N(f,k)\left[\delta^{(N)}_{(k_1,\dots,k_n)}\right] := \begin{cases} \delta^{(N)}_{(k)}(f) & \text{if } n = 0 \\ 0 & \text{if } n \ge 1 \end{cases}$$

and denote by $\boldsymbol{T}_N(f,k)$ the adjoint of $\boldsymbol{T}^+_N(f,k)$. The following result allows us to compute the value of $\boldsymbol{T}_N(f,k)$.

LEMMA 2.1. – *For any* $k = 1, \dots, N$, *for any* $f \in \mathcal{L}([0,1])$, *for any* $0 \le n \le N$ *and* $1 \le k_1 < \dots < k_n \le N$,

$$\boldsymbol{T}_N(f,k)\left[\delta^{(N)}_{(k_1,\dots,k_n)}\right] = \begin{cases} \delta^{(N)}_{\emptyset}(\overline{f}) & \textit{if } n = 1,\ k = k_1 \\ 0 & \textit{otherwise} \end{cases} \tag{2.2}$$

where $\delta^{(N)}_{\emptyset}(\overline{f}) := \overline{f}\left(\frac{k}{N}\right) \cdot \delta^{(N)}_{\emptyset}$.

PROOF. – We firstly observe that for any $n, m \in \mathbb{N}$, $1 \le k_1 < \dots < k_n \le N$, $1 \le h_1 < \dots < h_m \le N$

$$\left\langle \delta^{(N)}_{(h_1,\dots,h_m)}, \delta^{(N)}_{(k_1,\dots,k_n)} \right\rangle = \delta^n_m \prod_{j=1}^{n} \delta^{k_j}_{h_j}$$

where δ^n_m is the Kronecker symbol. Hence we have

$$\left\langle \delta^{(N)}_{(h_1,\dots,h_m)}, \boldsymbol{T}_N(f,k)\left[\delta^{(N)}_{(k_1,\dots,k_n)}\right] \right\rangle = \left\langle \boldsymbol{T}^+_N(f,k)\left[\delta^{(N)}_{(h_1,\dots,h_m)}\right], \delta^{(N)}_{(k_1,\dots,k_n)} \right\rangle$$

$$= \begin{cases} \overline{f}\left(\frac{k}{N}\right)\left\langle \delta^{(N)}_{(k)}, \delta^{(N)}_{(k_1,\ldots,k_n)} \right\rangle & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \overline{f}\left(\frac{k}{N}\right) & \text{if } m = 0,\, n = 1,\ k = k_1 \\ 0 & \text{otherwise} \end{cases}$$

From another hand

$$\begin{cases} \left\langle \delta^{(N)}_{(h_1,\ldots,h_m)}, \delta^{(N)}_{\emptyset}(\overline{f}) \right\rangle & \text{if } n = 1,\ k = k_1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \overline{f}\left(\frac{k}{N}\right)\left\langle \delta^{(N)}_{(h_1,\ldots,h_m)}, \delta^{(N)}_{\emptyset} \right\rangle & \text{if } n = 1,\ k = k_1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \overline{f}\left(\frac{k}{N}\right) & \text{if } m = 0,\ n = 1,\ k = k_1 \\ 0 & \text{otherwise} \end{cases}$$

□

For any $B \in \boldsymbol{B}\Big(\boldsymbol{L}^2([0,1],dx)\Big)$, we define a linear operator $\Lambda_N(B) : \left(\mathbb{R}^2\right)^{\otimes N} \to \left(\mathbb{R}^2\right)^{\otimes N}$ such that for any $0 \le n \le N$, for any $f_1, \ldots f_n \in \mathcal{L}([0,1])$ and $1 \le k_1 < \ldots < k_n \le N$

$$\Lambda_N(B)\left[\delta^{(N)}_{(k_1,\ldots,k_n)}(f_1,\ldots f_n)\right] := \begin{cases} \delta^{(N)}_{(k_1)}(Bf_1) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

One notices that if I is the identity operator on $\boldsymbol{L}^2([0,1],dx)$, then $\Lambda_N(I)$ is the identity on $\left(\mathbb{R}^2\right)^{\otimes N}$.

REMARK 2.3. – The operators $\boldsymbol{T}^+_N(f,k)$, $\boldsymbol{T}_N(f,k)$ and $\Lambda_N(B)$ above defined can be seen as creation, annihilation and preservation operators respectively, on $\left(\mathbb{R}^2\right)^{\otimes N}$.

PROPOSITION 2.1. – *For any* $B \in \boldsymbol{B}\big(\boldsymbol{L}^2([0,1],dx)\big)$, *for any* $h, k = 1, \ldots, N$, *for any* $f, g \in \mathcal{L}([0,1])$, *for any* $0 \le n \le N$ *and* $1 \le k_1 < \ldots < k_n \le N$,

$$\boldsymbol{T}_N(f,k)\boldsymbol{T}^+_N(g,h)\left[\delta^{(N)}_{\emptyset}\right] = \left(\overline{f}g\right)\left(\frac{k}{N}\right)\left[\delta^{(N)}_{\emptyset}\right]\delta^h_k \tag{2.3}$$

$$\Lambda_N(B)\boldsymbol{T}^+_N(f,k)\left[\delta^{(N)}_{(k_1,\ldots,k_n)}\right] = \boldsymbol{T}^+_N(Bf,k)\left[\delta^{(N)}_{(k_1,\ldots,k_n)}\right] \tag{2.4}$$

PROOF. – The result follows from simple computations. □

The previous results imply that any product of annihilation - creation - preservation operators can be reduced to a product of only annihilation - creation operators. For any $\varepsilon \in \{-1, 0, 1\}$, for any $B \in \boldsymbol{B}\Big(\boldsymbol{L}^2([0,1], dx)\Big)$, for any $1 \le k \le N$, and $f \in \mathcal{L}([0,1])$ we will denote

$$\boldsymbol{T}_N^{\varepsilon}(f, k, B) := \begin{cases} \boldsymbol{T}_N^{+}(f, k) & \text{if } \varepsilon = 1 \\ \Lambda_N(B) & \text{if } \varepsilon = 0 \\ \boldsymbol{T}_N(f, k) & \text{if } \varepsilon = -1 \end{cases}$$

3. – Moments of operators in discrete and Boolean case.

In our main result we need to know the value of the following joint expectations

$$\left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_n}(f_n, k_n, B_n) \cdots \boldsymbol{T}_N^{\varepsilon_1}(f_1, k_1, B_1)\delta_{\emptyset}^{(N)} \right\rangle$$
$$\langle A^{\varepsilon_n}(f_n, B_n) \cdots A^{\varepsilon_1}(f_1, B_1)\rangle$$

To this goal we present the following results for the discrete case. We observe that they hold in the Boolean case too. Their validity can be obtained just by replying all the cases $\mathfrak{H}$ to $\mathcal{L}([0,1])$, A^{ε} to $\boldsymbol{T}_N^{\varepsilon}$ and Φ to $\delta_{\emptyset}^{(N)}$.

LEMMA 3.1. – *For any $n \in \mathbb{N}$ and ε belongs to the set*

$$\{-1, 0, 1\}^n := \{\varepsilon = (\varepsilon(n), \ldots, \varepsilon(1)) : \varepsilon(i) \in \{-1, 0, 1\}, \ \forall i = 1, \ldots, n\}$$

i) if among $\left\{\boldsymbol{T}_N^{\varepsilon_n}(f_n, k_n, B_n), \ldots, \boldsymbol{T}_N^{\varepsilon_1}(f_1, k_1, B_1)\right\}$ there are more annihilators than creators, then

$$\boldsymbol{T}_N^{\varepsilon_n}(f_n, k_n, B_n) \cdots \boldsymbol{T}_N^{\varepsilon_1}(f_1, k_1, B_1)\delta_{\emptyset}^{(N)} = 0$$

ii) if the cardinality of the set $\{i : \varepsilon(i) = \pm 1\}$ is odd, then

$$\left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_n}(f_n, k_n, B_n) \cdots \boldsymbol{T}_N^{\varepsilon_1}(f_1, k_1, B_1)\delta_{\emptyset}^{(N)} \right\rangle = 0$$

iii) if either $\varepsilon(n) \in \{0, 1\}$ or $\varepsilon(1) \in \{-1, 0\}$, the scalar product

$$\left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_n}(f_n, k_n, B_n) \cdots \boldsymbol{T}_N^{\varepsilon_1}(f_1, k_1, B_1)\delta_{\emptyset}^{(N)} \right\rangle$$

is equal to zero.

iv) if among $\{\boldsymbol{T}_N^{\varepsilon_n}(f_n,k_n,B_n),\dots,\boldsymbol{T}_N^{\varepsilon_1}(f_1,k_1,B_1)\}$ *on has the same number of annihilators and creators, then there exists a constant c such that*

$$\boldsymbol{T}_N^{\varepsilon_n}(f_n,k_n,B_n)\cdots\boldsymbol{T}_N^{\varepsilon_1}(f_1,k_1,B_1)\delta_\emptyset^{(N)}=c\delta_\emptyset^{(N)} \tag{3.1}$$

PROOF. – The results above follow directly from (2.2), (2.3), (2.4). □

As in [3] we introduce the *depth function* of a given sequence $\varepsilon\in\{-1,0,1\}^n$.

DEFINITION 3.1. – *For* $n\in\mathbb{N}$ *and* $\varepsilon=(\varepsilon(n),\cdots,\varepsilon(1))\in\{-1,0,1\}^n$ *we define the depth function (of the string* ε*)* $d_\varepsilon:\{1,...,n\}\to\{0,\pm 1,...,\pm n\}$ *by*

$$\begin{aligned} d_\varepsilon(j) &= \sum_{k=1}^{j}\varepsilon(k) \\ &= |\{\varepsilon(k):\varepsilon(k)=1;\ k<j\}|-|\{\varepsilon(k):\varepsilon(k)=-1;\ k<j\}| \end{aligned}$$

Hence $d_\varepsilon(j)$ gives the relative number of creators (annihilators if negative) in the product $\boldsymbol{T}_N^{\varepsilon_n}(f_n,k_n,B_n)\cdots\boldsymbol{T}_N^{\varepsilon_1}(f_1,k_1,B_1)$ which are on the right side of $\boldsymbol{T}_N^{\varepsilon_j}(f_j,k_j,B_j)$.

DEFINITION 3.2. – $\{-1,0,1\}_+^n$ *is defined as the totality of all* $\{-1,0,1\}^n$ *satisfying the following conditions:*

i) $\sum_{k=1}^{n}\varepsilon(k)=d_\varepsilon(n)=0$;

ii) $\varepsilon(1)=1$ *and* $\varepsilon(n)=-1$;

iii) for all $i=1,\dots,n$, $d_\varepsilon(i)\geq 0$;

We denote by $\{-1,0,1\}_{+,B}^n$ the set $\{-1,0,1\}_+^n$ such that the following condition holds: for any $j=1,\dots,n$, if $\varepsilon(j)=1$ (respectively $\varepsilon(j)=-1$) then $\varepsilon(j+1)\neq 1$ (respectively $\varepsilon(j+1)\neq -1$).

LEMMA 3.2. – *For any* $n\in\mathbb{N}$, *and* $\varepsilon\in\{-1,0,1\}^n$, *the scalar product*

$$\left\langle\delta_\emptyset^{(N)},\boldsymbol{T}_N^{\varepsilon_n}(f_n,k_n,B_n)\cdots\boldsymbol{T}_N^{\varepsilon_1}(f_1,k_1,B_1)\delta_\emptyset^{(N)}\right\rangle \tag{3.2}$$

can be nonzero only if $\varepsilon\in\{-1,0,1\}_{+,B}^n$.

PROOF. – The conditions fulfilling Definition 3.2 can be obtained by using the same arguments developed in the proof of Lemma 3.2 in [3]. In order to prove that (3.2) does not vanish only if $\varepsilon\in\{-1,0,1\}_{+,B}^n$, we firstly observe that if $\varepsilon(2)=1$, then by definition of the creation operator on $(\mathbb{R}^2)^{\otimes N}$

$$\begin{aligned}(3.3)\qquad &\left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_n}(f_n,k_n,B_n)\cdots \boldsymbol{T}_N^{+}(f_2,k_2)\boldsymbol{T}_N^{+}(f_1,k_1)\delta_{\emptyset}^{(N)}\right\rangle \\ &= \left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_n}(f_n,k_n,B_n)\cdots \boldsymbol{T}_N^{+}(f_2,k_2)\delta_{(k)}^{(N)}(f)\right\rangle = 0\end{aligned}$$

Now we suppose that for all $h \le j-1$, if $\varepsilon(h)=1$, then $\varepsilon(h+1)\neq 1$ and prove it for $h=j$. If $\varepsilon(j+1)=1$, we have to compute

$$\left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_n}(f_n,k_n,B_n)\cdots \boldsymbol{T}_N^{+}(f_{j+1},k_{j+1})\boldsymbol{T}_N^{+}(f_j,k_j)\cdots \boldsymbol{T}_N(f_2,k_2)\boldsymbol{T}_N^{+}(f_1,k_1)\delta_{\emptyset}^{(N)}\right\rangle$$

After the reduction given by (2.4), on the right side of $\boldsymbol{T}_N^{+}(f_j,k_j)$ one finds only a sequence of creators and annihilators. Furthermore, from our hypothesis, the number of creators and annihilators, must be equal. So, from (3.1) it follows that there exists a constant c such that the quantity above becomes equal to

$$c\left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_n}(f_n,k_n,B_n)\cdots \boldsymbol{T}_N^{+}(f_{j+1},k_{j+1})\boldsymbol{T}_N^{+}(f_j,k_j)\delta_{\emptyset}^{(N)}\right\rangle$$

thus giving zero as in (3.3).

□

As a consequence of Lemma 3.1, the scalar product (3.2) is non zero only if the number of creation and annihilation operators is the same. Hence hereinafter, if m is the cardinality of the operators in (3.2), then $m=2n+j$, where n is the cardinality of creators (equivalently annihilators) and j the cardinality of preservation operators. Furthermore we denote $V := \{i=1,\dots,m : \varepsilon(i)=0\}$ (so $|V|=j$, where as usual $|\cdot|$ is the cardinality of the set), put $V =: \{i_1,\dots i_j\}$ and $\{h_1,\dots h_{2n}\} := \{1,\dots,m\}\backslash\{i_1,\dots i_j\}$.

The following result states that any scalar product of the form

$$\left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_m}(f_m,k_m,B_m)\cdots \boldsymbol{T}_N^{\varepsilon_1}(f_1,k_1,B_1)\delta_{\emptyset}^{(N)}\right\rangle$$

can be reduced to a product of factors, each of them formed by a scalar product containing exactly one annihilator and its following creator, in other words the *Boolean* condition of independence is here satisfied.

LEMMA 3.3. – *For any* $m\in\mathbb{N}$, *for any* $\varepsilon\in\{-1,0,1\}^m_{+,B}$, *for any* $f_1,\dots,f_m\in\mathcal{L}([0,1])$, $1\le k_1<\ldots<k_m\le N$ *and* $B_1,\dots,B_m\in\boldsymbol{B}(\boldsymbol{L}^2([0,1],dx))$

$$\begin{aligned}(3.4)\qquad &\left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N^{\varepsilon_m}(f_m,k_m,B_m)\cdots \boldsymbol{T}_N^{\varepsilon_1}(f_1,k_1,B_1)\delta_{\emptyset}^{(N)}\right\rangle \\ &= \prod_{\substack{p=1\\ p\in 2\mathbb{N}+1}}^{2n} \left\langle \delta_{\emptyset}^{(N)}, \boldsymbol{T}_N\Big(f_{h_{p+1}},k_{h_{p+1}}\Big)\boldsymbol{T}_N^{+}\Big(B_{V_{h_p}}f_{h_p},k_{h_p}\Big)\delta_{\emptyset}^{(N)}\right\rangle\end{aligned}$$

where for any $p = 1, \dots, 2n$, $p \in 2\mathbb{N} + 1$,

$$V_{h_p} := \{ i_r, \dots, i_q \in V : h_{p+1} < i_r < \dots < i_q < h_p \}$$

and $B_{V_{h_p}} := B_{i_r} \cdots B_{i_q}$, $i_r, \dots, i_q \in V_{h_p}$.

PROOF. – In the notations introduced above, let $h_1 := 1$, let h_l and i_1 be the indices relative respectively to the first annihilator and to the first preservation operator from the right in the left hand side of (3.4). From Lemma 3.2 we have that the left hand side of (3.4) is equal to

$$\Big\langle \delta_\emptyset^{(N)}, \boldsymbol{T}_N(f_m, k_m) \cdots \boldsymbol{T}_N^{\varepsilon_{h_{l+1}}}\big(f_{h_{l+1}}, k_{h_{l+1}}, B_{h_{l+1}}\big) \boldsymbol{T}_N\big(f_{h_l}, k_{h_l}\big) \boldsymbol{\Lambda}_N(B_{i_r}) \cdots$$
$$\cdots \boldsymbol{\Lambda}_N\big(B_{i_1}\big) \boldsymbol{T}_N^+(f_1, k_1) \delta_\emptyset^{(N)} \Big\rangle$$

Proposition 2.1 implies that the quantity above can be reduced to

$$\Big\langle \delta_\emptyset^{(N)}, \boldsymbol{T}_N(f_m, k_m) \cdots \boldsymbol{T}_N^{\varepsilon_{h_{l+1}}}\big(f_{h_{l+1}}, k_{h_{l+1}}, B_{h_{l+1}}\big) \boldsymbol{T}_N\big(f_{h_l}, k_{h_l}\big) \cdots \boldsymbol{T}_N^+\big(B_{i_1} f_1, k_1\big) \delta_\emptyset^{(N)} \Big\rangle$$

and, by using repeatedly Lemma 3.2, and Proposition 2.1, we obtain

$$\Big\langle \delta_\emptyset^{(N)}, \boldsymbol{T}_N(f_m, k_m) \cdots \boldsymbol{T}_N^{\varepsilon_{h_{l+1}}}\big(f_{h_{l+1}}, k_{h_{l+1}}, B_{h_{l+1}}\big) \boldsymbol{T}_N\big(f_{h_l}, k_{h_l}\big) \boldsymbol{\Lambda}_N(B_{i_r})$$
$$\cdots \boldsymbol{\Lambda}_N\big(B_{i_2}\big) \boldsymbol{T}_N^+\big(B_{i_1} f_1, k_1\big) \delta_\emptyset^{(N)} \Big\rangle$$
$$= \Big\langle \delta_\emptyset^{(N)}, \boldsymbol{T}_N(f_m, k_m) \cdots \boldsymbol{T}_N^{\varepsilon_{h_{l+1}}}\big(f_{h_{l+1}}, k_{h_{l+1}}, B_{h_{l+1}}\big)$$
$$\cdot \boldsymbol{T}_N\big(f_{h_l}, k_{h_l}\big) \boldsymbol{T}_N^+\big(B_{i_r} \cdots B_{i_2} B_{i_1} f_1, k_1\big) \delta_\emptyset^{(N)} \Big\rangle$$

Thus, necessarily $l = 2$ and consequently $B_{i_r} \cdots B_{i_2} B_{i_1} = B_{V_{h_1}}$. The scalar product above can be rewritten as

$$\Big\langle \delta_\emptyset^{(N)}, \boldsymbol{T}_N(f_m, k_m) \cdots \boldsymbol{T}_N^+\big(f_{h_3}, k_{h_3}\big) \delta_\emptyset^{(N)} \Big\rangle \Big\langle \delta_\emptyset^{(N)}, \boldsymbol{T}_N\big(f_{h_2}, k_{h_2}\big) \boldsymbol{T}_N^+\Big(B_{V_{h_1}} f_1, k_1\Big) \delta_\emptyset^{(N)} \Big\rangle$$

Iterating the same procedure for any $p = 3, \dots, 2n$, $p \in 2\mathbb{N} + 1$, the thesis follows. □

4. – Boolean Central Limit Theorem.

In this section we present the main result of the paper. From now on $\mathfrak{H} := \boldsymbol{L}^2([0,1], dx)$. In order to prove our central limit theorem, as in [3] and [2], we consider for any $\varepsilon \in \{-1, 0, 1\}$, for any $B \in \boldsymbol{B}(\mathfrak{H})$, for any $1 \le k \le N$, and $f \in \mathcal{L}([0,1])$ the "normalized family"

$$\widetilde{\boldsymbol{T}}^{\varepsilon}_N(f,k,B) := \begin{cases} \boldsymbol{T}^+_N(f,k) & \text{if } \varepsilon = 1 \\ c_k \boldsymbol{\Lambda}_N(B) & \text{if } \varepsilon = 0 \\ \boldsymbol{T}_N(f,k) & \text{if } \varepsilon = -1 \end{cases}$$

where $\{c_k\}$ is a bounded sequence in $\mathbb{R}$ satisfying

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} c_k = 1 \tag{4.1}$$

For any $\varepsilon \in \{-1,0,1\}$, we introduce the centered sum

$$S^{\varepsilon}_N(f,B) := \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \widetilde{\boldsymbol{T}}^{\varepsilon}_N(f,k,B)$$

and state the following central limit theorem.

THEOREM 4.1. – *For any $m \in \mathbb{N}$, for any $\varepsilon \in \{-1,0,1\}^m$, for any $f_1,\dots,f_m \in \mathcal{L}([0,1])$, and $B_1,\dots,B_m \in \boldsymbol{B}(\mathfrak{H})$, the limit, for $N \to \infty$, of*

$$\left\langle \delta^{(N)}_{\emptyset}, S^{\varepsilon_m}_N(f_m,B_m)\cdots S^{\varepsilon_1}_N(f_1,B_1)\delta^{(N)}_{\emptyset} \right\rangle \tag{4.2}$$

is equal to zero whenever $\varepsilon \in \{-1,0,1\}^m \backslash \{-1,0,1\}^m_{+,B}$ and if $\varepsilon \in \{-1,0,1\}^m_{+,B}$ it is equal to

$$\langle A^{\varepsilon_m}(f_m,B_m)\cdots A^{\varepsilon_1}(f_1,B_1)\rangle$$

PROOF. – In fact (4.2) can be rewritten as

$$\frac{1}{N^{\frac{m}{2}}} \sum_{k_1,\dots,k_m=1}^{N} \left\langle \delta^{(N)}_{\emptyset}, \widetilde{\boldsymbol{T}}^{\varepsilon_m}_N(f_m,k_m,B_m)\cdots \widetilde{\boldsymbol{T}}^{\varepsilon_1}_N(f_1,k_1,B_1)\delta^{(N)}_{\emptyset} \right\rangle$$

hence obtaining the first part of the thesis. If $\varepsilon \in \{-1,0,1\}^m_{+,B}$, then, from Lemma 3.3, this quantity is equal to

$$\left[\frac{1}{N^{\frac{j}{2}}} \prod_{q=1}^{j} \left(\sum_{k_{i_q}=1}^{N} c_{k_{i_q}} \right)\right]$$

$$\left[\frac{1}{N^n} \prod_{\substack{p=1 \\ p\in 2\mathbb{N}+1}}^{2n} \sum_{k_{h_p}=1}^{N} \left\langle \delta^{(N)}_{\emptyset}, \boldsymbol{T}_N\left(f_{h_{p+1}}, k_{h_{p+1}}\right) \boldsymbol{T}^+_N\left(B_{V_{h_p}} f_{h_p}, k_{h_p}\right) \delta^{(N)}_{\emptyset} \right\rangle\right]$$

where $m = 2n + j$ and, as defined above, $2n$ and j are respectively the cardinality of annihilator-creators and preservation operators. By definition of discrete annihilations, creation and preservation operators, we obtain

$$\left[\frac{1}{N^{\frac{j}{2}}}\prod_{q=1}^{j}\left(\sum_{k_{i_q}=1}^{N} c_{k_{i_q}})\right]\cdot\left[\frac{1}{N^n}\prod_{\substack{p=1\\ p\in 2\mathbb{N}+1}}^{2n}\sum_{k_{h_p}=1}^{N}\overline{f}_{h_{p+1}}\left(\frac{k_{h_p}}{N}\right)B_{V_{h_p}}f_{h_p}\left(\frac{k_{h_p}}{N}\right)\right]$$

$$=\left[\frac{1}{N^{\frac{j}{2}}}\prod_{q=1}^{j}\left(\sum_{k_{i_q}=1}^{N} c_{k_{i_q}}\right)\right]\cdot\left[\frac{1}{N}\sum_{k_{h_1}=1}^{N}\overline{f}_{h_2}\left(\frac{k_{h_1}}{N}\right)B_{V_{h_1}}f_{h_1}\left(\frac{k_{h_1}}{N}\right)\right]$$

$$\times\ldots\times\left[\frac{1}{N}\sum_{k_{h_{2n-1}}=1}^{N}\overline{f}_{h_{2n}}\left(\frac{k_{h_{2n-1}}}{N}\right)B_{V_{h_{2n-1}}}f_{h_{2n-1}}\left(\frac{k_{h_{2n-1}}}{N}\right)\right]$$

Taking the limit for $N\to\infty$, the first factor converges to 1, thanks to (4.1), while, for the remaining others, we recognize in each of them, a Riemann-Lebesgue sum. Hence we have

$$\prod_{\substack{p=1\\ p\in 2\mathbb{N}+1}}^{2n}\left(\int_0^1\overline{f}_{h_{p+1}}(x)B_{V_{h_p}}f_{h_p}(x)dx\right)=\prod_{\substack{p=1\\ p\in 2\mathbb{N}+1}}^{2n}\left\langle f_{h_{p+1}},B_{V_{h_p}}f_{h_p}\right\rangle_{\mathfrak{H}}$$

By definition of annihilation, creation and preservation operators, the quantity above is equal to

$$\prod_{\substack{p=1\\ p\in 2\mathbb{N}+1}}^{2n}\left\langle A\left(f_{h_{p+1}}\right)\Lambda\left(B_{V_{h_p}}\right)A^+\left(f_{h_p}\right)\right\rangle=\prod_{\substack{p=1\\ p\in 2\mathbb{N}+1}}^{2n}\left\langle A\left(f_{h_{p+1}}\right)A^+\left(B_{V_{h_p}}f_{h_p}\right)\right\rangle$$

By applying the Boolean counterpart of Lemma 3.3, we finally obtain

$$\langle A^{\varepsilon_m}(f_m,B_m)\cdots A^{\varepsilon_1}(f_1,B_1)\rangle$$

□

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