BOLLETTINO
UNIONE MATEMATICA ITALIANA

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 1149–1182.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_1149_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 1149-1182

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Sunto. – *Usando la teoria delle aggiunte e aggiunte pluricanoniche si costruiscono tre varietà tridimensionali, come desingolarizzazioni di ipersuperficie di ordine* 6 *in* $\mathbb{P}^4$, *aventi le irregolarità* $q_1 = q_2 = 0$ *e, rispettivamente, le seguenti sequenze periodiche di plurigeneri*

$$(p_g, P_2, P_3, \ldots, P_m, \ldots) = (0,0,1,0,0,1,\ldots), (0,0,0,1,0,0,0,1,\ldots),$$
$$(0,0,0,0,1,0,0,0,0,1,\ldots).$$

Nell'Appendice, a partire dal secondo esempio di sopra, si costruisce una varietà di tipo generale con $q_1 = q_2 = 0$, $p_g = 1$, $P_2 = 2$ *la cui trasformazione* m*-canonica è birazionale se e solo se* $m \geq 11$.

Summary. – *Using the theory of adjoints and pluricanonical adjoints, we construct three nonsingular threefolds, as desingularizations of degree six hypersurfaces in* $\mathbb{P}^4$, *having the irregularities* $q_1 = q_2 = 0$ *and the following periodical sequences of plurigenera respectively*

$$(p_g, P_2, P_3, \ldots, P_m, \ldots) = (0,0,1,0,0,1,\ldots), (0,0,0,1,0,0,0,1,\ldots),$$
$$(0,0,0,0,1,0,0,0,0,1,\ldots).$$

In the Appendix, starting from the second above-mentioned example, we construct a threefold of general type with $q_1 = q_2 = 0$, $p_g = 1$, $P_2 = 2$ *whose* m*-canonical transformation is birational if and only if* $m \geq 11$.

Introduction.

L. Godeaux constructed nonsingular, algebraic threefolds Y_2, Y_3, Y_5 such that $2K_{Y_2} \equiv 0$ and $K_{Y_2} \not\equiv 0$, $3K_{Y_3} \equiv 0$ and $K_{Y_3} \not\equiv 0$, $5K_{Y_5} \equiv 0$ and $K_{Y_5} \not\equiv 0$, where K_{Y_i} is a canonical divisor on Y_i, $i = 2, 3, 5$, and "$\equiv$" denotes linear equivalence (cf. [G_1, G_2, G_3]).

By the Riemann-Roch theorem, it is not difficult to see that the first irregularity is $q_1(Y_i) = \dim_k H^1(Y_i, \mathcal{O}_{Y_i}) > 0$ for the three examples, $i = 2, 3, 5$. The Kodaira dimension of these threefolds is zero. As for the m-genus $P_m(Y_i) = \dim_k H^0(Y_i, \mathcal{O}_{Y_i}(mK_{Y_i}))$, $i = 2, 3, 5$, of the above threefolds, we find that

- $P_{2j}(Y_2) = 1, \forall\, j$, and $P_n(Y_2) = 0$ for $n \neq 2j$;
- $P_{3j}(Y_3) = 1, \forall\, j$, and $P_n(Y_3) = 0$ for $n \neq 3j$;
- $P_{5j}(Y_5) = 1, \forall\, j$, and $P_n(Y_5) = 0$ for $n \neq 5j$.

This prompts the question of whether nonsingular threefolds Z_2, Z_3, Z_5 having the first irregularity $q_1(Z_i) = 0$, $i = 2, 3, 5$ and the above respective plurigenera actually exist, i.e.

- $P_{2j}(Z_2) = 1, \forall\, j$, and $P_n(Z_2) = 0$ for $n \neq 2j$;
- $P_{3j}(Z_3) = 1, \forall\, j$, and $P_n(Z_3) = 0$ for $n \neq 3j$;
- $P_{5j}(Z_5) = 1, \forall\, j$, and $P_n(Z_5) = 0$ for $n \neq 5j$.

It should be noted that they again have Kodaira dimension zero.

Among many other constructions, in $[U]$ K. Ueno presented a threefold Z_2 with the above properties, as well as $q_2(Z_2) = \dim_k H^2(Z_2, \mathcal{O}_{Z_2}) = 0$, thus answering in this way to the above question relating to the existence of the first threefold. Other threefolds Z_2, with the same properties as Ueno's example, were subsequently constructed by S. Chiaruttini and M. C. Ronconi (cf. $[CR]$), and by M. C. Ronconi (cf. ICM-1998, Short Comm.).

In the present paper, we affirmatively answer the question of whether threefolds Z_3 and Z_5 with the above properties and also having $q_2(Z_i) = 0$, $i = 3, 5$ exist. In addition, we fill the gap between Z_3 and Z_5 by constructing a nonsingular threefold Z_4 having $q_1(Z_4) = q_2(Z_4) = 0$, $P_{4j}(Z_4) = 1, \forall\, j$, and $P_n(Z_4) = 0$ for $n \neq 4j$, which is missing in the parallel constructions by Godeaux. Moreover, to our knowledge, there are no examples in the literature of nonsingular threefolds Y_4 with $4K_{Y_4} \equiv 0$ and $2K_{Y_4} \not\equiv 0$.

Concerning the above examples Z_3, Z_4 and Z_5, there remains the problem of how to establish explicitly their structure in terms of the existence of Iitaka fibrations on them.

The only explicit result, with regard to Iitaka fibrations, is the existence of a net of elliptic curves on Z_3 (see section 7).

On the matter of the existence of Iitaka fibrations, M. C. Ronconi is studying which properties of Enriques surfaces have analogues among the threefolds with $q_1 = q_2 = p_g = 0, P_{2i} = 1$ and $P_{2i+1} = 0$ (cf. ICM-1998, Short Comm.).

In the construction of our threefolds Z_3, Z_4 and Z_5, we use the theory of adjoints and pluricanonical adjoints developed in $[S_1]$. Said theory enables us to construct the three nonsingular threefolds as desingularizations of degree six hypersurfaces in $\mathbb{P}^4$ endowed with suitable singularities. We can apply said theory because the singularities on the three hypersurfaces satisfy the hypotheses of $[S_1]$, i.e. it must be possible to resolve the singularities on the threefolds with local blow-ups along linear affine subspaces; moreover, the degree six hypersurfaces in $\mathbb{P}^4$ must have singularities of codimension ≥ 2 (that is the hypersurfaces must be normal). In the construction, we have to solve two problems: the first is to find suitable singularities such that we have $p_g = P_2 = 0$ and $P_3 = 1$, $p_g = P_2 = P_3 = 0$ and $P_4 = 1$, $p_g = P_2 = P_3 = P_4 = 0$ and $P_5 = 1$ respectively; the second is to prove that the above sequences of plurigenera are periodical (see sections 5, 12 and 18).

The ground field k is an algebraically closed field of characteristic zero, which we may assume to be the field of complex numbers.

The example Z_5 partially answers the following question on nonsingular threefolds: which is the minimum integer m_0 such that $q_1 = q_2 = p_g = P_2 = P_3 = \cdots = P_{m_0} = 0 \Rightarrow P_m = 0, \forall m$? That is to say, the example tells us that $m_0 \geq 6$. The solution to the above problem is still unknown; for instance, we do not know whether a threefold with $q_1 = q_2 = p_g = 0$ and $P_{6i} = 1$, $P_n = 0$ if $n \neq 6i, i \geq 1$ exists. All the examples we have constructed in this direction (either published or not) satisfy the implication $q_1 = q_2 = p_g = P_2 = \cdots = P_5 = 0 \Rightarrow P_m = 0, \forall m$.

In the Appendix, with a construction similar to that of Z_4, but imposing only four of the five singularities imposed on the hypersurface in $\mathbb{P}^4$, we construct a nonsingular threefold X of general type having $q_1(X) = q_2(X) = 0$, $p_g(X) = 1$, $P_2(X) = 2$, whose m-canonical transformation is birational if and only if $m \geq 11$. From a result provided by M. Chen [C], we know that a threefold, with the bigenus $P_2 \geq 2$, has the m-canonical transformation which is birational for $m \geq 16$. As a consequence, the sharpest limitation, for the birationality of the m-canonical transformation for threefolds with $P_2 = 2$, is now between 11 and 16.

We note that the threefolds constructed here have no analogues among surfaces, in the sense that there are no regular nonsingular surfaces having one of the above sequences of plurigenera; in fact, according to Castelnuovo's criterion of rationality, a regular nonsingular surface with the bigenus $P_2 = 0$ is rational. Moreover, to the best of the author's knowledge, no examples of threefolds with the same above birational invariants of Z_3, Z_4, Z_5 are available in the literature.

This paper is organized as follows: we construct Z_3 in sections 1-7, Z_4 in sections 8-13 and Z_5 in sections 14-19.

CONSTRUCTION OF THE FIRST THREEFOLD Z_3.

1. – Imposing singularities on a degree six hypersurface V in $\mathbb{P}^4$.

Let $(x_0, x_1, x_2, x_3, x_4)$ be homogeneous coordinates in $\mathbb{P}^4$ and let us indicate as $f_6(X_0, X_1, X_2, X_3, X_4)$ a form (homogeneous polynomial) of degree 6, in the variables X_0, X_1, X_2, X_3, X_4, defining a hypersurface $V \subset \mathbb{P}^4$ of degree six. We impose a triple point on V at each of the three vertices $A_0 = (1,0,0,0,0)$, $A_1 = (0,1,0,0,0)$ and $A_3 = (0,0,0,1,0)$, and an ordinary 4-ple (quadruple) point at each of the remaining two vertices $A_2=(0,0,1,0,0,)$, $A_4=(0,0,0,0,1)$ of the fundamental tetrahedron $X_0X_1X_2X_3X_4=0$.

The equation for V, with the imposed singularities, is of the following type

$$\begin{aligned} V: & f_6(X_0,X_1,X_2,X_3,X_4) \\ & = X_0^3(a_{33000}X_1^3+\ldots)+X_1^3(a_{23100}X_0^2X_2+\ldots)+X_2^2(\ldots)+X_3^3(\ldots)+X_4^2(\ldots) \\ & + a_{22200}X_0^2X_1^2X_2^2+a_{22110}X_0^2X_1^2X_2X_3+\ldots+a_{00222}X_2^2X_3^2X_4^2=0, \end{aligned}$$

where $a_{ijkhl}\in k$ denotes the coefficient of the monomial $X_0^iX_1^jX_2^kX_3^hX_4^l$.

We impose an infinitely near triple line r_i at the point A_i, $i=0,1,3$, in the first neighbourhood. We follow the same method as in [S_1], section 5, and impose the same triple line r_0 infinitely near A_0. To be more precise, let us consider the affine open set $U_0\ni A_0$ in $\mathbb{P}^4$ given by $X_0\neq 0$ of affine coordinates $\left(\frac{x_1}{x_0},\frac{x_2}{x_0},\frac{x_3}{x_0},\frac{x_4}{x_0}\right)$. The affine equations of $V\cap U_0$ is given by $f_6(1,x,y,z,t)=0$, where $x=\frac{X_1}{X_0}$, $y=\frac{X_2}{X_0}$, $z=\frac{X_3}{X_0}$, $t=\frac{X_4}{X_0}$.

The affine coordinates of A_0 are $(0,0,0,0)$. So, the blow-up of $\mathbb{P}^4$ at the point A_0 is locally given by the formulas:

$$\mathcal{B}_{x_1}:\begin{cases}x=x_1\\ y=x_1y_1\\ z=x_1z_1\\ t=x_1t_1\end{cases};\quad \mathcal{B}_{y_2}:\begin{cases}x=x_2y_2\\ y=y_2\\ z=y_2z_2\\ t=y_2t_2\end{cases};\quad \mathcal{B}_{z_3}:\begin{cases}x=x_3z_3\\ y=y_3z_3\\ z=z_3\\ t=z_3t_3\end{cases};\quad \mathcal{B}_{t_4}:\begin{cases}x=x_4t_4\\ y=y_4t_4\\ z=z_4t_4\\ t=t_4\end{cases}$$

and we consider $\mathcal{B}_{t_4}$. The strict (or proper) transform V' of V with respect to the local blow-up $\mathcal{B}_{t_4}$ has an affine equation given by

$$\frac{1}{t_4^3}f_6(1,x_4t_4,y_4t_4,z_4t_4,t_4)=a_{31200}x_4y_4^2+\ldots+a_{00222}y_4^2z_4^2t_4^3=0.$$

On this threefold V' we impose the triple line given affinely by $\begin{cases}x_4=0\\ y_4=0\\ t_4=0\end{cases}$ (i.e. we make this line a locus of triple points on V'). Therefore, the conditions on the coefficients a_{ijkhl}, for V to have the triple line r_0 infinitely near are the same as in [S_1], p. 176, given by

$a_{32010}=0$	$a_{30201}=0$	$a_{30003}=0$	$a_{20013}=0$
$a_{32001}=0$	$a_{30120}=0$	$a_{21030}=0$	$a_{10023}=0$
$a_{31110}=0$	$a_{30111}=0$	$a_{20130}=0$	$a_{21021}=0$
$a_{31101}=0$	$a_{30102}=0$	$a_{20031}=0$	$a_{21012}=0$
$a_{31020}=0$	$a_{30030}=0$	$a_{10032}=0$	$a_{20121}=0$
$a_{31011}=0$	$a_{30021}=0$	$a_{21003}=0$	$a_{20112}=0$
$a_{31002}=0$	$a_{30012}=0$	$a_{20103}=0$	$a_{20022}=0$
$a_{30210}=0$			

To tell the truth, some of these coefficients are already zero, having imposed two 4-ple points on V; they are the coefficients of the type a_{ij3hl} or a_{ijkh3}. Neverthless, we have written said coefficients here to ensure that we have correct results in the following rotations of indices and variables.

Again according to [S_1], we impose an infinitely near triple line r_i at A_i, for i=1,3, with suitable rotations of the indices $ijkhl$ of the coefficient a_{ijkhl} and of the corresponding variables $a_{ijkhl}X_0^iX_1^jX_2^kX_3^hX_4^l$.

The rotations of the indices and variables passing from A_0 to A_1 and from A_1 to A_3, are as follows.

Rotations of indices (and variables)

$$A_0 \longmapsto A_1 \longmapsto A_3$$
$$ijkhl \longmapsto lijkh \longmapsto khlij$$

We give the final equation for our hypersurface V, after imposing all the above-mentioned singularities.

$$\begin{aligned}
&V : f_6(X_0, X_1, X_2, X_3, X_4) = \\
&a_{31200}X_0^3X_1X_2^2+ \\
&a_{03210}X_1^3X_2^2X_3+ \\
&a_{10230}X_0X_2^2X_3^3+ \\
&a_{22200}X_0^2X_1^2X_2^2 + a_{22110}X_0^2X_1^2X_2X_3 + a_{22020}X_0^2X_1^2X_3^2 + a_{21210}X_0^2X_1X_2^2X_3+ \\
&a_{21201}X_0^2X_1X_2^2X_4 + a_{21120}X_0^2X_1X_2X_3^2 + a_{21111}X_0^2X_1X_2X_3X_4 + a_{21102}X_0^2X_1X_2X_4^2+ \\
&a_{20220}X_0^2X_2^2X_3^2 + a_{20211}X_0^2X_2^2X_3X_4 + a_{20202}X_0^2X_2^2X_4^2 + a_{12210}X_0X_1^2X_2^2X_3+ \\
&a_{12201}X_0X_1^2X_2^2X_4 + a_{12120}X_0X_1^2X_2X_3^2 + a_{12111}X_0X_1^2X_2X_3X_4 + a_{11220}X_0X_1X_2^2X_3^2+ \\
&a_{11211}X_0X_1X_2^2X_3X_4 + a_{11202}X_0X_1X_2^2X_4^2 + a_{11121}X_0X_1X_2X_3^2X_4 + a_{11112}X_0X_1X_2X_3X_4^2+ \\
&a_{10221}X_0X_2^2X_3^2X_4 + a_{10212}X_0X_2^2X_3X_4^2 + a_{10122}X_0X_1X_3^2X_4^2 + a_{02220}X_1^2X_2^2X_3^2+ \\
&a_{02211}X_1^2X_2^2X_3X_4 + a_{02202}X_1^2X_2^2X_4^2 + a_{02112}X_1^2X_2X_3X_4^2 + a_{01221}X_1X_2^2X_3^2X_4+ \\
&a_{01212}X_1X_2^2X_3X_4^2 + a_{00222}X_2^2X_3^2X_4^2 = 0.
\end{aligned}$$

Several coefficients can be chosen as equal to zero because they are inessential for the computation of the birational invariants of a desingularization Z_3 of V. The shortest equation with the essential coefficients is

$$\begin{aligned}
&V : f_6(X_0, X_1, X_2, X_3, X_4) = \\
&a_{31200}X_0^3X_1X_2^2+ \\
&a_{03210}X_1^3X_2^2X_3+ \\
&a_{10230}X_0X_2^2X_3^3+ \\
&a_{22200}X_0^2X_1^2X_2^2 + a_{22020}X_0^2X_1^2X_3^2 + a_{21102}X_0^2X_1X_2X_4^2 + a_{20220}X_0^2X_2^2X_3^2+ \\
&a_{20202}X_0^2X_2^2X_4^2 + a_{10122}X_0X_1X_3^2X_4^2 + a_{02220}X_1^2X_2^2X_3^2 + a_{02202}X_1^2X_2^2X_4^2+ \\
&a_{02112}X_1^2X_2X_3X_4^2 + a_{00222}X_2^2X_3^2X_4^2 = 0.
\end{aligned}$$

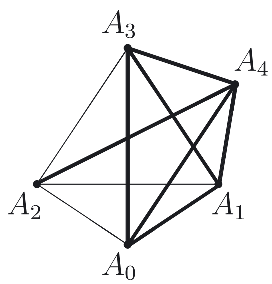
From here on, V denotes this last hypersurface defined by the above last form $f_6(X_0, X_1, X_2, X_3, X_4)$ for a generic choice of the parameters a_{ijkhl}.

2. – Imposed and unimposed singularities of *V*: the actual singularities.

We consider the hypersurface V at the end of section 1.

New singularities appear on the generic V close to the singularities imposed on V; they are actual or infinitely near singularities. We call a singularity on V *actual* to distinguish it from those infinitely near.

There are seven actual double (straight) lines on V given by A_0A_1, A_0A_3, A_0A_4, A_1A_3, A_1A_4, A_2A_4 and A_3A_4, according to the following picture, where the double lines are drawn in bold type.



The generic V has no other actual singularities. It follows that the generic V is reduced, irreducible and normal.

3. – The infinitely near singularities of *V*.

In section 2, we described the actual singularities on V; in the present section, we describe the infinitely near singularities (whether they are imposed or not). To do so, we need the

RESOLUTION OF SINGULARITIES OF V

The desingularization of V is very long, but also very easy. In section 1, we imposed a triple line infinitely near the triple point A_0. Said computation can be interpreted as the beginning of a desingularization of A_0 and of the singularities infinitely near A_0.

As an example, we solve only the singularities of V belonging to the affine open set $U_2 = \{X_2 \neq 0\}$.

Blow-up of $\mathbb{P}^4$ at A_2

Here, we can assume that the blow-up of A_2 is the first that we perform. So, let $\pi : \mathbb{P}_1 \longrightarrow \mathbb{P}^4$ be the blow-up of $\mathbb{P}^4$ at A_2 and let $U_2 \ni A_2$ be the affine open set in $\mathbb{P}^4$ given by $X_2 \neq 0$ of affine coordinates $\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_3}{x_2}, \frac{x_4}{x_2}\right)$.

We use V_{U_2} to indicate the affine threefold $V \cap U_2$ of the affine equation $f_6(x, y, 1, z, t) = 0$, where $x = \frac{X_0}{X_2}, y = \frac{X_1}{X_2}, z = \frac{X_3}{X_2}, t = \frac{X_4}{X_2}$.

The point A_2 has affine coordinates $(0, 0, 0, 0)$.

The blow-up π at the point A_2 is locally given by the same formulas as in section 1: $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$.

- *The strict* (or proper) *transform of* V_{U_2}*, with respect to* $\mathcal{B}_{x_1}$*, is given by*

$$V_{x_1} : \frac{1}{x_1^4} f_6(x_1, x_1 y_1, 1, x_1 z_1, x_1 t_1)$$
$$= a_{31200} y_1 + a_{03210} y_1^3 z_1 + a_{10230} z_1^3 + a_{22200} y_1^2 + \cdots + a_{20202} t_1^2 + \cdots = 0.$$

We are interested in the singularities infinitely near A_2, i.e. we focus on the singularities on V_{x_1} belonging to the exceptional divisor E_1 of the blow-up π. Locally, an equation of E_1 is given by $x_1 = 0$.

The (incomplete) linear system defining V_{x_1} has the base points on $x_1 = 0$ given by the unique point $O = (0, 0, 0, 0)$. According to Bertini, *the singularities on* $x_1 = 0$ *of the* (generic) V_{x_1} *are only on the base points of the linear system*, i.e. only O can be a singular point. But we have

$$\left(\frac{\partial V_{x_1}}{\partial y_1}\right)_O = a_{31200} \neq 0.$$

So, there are no singularities on V_{x_1} infinitely near A_2.

The base points outside the exceptional divisor $x_1 = 0$ are given by the line $\begin{cases} y_1 = 0 \\ z_1 = 0 \\ t_1 = 0 \end{cases}$. According to Bertini, the possible singularities on V_{x_1} belong to this line. But we have

$$\left(\frac{\partial V_{x_1}}{\partial y_1}\right)_{y_1 = z_1 = t_1 = 0} = a_{31200} \neq 0.$$

In conclusion, V_{x_1} *is nonsingular.*

Likewise, we see that

- *the strict transform of* V_{U_2} *with respect to* $\mathcal{B}_{y_2}$ *is again nonsingular*;
- *the strict transform of* V_{U_2} *with respect to* $\mathcal{B}_{z_3}$ *is also nonsingular.*

Next, we find that

- *the strict transform of V_{U_2} with respect to $\mathcal{B}_{t_4}$ is given by*

$$V_{t_4} : \frac{1}{t_4^4} f_6(x_4t_4, y_4t_4, 1, z_4t_4, t_4)$$
$$= a_{31200}x_4^3y_4 + a_{03210}y_4^3z_4 + a_{10230}x_4z_4^3 + a_{22200}x_4^2y_4^2 + \cdots + a_{20202}x_4^2 + \cdots + a_{02202}y_4^2$$
$$+ \cdots + a_{00222}z_4^2 = 0$$

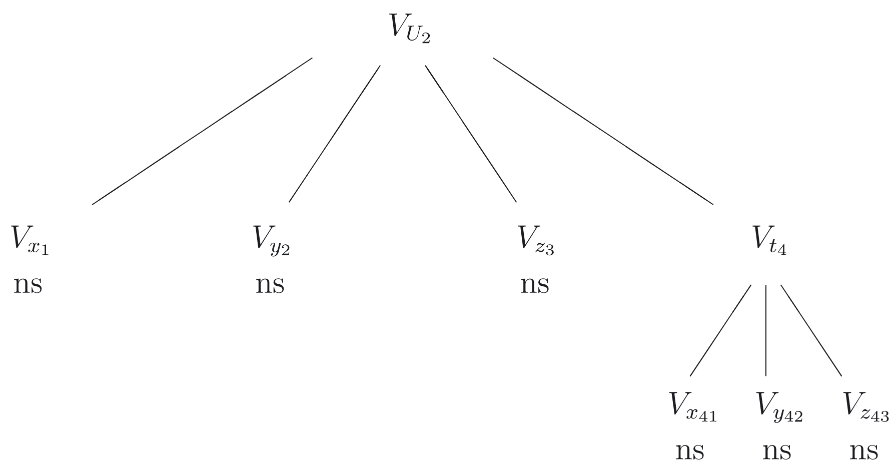
and it has the double singular line $\ell : \begin{cases} x_4 = 0 \\ y_4 = 0 \\ z_4 = 0 \end{cases}$ as unique singularity. This line ℓ is the strict transform of the actual line $A_2A_4 \cap U_2$ on V_{U_2}.

It is easy to check that, thanks to the presence in the equation of the addenda $a_{20202}x_4^2$, $a_{02202}y_4^2$, $a_{00222}z_4^2$, the line ℓ is resolved with only one blow-up.

By patching $V_{x_1}, V_{y_2}, V_{z_3}, V_{t_4}$ together, from general blow-up theory, we obtain a threefold $V' \cap U'$ defined on $U' = \pi^{-1}(U_2)$, where V' is the strict transform of V with respect to the blow-up $\pi : \mathbb{P}_1 \longrightarrow \mathbb{P}^4$ of $\mathbb{P}^4$ at A_2. We can cover $V' \cap U'$ with $V_{x_1}, V_{y_2}, V_{z_3}, V_{t_4}$. Thus, locally, we can blow up ℓ on V_{t_4}.

Conclusion. *We have resolved the singularities of $V_{U_2} = V \cap U_2$.*

The tree of the blow-ups is as follows



where “ns” means “nonsingular”.

By doing long and tedious calculations similar to those above and in $[S_1]$, we obtain a desingularization of $V \subset \mathbb{P}^4$. In this desingularization, we can see that new unimposed infinitely near singularities also appear on V among the imposed infinitely near singularities. They are only double singular curves and isolated double points. So, none of the unimposed singularities affect the birational invariants of a desingularization Z_3 of V, such as the irregularities and the plurigenera of Z_3. This means that, in calculating these invariants, we can assume that there are only the imposed singularities on V.

Having said as much, we consider the desingularization of V as achieved.

4. – The m-canonical adjoints to V $\subset \mathbb{P}^4$

Let

$$\mathbb{P}_r \xrightarrow{\pi_r} \ldots \xrightarrow{\pi_3} \mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}_0 = \mathbb{P}^4$$

be a sequence of blow-ups solving the singularities of V.

If we call $V_i \subset \mathbb{P}_i$ the *strict transform* of V_{i-1} with respect to π_i, then we obtain from the above sequence

$$Z_3 = V_r \xrightarrow{\pi'_r} \cdots \xrightarrow{\pi'_3} V_2 \xrightarrow{\pi'_2} V_1 \xrightarrow{\pi'_1} V_0 = V,$$

where $\pi'_i = \pi_{i|V_i} : V_i \longrightarrow V_{i-1}$ and $\sigma_{|_{Z_3}} : Z_3 \longrightarrow V$, $\sigma = \pi_r \circ \cdots \circ \pi_1$, is a desingularization of $V \subset \mathbb{P}^4$.

Let us assume that π_i is a blow-up along a subvariety Y_{i-1} of $\mathbb{P}_{i-1}$, of dimension j_{i-1}, which can be either a singular or a nonsingular subvariety of $V_{i-1} \subset \mathbb{P}_{i-1}$ (i.e. Y_{i-1} is a locus of singular or simple points of V_{i-1}). Let m_{i-1} be the multiplicity of the variety Y_{i-1} on V_{i-1}.

Let us set $n_{i-1} = -3 + j_{i-1} + m_{i-1}$, for $i = 1, \ldots, r$ and $\deg(V) = d$.

A hypersurface $\Phi_{m(d-5)}$ of degree $m(d-5)$ in $\mathbb{P}^4$ is an *m-canonical adjoint* to V (with respect to the sequence of blow-ups $\pi_1, \ldots, \pi_r$) if the restriction to Z_3 of the divisor

$$D_m = \pi_r^*\{\pi_{r-1}^*[\ldots\pi_1^*(\Phi_{m(d-5)}) - mn_0E_1\ldots] - mn_{r-2}E_{r-1}\} - mn_{r-1}E_r$$

is effective, i.e. $D_{m|_{Z_3}} \geq 0$, where $E_i = \pi^{-1}(Y_{i-1})$ is the exceptional divisor of π_i and $\pi_i^* : Div(\mathbb{P}_{i-1}) \longrightarrow Div(\mathbb{P}_i)$ is the homomorphism of the Cartier (or locally principal) divisor groups (cf. [S_1], sections 1,2).

An m-canonical adjoint $\Phi_{m(d-5)}$ is a *global* m-canonical adjoint to V (with respect to $\pi_1, \ldots, \pi_r$) if the divisor D_m is effective on $\mathbb{P}_r$, i.e. $D_m \geq 0$ (loc. cit.).

Note that, if $\Phi_{m(d-5)}$ is an m-canonical adjoint to V, then $D_{m|_{Z_3}} \equiv mK$, where '$\equiv$' denotes linear equivalence and K denotes a canonical divisor on Z_3.

In our above example, an order can be established in the sequence of blow-ups, e.g. let us assume that the blow-up π_1 is the blow-up at the 4-ple point A_2, π_2 is the blow-up at the 4-ple point A_4, π_3 is the blow-up at the triple point A_0, π_4 is the blow-up along the triple curve infinitely near A_0, π_5 is the blow-up at the triple point A_1, π_6 is the blow-up along the triple curve infinitely near A_1, π_7 is the blow-up at the triple point A_3, and π_8 is the blow-up along the triple curve infinitely near A_3.

The example V has degree $d = 6$ and D_m is given by:

$$(*)\ D_m = \pi_r^*\ldots\{\pi_2^*[\pi_1^*(\Phi_m) - mE_1] - mE_2\} - mE_4 - mE_6 - mE_8 + \sum mE,$$

where E_i is the exceptional divisor of the blow-up π_i and, to be more specific, E_1 is the exceptional divisor of the blow-up π_1 at the 4-fold point A_2 , ... and E_8 is the exceptional divisor of the blow-up π_8 along the triple curve infinitely near A_3.

No other exceptional divisors are subtracted in D_m because, as we said, the unimposed singularities are either actual or infinitely near double singular curves or isolated double points on our (generic) V. Put more precisely, the exceptional divisors of the blow-ups along the curves appear with coefficient $n_h = 0$ in the above expression of D_m, whereas the exceptional divisors of the blow-ups at double points appear with coefficient $n_h = -1$: we have indicated these divisors as $\sum mE$. In addition, note that the exceptional divisor of a blow-up at a triple point also appears with coefficient $n_h = 0$ in D_m. From here on, we omit writing $\sum mE$, because they are not essential to the computation of the birational invariants that we shall consider.

5. – The plurigenera of a desingularization $\mathbf{Z_3}$ of V.

Let $\sigma_{|_{Z_3}} : Z_3 \longrightarrow V$ be a desingularization of our hypersurface $V \subset \mathbb{P}^4$, where $\sigma = \pi_r \circ \cdots \circ \pi_1$ (section 4).

PROPOSITION 1. – *The plurigenera of* Z_3 *are given by* $P_{3i} = 1$, $\forall i \geq 1$, *and* $P_m = 0$ *if* $m \neq 3i$.

PROOF. – Let us consider the equation of V: $f_6(X_0, X_1, X_2, X_3, X_4) = 0$ at the end of section 1, and we arrange the form f_6 according to the powers of X_2.

$$f_6 = \varphi_4(X_0, X_1, X_3, X_4)X_2^2 + \varphi_5(X_0, X_1, X_3, X_4)X_2 + \varphi_6(X_0, X_1, X_3, X_4) = 0,$$

where $\varphi_i(X_0, X_1, X_3, X_4)$ is a form of degree i in X_0, X_1, X_3, X_4.

Next, let us consider the hypersurface Φ_m, appearing in (*) section 4 and assume that its equation is $F_m(X_0, X_1, X_2, X_3, X_4) = 0$, of degree m. Arranging the form F_m according to the powers of X_2, we can write

$$\begin{aligned} &F_m(X_0, X_1, X_2, X_3, X_4) \\ &= \psi_s(X_0, X_1, X_3, X_4)X_2^{m-s} + \psi_{s+1}(X_0, X_1, X_3, X_4)X_2^{m-s-1} + \cdots \\ &\quad + \psi_m(X_0, X_1, X_3, X_4), \end{aligned}$$

where $\psi_j(X_0, X_1, X_3, X_4)$ is a form of degree j in X_0, X_1, X_3, X_4 and s is an integer satisfying $0 \leq s \leq m$. So, Φ_m has at A_2 an s-ple point, with $0 \leq s \leq m$.

We need the preliminary result, concerning Φ_m, given by the following

LEMMA 1. – *If* Φ_m *is an m-canonical adjoint* (either global or not), *then, modulo* $V : f_6 = 0$, *we can assume* $s \geq m - 1$; i.e *if* Φ_m *is an m-canonical* *adjoint, then we can assume that its equation, modulo* $f_6 = 0$*, is defined by the form*

$$F_m = \psi_{m-1}(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4).$$

The idea for the proof is due to M. C. Ronconi [*CR*], [*Ro*].

PROOF OF LEMMA 1. For $m = 1$, the lemma is trivial, so we assume $m \geq 2$. For the same reason, considering $\Phi_m : \psi_s X_2^{m-s} + \psi_{s+1} X_2^{m-s-1} + \cdots + \psi_m$, we assume $s \leq m - 2$.

Let us consider the affine open set U_2 given by $X_2 \neq 0$ of affine coordinates $\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_3}{x_2}, \frac{x_4}{x_2}\right)$. Let us write the affine equation of $V \cap U_2 = V_{U_2}$ as follows:

$$V_{U_2} : \; f_6(x, y, 1, z, t) = \varphi_4(x, y, z, t) + \varphi_5(x, y, z, t) + \varphi_6(x, y, z, t) = 0$$

and the local equation of Φ_m:

$$\Phi_{mU_2} : F_m(x, y, 1, z, t) = \psi_s(x, y, z, t) + \cdots + \psi_m(x, y, z, t) = 0,$$

where $x = \dfrac{X_0}{X_2}, y = \dfrac{X_1}{X_2}, z = \dfrac{X_3}{X_2}, t = \dfrac{X_4}{X_2}$.

Let us consider the first blow-up π_1 at A_2. One of the local espressions of π_1 is given by $\mathcal{B}_{z_3} : \begin{cases} x = x_3 z_3 \\ y = y_3 z_3 \\ z = z_3 \\ t = z_3 t_3 \end{cases}$. The strict transform V_{z_3} of $V_{U_2} = V \cap U_2$ with respect to $\mathcal{B}_{z_3}$ is given by

$$V_{z_3} : \frac{1}{z_3^4} f_6(x_3 z_3, y_3 z_3, 1, z_3, z_3 t_3) = f_{U_2 z_3}(x_3, y_3, z_3, t_3)$$
$$= \varphi_4(x_3, y_3, 1, t_3) + \varphi_5(x_3, y_3, 1, t_3) z_3 + \varphi_6(x_3, y_3, 1, t_3) z_3^2 = 0$$

Now, let us consider the **total** transform of Φ_{mU_2} with respect to $\mathcal{B}_{z_3}$. We note that $\mathcal{B}_{z_3}$ locally coincides, up to isomorphisms, with its total transform with respect to the desingularization $\sigma_{|_{Z_3}}$ (because V_{z_3} is nonsingular, see the tree of blow-ups in section 3). This total transform of Φ_{mU_2} is

$$\Phi^*_{mU_2} : \; z_3^s[\psi_s(x_3, y_3, 1, t_3) + \cdots + \psi_m(x_3, y_3, 1, t_3) z_3^{m-s}] = 0.$$

Next, if Φ_m is an m-canonical adjoint to V, then, from (*) and the definition of m-canonical adjoint to V (section 4), we can deduce in particular

$$\left[\Phi^*_{mU_2} - m(E_{1|_{V_{z_3}}})\right]_{|_{V_{z_3}}} \geq 0,$$

where E_1 is the exceptional divisor of π_1. This inequality, translated in terms of polynomials, gives us

$$z_3^s[\psi_s(x_3,y_3,1,t_3)+\cdots+\psi_m(x_3,y_3,1,t_3)z_3^{m-s}]$$

$$+B(x_3,y_3,z_3,t_3)[\varphi_4(x_3,y_3,1,t_3)+\varphi_5(x_3,y_3,1,t_3)z_3+\varphi_6(x_3,y_3,1,t_3)z_3^2]=z_3^m(\cdots),$$

where $B(x_3,y_3,z_3,t_3)$ is a suitable polynomial. In this equality of polynomials, we have $B(x_3,y_3,z_3,t_3)=z_3^sB'(x_3,y_3,z_3,t_3)$, so, we can simplify z_3^s and we put $z_3=0$ in the remaining equality, obtaining the equality of polynomials

$$(**)\qquad \psi_s(x_3,y_3,1,t_3)=B'(x_3,y_3,z_3,0)\varphi_4(x_3,y_3,1,t_3).$$

Multiplying both sides of (**) by z_3^s, from the equalities in $\mathcal{B}_{z_3}$, we obtain

$$\psi_s(x,y,z,t)=B''(x,y,z,t)\varphi_4(x,y,z,t);$$

and by homegenizing, we return to the forms

$$(***)\qquad \psi_s(X_0,X_1,X_3,X_4)=B''(X_0,X_1,X_3,X_4)\varphi_4(X_0,X_1,X_3,X_4),$$

where $B''(X_0,X_1,X_3,X_4)$ is a form of degree $s-4$ in X_0,X_1,X_3,X_4.

At this point, we consider [S_1], Corollary 8, section 3: if V is normal, there is an isomorphism of projective spaces for any $m\geq 1$

$$\begin{pmatrix} \textit{linear system of} \\ \textit{m-canonical adjoints to } V \end{pmatrix}_{|_V} \longrightarrow |mK_{Z_3}|$$

$$\Phi_{m|_V} \longmapsto D_{m|_{Z_3}}\ .$$

Bearing in mind that our purpose is to compute the m-canonical genus $P_m=\dim|mK_{Z_3}|+1=\dim\ (\textit{linear system of m-canonical adjoints})_{|_V}+1$, we can substitute Φ_m with Φ'_m if $\Phi'_{m|_V}=\Phi_{m|_V}$. This is equivalent to saying that the form F'_m defining Φ'_m must be of the type $F'_m=F_m+Af_6$, where A is a form of degree $m-6$, F_m is the form defining Φ_m and f_6 is the form defining V. To be more precise, we can take F'_m given by

$$(*v)\qquad F'_m=F_m-B''(X_0,X_1,X_3,X_4)X_2^{m-s-2}f_6,$$

where $B''(X_0,X_1,X_3,X_4)$ is the form in (***). Of course, we need $m-s-2\geq 0$ for this substitution. Note that we can assume $m-s-2\geq 0$; otherwise, if $m-s-2<0$, then the Lemma is true. From (***) we obtain

$$F'_m:\psi_{s+1}(X_0,X_1,X_3,X_4)X_2^{m-s-1}+\cdots+\psi_m(X_0,X_1,X_3,X_4)$$
$$-B''(X_0,X_1,X_3,X_4)(\varphi_5X_2^{m-s-1}+\varphi_6X_2^{m-s-2})=0.$$

$\Phi'_m:F'_m=0$ has an $(s+1)$-ple point at A_2. So we can iterate the process and substitute Φ'_m with a new m-canonical adjoint Φ''_m having an $(s+2)$-ple point at

A_2, and so on. Since the inequality $m - s - 2 \geq 0$ in ($*^v$) must hold, the iteration stops when $s > m - 2$.

This proves Lemma 1.

PROOF OF PROPOSITION 1 (continuation) (see also [*CR*]). In Lemma 1, we established that we can assume $\Phi_m : \psi_{m-1}X_2 + \psi_m = 0$. Starting from this Φ_m, and using the same arguments as in the proof of Lemma 1, we obtain a formula similar to (***), i.e.

$$(^v) \qquad \psi_{m-1}(X_0, X_1, X_3, X_4) = B'''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4),$$

where $B'''(X_0, X_1, X_3, X_4)$ is a form of degree $m - 5$. In particular, the equality (^{v}) tells us that $\psi_{m-1}(X_0, X_1, X_3, X_4)$ can be divided by $\varphi_4(\ldots)$.

Next, we order the forms f_6 and Φ_m according to the powers of X_4,

$$f_6 = \varphi_4^*(X_0, X_1, X_2, X_3)X_4^2 + \varphi_5^*(X_0, X_1, X_2, X_3)X_4 + \varphi_6^*(X_0, X_1, X_2, X_3),$$

where $\varphi_i^*(X_0, X_1, X_3, X_4)$ is a form of degree i in X_0, X_1, X_2, X_3.

$$\Phi_m : \psi_t^*(X_0, X_1, X_2, X_3)X_4^{m-t} + \psi_{t+1}^*(X_0, X_1, X_2, X_3)X_4^{m-t-1} + \cdots + \psi_m^*(X_0, X_1, X_2, X_3),$$

where $\psi_j^*(X_0, X_1, X_2, X_3)$ is a form of degree j in X_0, X_1, X_2, X_3.

Then we apply Lemma 1 to the m-canonical adjoint Φ_m and we consider its behaviour under the blow-up π_2 at the point A_4. In this case, if we change X_2 with X_4, the result of Lemma 1 or, to be more precise, the analogous equality of (^{v}), tells us that the form $\psi_t^*(X_0, X_1, X_2, X_3)$ must be divisible by $\varphi_4^*(X_0, X_1, X_2, X_3)$. But $\varphi_4^*(X_0, X_1, X_2, X_3)$ contains X_2^2, whereas $\Phi_m : \psi_{m-1}X_2 + \psi_m = 0$ contains X_2 to the power of 1. Iterating the process, this implies that $t = m$. Comparing

$$\Phi_m : B'''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4) = 0$$

and $\Phi_m : \psi'_m(X_0, X_1, X_2, X_3) = 0$ [since $\varphi_4(X_0, X_1, X_3, X_4)$ contains X_4], we obtain

$$\Phi_m : \psi''_m(X_0, X_1, X_3) + Af_6 = 0,$$

where $\psi''_m(X_0, X_1, X_3)$s is a form of degree m in the variables X_0, X_1, X_3.

We continue the proof considering the blow-up at A_0 performed at the beginning of section 1 and let $U_0 \ni A_0$ be the affine open set in $\mathbb{P}^4$ given by $X_0 \neq 0$.

We consider the local blow-up of $\mathbb{P}^4$ at the point A_0 given by $\mathcal{B}_{z_3} : \begin{cases} x = x_3 z_3 \\ y = y_3 z_3 \\ z = z_3 \\ t = z_3 t_3 \end{cases}$.

For the sake of brevity, we consider this blow-up as the first in solving the singularities of V; we can do so because the blow-ups π_i and π_j are interchangeable. The total transform of Φ_m with respect to $\mathcal{B}^*_{z_3}$ is given by $\mathcal{B}^*_{z_3}(\Phi_m) : \psi''_m(1, x_3 z_3, z_3) = 0$. Now we consider the affine triple line infinitely near A_0, given by $\begin{cases} x_3 = 0 \\ y_3 = 0 \\ z_3 = 0 \end{cases}$ and we consider the local blow-up given by $\mathcal{B}_{x_{31}}: \begin{cases} x_3 = x_{31} \\ y_3 = x_{31}y_{31} \\ z_3 = x_{31}z_{31} \\ t_3 = t_{31} \end{cases}$. The total transform of $\mathcal{B}^*_{z_3}(\Phi_m)$ with respect to $\mathcal{B}_{x_{31}}$ is given by $\mathcal{B}^*_{x_{31}}(\mathcal{B}^*_{z_3}(\Phi_m)) : \psi''_m(1, x_{31}^2 z_{31}, x_{31}z_{31}) = 0$.

Next, we write the form $\psi''_m(X_0, X_1, X_3)$ as follows

$$\psi''_m(X_0, X_1, X_3) = \sum_{i+j+h=m} c_{ijh} X_0^i X_1^j X_3^h.$$

where $c_{ijh} \in k$. We thus obtain the total transform

$$\mathcal{B}^*_{x_{31}}(\mathcal{B}^*_{z_3}(\Phi_m)) : \sum_{i+j+h=m} c_{ijh} x_{31}^{2j+h} z_{31}^{j+h} = 0.$$

If we want Φ_m to be an m-canonical adjoint, from the expression of D_m in (*), section 4, we must put x_{31}^m in evidence, modulo $V : f_6 = 0$, in the latter total transform. But, considering V_{U_0} given by

$$f_{6U_0}(x, y, z, t) = a_{31200}xy^2 + \varphi''_4(x, y, z, t) + \varphi''_5(x, y, z, t) + \varphi''_6(x, y, z, t) = 0,$$

where $\varphi''_i(x, y, z, t)$ is a form of degree i in x, y, z, t, we obtain that the strict trasform of f_{6U_0} with respect to $\mathcal{B}_{z_3}$ and $\mathcal{B}_{x_{31}}$ is

$$a_{31200}y_{31}^2 + \varphi''_4(\cdots)x_{31} + \varphi''_5(\cdots)x_{31}^2 + \varphi''_6(\cdots)x_{31}^3.$$

Given the presence of $a_{31200}y_{31}^2$, we deduce that putting x_{31}^m in evidence, modulo $V : f_6 = 0$, in $\mathcal{B}^*_{x_{31}}(\mathcal{B}^*_{z_3}(\Phi_m)) : \sum_{i+j+h=m} c_{ijh} x_{31}^{2j+h} z_{31}^{j+h} = 0$, is equivalent to put x_{31}^m in evidence without "modulo $V : f_6 = 0$". Here, as in the proof of Lemma 1, we use the fact that $\mathcal{B}_{x_{31}} \circ \mathcal{B}_{z_3}$ coincides with the desingularization $\sigma_{|_{Z_3}}$ on the affine open set $V_{x_{31}}$.

So, it remains for us to establish when we can put x_{31}^m in evidence in

$$\sum_{i+j+h=m} c_{ijh} x_{31}^{2j+h} z_{31}^{j+h} = 0,$$

without "modulo $V : f_6 = 0$" and this can be done immediately.

Conclusion. In the above polynomial, we can put x_{31}^m in evidence if and only if $2j + h \geq m$.

Finally, if we consider the triple points A_1 and A_3 as both having an infinitely near triple line, then for the singularities given by A_1 and A_3, we also get much the same results as in the above Conclusion, that we obtained for the singularity given by A_0. That is to say, we obtain $2h + i \geq m$ in the case of A_1, and $2i + j \geq m$

in the case of A_3. Adding the three inequalities, we obtain $3(i+j+h) \geq 3m$. Since $i+j+h=m$, we deduce that all the inequalities are equalities. Therefore $i=j=h$, i.e.

$$\psi''_m(X_0, X_1, X_3) = \sum_{3i=m} c_{iii} X_0^i X_1^i X_3^i = c_{iii} X_0^i X_1^i X_3^i.$$

This shows that the m-canonical adjoints to V are of the type

$$\Phi_m : c_{iii}(X_0 X_1 X_3)^i = 0,$$

for $i > 0$, $3i = m$ and Proposition 1 is proved. □

6. – Computing the irregularities of $\mathbf{Z}_3$.

There remains for us to prove that $q_i = \dim_k H^i(Z_3, \mathcal{O}_{Z_3}) = 0$, for $i = 1, 2$. We know that $q_1 = \dim_k H^1(Z_3, \mathcal{O}_{Z_3}) = q(S_r) = \dim_k H^1(S_r, \mathcal{O}_{S_r})$, where $S_r \subset Z_3$ is the strict transform of a generic hyperplane section S of V (cf. $[S_1]$, section 4, for instance). S has several isolated (actual or infinitely near) double points and no other singularities. This follows from the fact that the hypersurface V, outside the points A_0, A_1, A_2, A_3 and A_4, only has actual or infinitely near double curves. So, $q_1 = 0$.

To prove that $q_2 = 0$, we use the formula (36), section 4 in $[S_1]$, which states that:

$$q_2 = p_g(X) + p_g(S_r) - \dim_k (W_2),$$

where W_2 is the vector space of the degree 2 forms defining global adjoints Φ_2 to V, i.e. defining hyperquadrics Φ_2 such that

$$\pi_r^* \ldots \pi_2^*[\pi_1^*(\Phi_2)] - E_1 - E_2 - E_4 - E_6 - E_8 \geq 0,$$

(cf. the expression of D_m in (*), section 4). So the above hyperquadrics Φ_2 are those passing through the points A_0, A_1, A_2, A_3 and A_4. Thus, we have: $\dim_k(W_2) = 15 - 5 = 10$. It follows from $p_g(S_r) = 10$ and $p_g(X) = 0$ (cf. Proposition 1 in section 5), that $q_2 = 0$.

7. – A net of elliptic curves on $\mathbf{Z}_3$.

Let us consider the two 4-ple points A_2 and A_4 and the double line A_2A_4 on $V \subset \mathbb{P}^4$. Clearly, there is a net of planes passing through the two points (and the double line). The generic plane of the net cuts out a degree six plane curve on V, which is split into the line A_2A_4 counted twice, and into an irreducible quartic C_4 having exactly two nodes (ordinary double points) (according to Bertini). If

$\widetilde{C_4} \longrightarrow C_4$ is a desingularization of C_4, then $\widetilde{C_4}$ is an elliptic curve and we can assume $\widetilde{C_4} \subset Z_3$. This shows that *on Z_3 there is a net of elliptic curves.*

THIS COMPLETES THE CONSTRUCTION AND THE DESCRIPTION OF THE FIRST THREEFOLD Z_3.

CONSTRUCTION OF THE THREEFOLD Z_4.

8. – Imposing four triple points with an infinitely near double surface and a 4-ple point on a degree six hypersurface V′ in $\mathbb{P}^4$.

Let $(x_0, x_1, x_2, x_3, x_4)$ be homogeneous coordinates in $\mathbb{P}^4$ and let us indicate as $f_6'(X_0, X_1, X_2, X_3, X_4)$ a form (homogeneous polynomial) of degree 6, in the variables X_0, X_1, X_2, X_3, X_4, defining a hypersurface of degree six $V' \subset \mathbb{P}^4$. We impose a triple point with an infinitely near double (singular) surface on V' at each of the four vertices $A_0 = (1, 0, 0, 0, 0)$, $A_1 = (0, 1, 0, 0, 0)$, $A_3 = (0, 0, 0, 1, 0)$ and $A_4 = (0, 0, 0, 0, 1)$, with an ordinary 4-ple (quadruple) point at the remaining vertex $A_2 = (0, 0, 1, 0, 0,)$ of the fundamental tetrahedron $X_0X_1X_2X_3X_4 = 0$. We have already considered the singularity given by a triple point with an infinitely near double surface in [S_2].

The equation of V' with the above imposed singularities contains 27 coefficients, but the essential coefficients for our purposes are fewer (as in the case of V, section 1); in fact, we only need 12 coefficients. We write the equation of V' directly, with the imposed singularities with the 12 essential coefficients.

$$
\begin{aligned}
&V' : f_6'(X_0, X_1, X_2, X_3, X_4) = \\
&a_{31002}X_0^3X_1X_4^2 + \\
&a_{13020}X_0X_1^3X_3^2 + \\
&a_{20031}X_0^2X_3^3X_4 + \\
&a_{02013}X_1^2X_3X_4^3 + \\
&a_{21201}X_0^2X_1X_2^2X_4 + a_{20211}X_0^2X_2^2X_3X_4 + a_{12210}X_0X_1^2X_2^2X_3 + a_{11220}X_0X_1X_2^2X_3^2 + \\
&a_{11202}X_0X_1X_2^2X_4^2 + a_{10221}X_0X_2^2X_3^2X_4 + a_{02211}X_1^2X_2^2X_3X_4 + a_{01212}X_1X_2^2X_3X_4^2 = 0.
\end{aligned}
$$

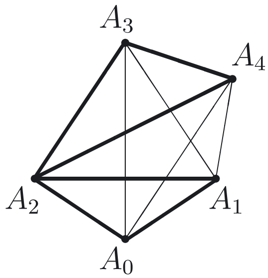
From here on, V' denotes this last hypersurface defined by the above last form $f_6'(X_0, X_1, X_2, X_3, X_4)$ for a generic choice of the parameters a_{ijkhl}.

9. – The unimposed actual singularities on V′.

We consider the hypersurface V' at the end of section 8.

Close to the singularities imposed on V', new singularities appear on the generic V'; they are actual or infinitely near singularities. The actual unimposed singularities are given by six actual double (straight) lines on V' given by A_0A_1,

$A_0A_2, A_1A_2, A_2A_3, A_2A_4$ and A_3A_4, according to the following picture, where the double lines are drawn in bold type.



The generic V' has no other actual singularities, so the generic V' is reduced, irreducible and normal.

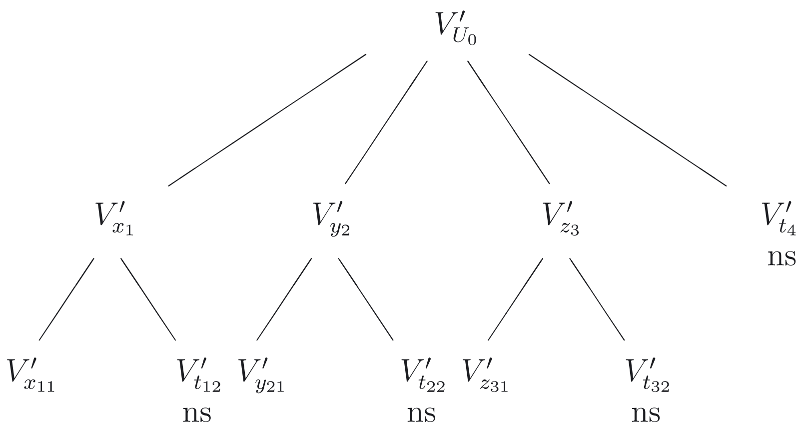
10. – The infinitely near singularities of V′.

RESOLUTION OF SINGULARITIES OF V'

Here again, the desingularization of V' is very long but very easy. New unimposed infinitely near singularities appear on the generic V', close to the imposed infinitely near singularities. They are double singular curves. Here, there are none of the infinitely near isolated double points seen in the case of V.

So, none of the unimposed singularities affect the birational invariants of a desingularization Z_4 of V', such as the irregularities and the plurigenera of Z_4, i.e. in calculating these invariants, we can assume that there are only the imposed singularities on V'.

The desingularization of V' is, more or less, a repetition of the one in [S_1], [S_2] and in section 3, so only the tree of the first blow-ups of $V'_{U_0} = V' \cap U_0$, where $U_0 = \{X_0 \neq 0\}$, is reproduced here.



11. – The m-canonical adjoints to $\mathbf{V}' \subset \mathbb{P}^4$.

Following the notations in section 4, an order can be established in the sequence of blow-ups in the example V': let us assume that π_1 is the blow-up at the triple point A_0 and π_2 is the blow-up along the double surface infinitely near A_0, π_3 is the blow-up at the triple point A_1 and π_4 is the blow-up along the double surface infinitely near A_1, π_5 is the blow-up at the 4-ple point A_2, π_6 is the blow-up at the triple point A_3, π_7 is the blow-up along the double surface infinitely near A_3, π_8 is the blow-up at the triple point A_4, and π_9 is the blow-up along the double surface infinitely near A_4. Let $\sigma = \pi_r \circ \cdots \circ \pi_1$ be a sequence of blow-ups solving the singularities of V'.

The equivalent formula of (*), section 4, is given here by:

$$(^\circ) \qquad D'_m \quad = \quad \pi_r^* ... \{\pi_2^*[\pi_1^*(\Phi_m)] - mE_2\} - mE_4 - mE_5 - mE_7 - mE_9,$$

where E_i is the exceptional divisor of the blow-up π_i, i.e. E_2 is the exceptional divisor of the blow-up π_2 along the double surface infinitely near $A_0, ..., E_5$ is the exceptional divisor of the blow-up at the 4-ple point A_2 , ... and E_9 is the exceptional divisor of the blow-up π_9 along the double surface infinitely near A_4.

No other exceptional divisors appear in D'_m because the unimposed singularities are either actual or infinitely near double singular curves on our (generic) V'.

12. – The plurigenera of a desingularization $\mathbf{Z}_4$ of $\mathbf{V}'$.

Let $\sigma_{|_{Z_4}} : Z_4 \longrightarrow V'$ be a desingularization of the hypersurface $V' \subset \mathbb{P}^4$, where $\sigma = \pi_r \circ \cdots \circ \pi_1$ (section 11).

PROPOSITION 2. – *The plurigenera of* Z_4 *are given by* $P_{4i} = 1$, $\forall i \geq 1$, *and* $P_m = 0$ *if* $m \neq 4i$.

PROOF. – Let us consider the equation of V': $f'_6(X_0, X_1, X_2, X_3, X_4) = 0$, and we arrange the form f'_6 to the powers of X_2.

$$f'_6 = \varphi_4(X_0, X_1, X_3, X_4)X_2^2 + \varphi_5(X_0, X_1, X_3, X_4)X_2 + \varphi_6(X_0, X_1, X_3, X_4) = 0,$$

where $\varphi_i(X_0, X_1, X_3, X_4)$ is a form of degree i in X_0, X_1, X_3, X_4.

Next, let us consider the hypersurface Φ_m appearing in $(^\circ)$ section 11, assuming that its equation is $F_m(X_0, X_1, X_2, X_3, X_4) = 0$, of degree m. Arranging the form F_m to the powers of X_2, we can write

$$F_m(X_0, X_1, X_2, X_3, X_4)$$
$$= \psi_s(X_0, X_1, X_3, X_4)X_2^{m-s} + \psi_{s+1}(X_0, X_1, X_3, X_4)X_2^{m-s-1} + \ldots + \psi_m(X_0, X_1, X_3, X_4),$$

where $\psi_j(X_0, X_1, X_3, X_4)$ is a form of degree j in X_0, X_1, X_3, X_4 and s is an integer satisfying $0 \leq s \leq m$. So, Φ_m has an s-ple point at A_2, with $0 \leq s \leq m$.

Let us assume that Φ_m is an m-canonical adjoint to V', i.e. $D'_{m|_{Z_4}} \geq 0$. From Lemma 1, section 5, we have the following result: modulo $f'_6 = 0$, we can assume $s \geq m - 1$; i.e that Φ_m, modulo $f'_6 = 0$, is defined by the form

$$F_m = \psi_{m-1}(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4).$$

From the (^{v}) in the proof of Proposition 1, section 5, we have the following equality:

$\psi_{m-1}(X_0, X_1, X_3, X_4) = B'''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)$,

where $B'''(X_0, X_1, X_3, X_4)$ is a form of degree $m - 5$ in X_0, X_1, X_3, X_4 or zero-form.

LEMMA 2. – *The m-canonical adjoint to V' given by*

$$\Phi_m : \psi_{m-1}(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4) = 0,$$

where $\psi_{m-1}(X_0, X_1, X_3, X_4) = B'''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)$, *has the following property*

$$D'_{m|_{Z_4}} \geq 0 \Longleftrightarrow D'_m + E_5 \geq 0,$$

where $D'_m = \pi_r^* ... \{\pi_2^*[\pi_1^*(\Phi_m)] - mE_2\} - mE_4 - mE_5 - mE_7 - mE_9$, *is defined in* ($^\circ$), *section* 11.

PROOF OF LEMMA 2. Let us consider the affine open set $U_0 = \{X_0 \neq 0\}$ as in section 1. Locally, the blow-up π_1 of $\mathbb{P}^4$ at A_0 is given by $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$ (cfr. section 1).

The total transform of $\Phi_m \cap U_0$ with respect to $\mathcal{B}_{x_1}$ is given by

$$\mathcal{B}_{x_1}^*(\Phi_m \cap U_0) : \psi_{m-1}(1, x_1, x_1z_1, x_1t_1)x_1y_1 + \psi_m(1, x_1, x_1z_1, x_1t_1) = 0.$$

The double surface S_0 infinitely near A_0 in affine coordinates (x_1, y_1, z_1, t_1) is given by $\begin{cases} x_1 = 0 \\ t_1 = 0 \end{cases}$ and the blow-up π_2 along S_0 is locally given by the formulas:

$$\mathcal{B}_{x_{11}} : \begin{cases} x_1 = x_{11} \\ y_1 = y_{11} \\ z_1 = z_{11} \\ t_1 = x_{11}t_{11} \end{cases} ; \quad \mathcal{B}_{t_{12}} : \begin{cases} x_1 = x_{12}t_{12} \\ y_1 = y_{12} \\ z_1 = z_{12} \\ t_1 = t_{12} \end{cases} .$$

The total transform of $\mathcal{B}_{x_1}^*(\Phi_m \cap U_0)$ with respect to $\mathcal{B}_{t_{12}}$ is given by

$$\mathcal{B}_{t_{12}}^*[\mathcal{B}_{x_1}^*(\Phi_m \cap U_0)] : \psi_{m-1}(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2)x_{12}y_{12}t_{12}$$
$$+ \psi_m(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2) = 0.$$

Since Φ_m is an m-canonical adjoint to V', the latter equation, modulo

$$\frac{1}{t_{12}^5} f_6'(1, x_{12}t_{12}, x_{12}y_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2) = 0,$$

must be of the type $t_{12}^m(...) = 0$.

The latter statement follows, as in the proof of Lemma 1, from the fact that $\mathcal{B}_{t_{12}} \circ \mathcal{B}_{x_1}$ coincides with the desingularization $\sigma_{|_{Z_4}}$ on the affine open set $V'_{t_{12}}$ (see the tree of blow-ups).

In other words, the following equality of polynomials must hold

$$(^{\circ\circ}) \quad \begin{aligned} &\psi_{m-1}(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_1t_{12}^2)x_{12}y_{12}t_{12} + \psi_m(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2) \\ &\quad + A(x_{12}, y_{12}, z_{12}, t_{12})[\cdots + a_{21201}x_{12}y_{12}^2 + a_{20211}x_{12}y_{12}^2z_{12} + \cdots] \\ &\quad = t_{12}^m B(x_{12}, y_{12}, z_{12}, t_{12}), \end{aligned}$$

where A and B are suitable polynomials in the variables $x_{12}, y_{12}, z_{12}, t_{12}$ and $[\cdots + a_{21201}x_{12}y_{12}^2 + a_{20211}x_{12}y_{12}^2z_{12}t_{12} + \cdots] = \frac{1}{t_{12}^5} f_6'$.

Note that the variable y_{12} in the equality $(^{\circ\circ})$ appears in

$$\psi_{m-1}(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_1t_{12}^2)x_{12}y_{12}t_{12} + \psi_m(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2)$$

with power one and in $[\cdots + a_{21201}x_{12}y_{12}^2 + a_{20211}x_{12}y_{12}^2z_{12} + \cdots]$ with power two. It follows that the equality $(^{\circ\circ})$ holds if and only if

$$(^{\circ\circ\circ}) \quad \psi_{m-1}(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_1t_{12}^2)x_{12}y_{12}t_{12} + \psi_m(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2) = t_{12}^m(...)$$

and moreover $A(x_{12}, y_{12}, z_{12}, t_{12}) = t_{12}^m(...)$ or $A(x_{12}, y_{12}, z_{12}, t_{12})$ is zero.

We obtain the result given by the equality $(^{\circ\circ\circ})$, obtained in the affine open set U_0, in the affine open sets U_1, U_3 and U_4 ($U_i = \{X_i \neq 0\}$) too. So the equality $(^{\circ\circ\circ})$, and the analogous equalities in U_1, U_3 and U_4, tell us that

$$\Phi_m : \psi_{m-1}(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4) = 0,$$

where $\psi_{m-1}(X_0, X_1, X_3, X_4) = B'''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)$ satisfies $D'_{m|_{Z_4}} \geq 0 \Longleftrightarrow D'_m + mE_5 \geq 0$.

Finally, if we consider $U_2 = \{X_2 \neq 0\}$, then we obtain

$$D'_{m|_{Z_4}} \geq 0 \Longleftrightarrow D'_m + E_5 \geq 0.$$

This proves Lemma 2.

Proof of Proposition 2 (continuation). From the result of Lemma 2, to compute the plurigenera of Z_4, we consider

$$F_m = B'''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4).$$

where

$$\varphi_4(X_0, X_1, X_3, X_4) = a_{21201}X_0^2X_1X_4 + a_{20211}X_0^2X_3X_4 + a_{12210}X_0X_1^2X_3$$
$$+ a_{11220}X_0X_1X_3^2 + a_{11202}X_0X_1X_4^2 + a_{10221}X_0X_3^2X_4 + a_{02211}X_1^2X_3X_4 + a_{01212}X_1X_3X_4^2.$$

Let us write

$$B'''(X_0, X_1, X_3, X_4)X_2 = \Big(\sum_{i+j+h+l=m-5} b_{ijhl}X_0^iX_1^jX_3^hX_4^l\Big)X_2,$$

$$\psi_m(X_0, X_1, X_3, X_4) = \sum_{i'+j'+h'+l'=m} c_{ijhl}X_0^{i'}X_1^{j'}X_3^{h'}X_4^{l'},$$

where $b_{ijhl}, c_{ijhl} \in k$.

With these notations, and considering $\Phi_m \cap U_0$, the equality in $(^{\circ\circ\circ})$ becomes

$$\Big(\sum_{i+j+h+l=m-5} b_{ijhl}x_{12}^{j+h+l}z_{12}^h t_{12}^{j+h+2l}\Big)\varphi_4(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2)(x_{12}y_{12}t_{12})$$
$$+ \sum_{i'+j'+h'+l'=m} c_{ijhl}x_{12}^{j'+h'+l'}z_{12}^{h'}t_{12}^{j'+h'+2l'} = t_{12}^m(\ldots).$$

The latter equality is equivalent to the inequalities $\begin{cases} j+h+2l+4 \geq m \\ j'+h'+2l' \geq m \end{cases}.$

Similarly, if we consider $\Phi_m \cap U_1$, the analogous equality of $(^{\circ\circ\circ})$ provides the inequalities $\begin{cases} i+2h+l+4 \geq m \\ i'+2h'+l' \geq m \end{cases}.$

Again, $\Phi_m \cap U_3$ and $\Phi_m \cap U_4$ provide the inequalities

$$\begin{cases} 2i+j+l+4 \geq m \\ 2i'+j'+l' \geq m \end{cases} \quad \text{and} \quad \begin{cases} i+2j+h+4 \geq m \\ i'+2j'+h' \geq m \end{cases}.$$

Combining all the inequalities gives us $\begin{cases} l \geq i+1 \geq h+2 \geq j+3 \geq l+4 \\ l' \geq i' \geq h' \geq j' \geq l' \end{cases}.$

The first line tells us that $B'''(X_0, X_1, X_3, X_4)$ is the zero-form and the second line shows that $i' = j' = h' = l'$, i.e. the form defining Φ_m is

$$c_{rrrr}X_0^rX_1^rX_3^rX_4^r, \ \forall r \geq 1.$$

This proves Proposition 2. □

13. – The irregularities of $\mathbf{Z_4}$.

With the same proof as in the case of Z_3, cf. section 6, the irregularities of Z_4 are $q_1 = q_2 = 0$.

THIS COMPLETES THE CONSTRUCTION OF THE SECOND THREEFOLD Z_4.

CONSTRUCTION OF THE THREEFOLD Z_5.

14. – Imposing five triple points with an infinitely near double surface on a degree six hypersurface V″ in $\mathbb{P}^4$.

The simplest equation of a degree six hypersurface V'' having five triple points with an infinitely near double surface is given by

$$V'' : f_6''(X_0, X_1, X_2, X_3, X_4) =$$
$$a_{30201}X_0^3X_2^2X_4 +$$
$$a_{13020}X_0X_1^3X_3^2 +$$
$$a_{01302}X_1X_2^3X_4^2 +$$
$$a_{20130}X_0^2X_2X_3^3 +$$
$$a_{02013}X_1^2X_3X_4^3 = 0.$$

The rotations of indices and variables passing from A_i to A_{i+1} and returning to A_0 are as follows.

Rotations of indices (and variables)

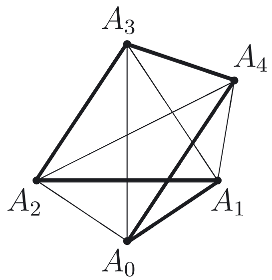
$$A_0 \mapsto A_1 \mapsto A_2 \mapsto A_3 \mapsto A_4 \mapsto A_0$$
$$ijkhl \mapsto lijkh \mapsto hlijk \mapsto khlij \mapsto jkhli \mapsto ijkhl$$

Here, the equation $f_6''(X_0, X_1, X_2, X_3, X_4) = 0$ is *invariant* with respect to the five rotations. So, a statement on the equation that is true for the point A_i, for the affine open set U_i, ... holds true for any other point A_j, for any other affine open set U_j, ...

From now on, V'' denotes the degree six hypersurface defined by the $f_6''(X_0, X_1, X_2, X_3, X_4)$ for a generic choice of parameters a_{ijkhl}.

15. – The unimposed actual singularities on V″.

There are five unimposed actual double (straight) lines on V'': A_0A_1, A_0A_4, A_1A_2, A_2A_3, A_3A_4; see below, where the double lines are drawn in bold type.



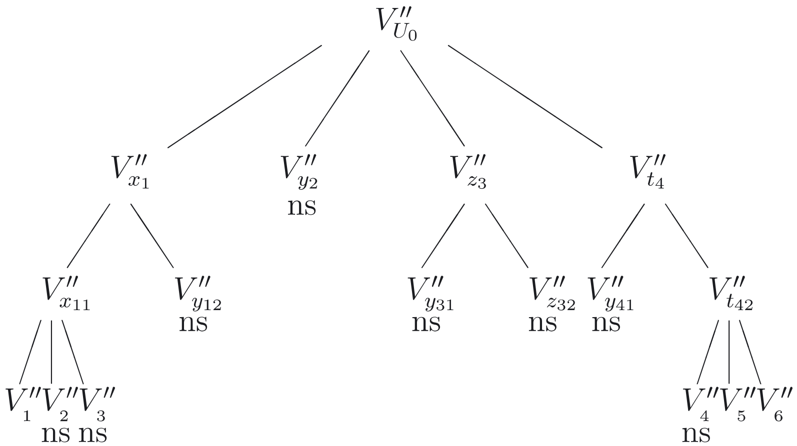
The (generic) V'' has no other actual singularities, so V'' is reduced, irreducible and normal.

16. – The infinitely near singularities of $\mathbf{V}''$.

Here, since the equation of V'' is invariant with respect to the rotations of indices and variables, all we have to do is resolve the singularities in just one of the affine open sets $U_i = \{X_i \neq 0\}$.

New unimposed infinitely near singularities appear on the generic V'' close to the imposed infinitely near singularities; they are only double singular curves. So, here again, none of the unimposed singularities affect the birational invariants of a desingularization Z_5 of V''.

The desingularization of V'' is, more or less, the same as in [S_1], [S_2] and section 3, so only a part of the tree of the blow-ups of $V''_{U_0} = V'' \cap U_0$ is given here.



The affine threefolds V''_1, V''_5 and V''_6 are singular along (locally) double straight lines. Here again, they have local double lines infinitely near and, after some blow-ups, we can resolve all the singularities.

17. – The m-canonical adjoints to $\mathbf{V}'' \subset \mathbb{P}^4$.

As in the previous constructions, an order can be established in the sequence of blow-ups, e.g. let us assume that π_1 is the blow-up at the triple point A_0, π_2 is the blow-up along the double surface infinitely near A_0, π_3 is the blow-up at A_1, π_4 is the blow-up along the double surface infinitely near A_1, ... , and π_{10} is the blow-up along the double surface infinitely near A_4.

The equivalent formula of (*), section 4, is given by:

$$(\diamondsuit) \qquad D''_m \quad = \quad \pi_r^* ... \{\pi_2^*[\pi_1^*(\Phi_m)] - mE_2\} - mE_4 - mE_6 - mE_8 - mE_{10}\,,$$

where E_i is the exceptional divisor of the blow-up π_i, i.e. E_2 is the exceptional divisor of the blow-up π_2 along the double surface infinitely near A_0, and so on.

No other exceptional divisors appear in D''_m because the unimposed singularities are either actual or infinitely near double singular curves on our (generic) V''.

18. – The plurigenera of a desingularization $\mathbf{Z_5}$ of $\mathbf{V''}$.

Let $\sigma_{|_{Z_5}} : Z_5 \longrightarrow V''$ be a desingularization of the hypersurface $V'' \subset \mathbb{P}^4$, where $\sigma = \pi_r \circ \cdots \circ \pi_1$ (section 17).

PROPOSITION 3. – *The plurigenera of* Z_5 *are given by* $P_{5i} = 1$, $\forall i \geq 1$, *and* $P_m = 0$ *if* $m \neq 5i$.

The statement of Proposition 3 is much the same as those of Propositions 1 (section 5) and 2 (section 12), but the proof is completely different, essentially because the hypersurface V'' has no 4-ple points in this case.

To prove Proposition 3, we need some preliminary results on global and non-global m-canonical adjoints to V'' (cf. section 4, or $[S_1]$ for the definitions).

LEMMA 3. – *The global m-canonical adjoints to* V'' *are given by*

$$\Phi_m : c_i X_0^i X_1^i X_2^i X_3^i X_4^i = 0, \ \forall i \geq 0,$$

where $c_i \in k$.

In particular, the global m-canonical adjoints exist if and only if $m = 5i$, $\forall i$ and there is only one of them for $m = 5i$.

PROOF OF LEMMA 3. Let us consider a global m-canonical adjoint to V''

$$\Phi_m : \sum_{i+j+k+h+l=m} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = 0,$$

Locally, the blow-up π_1 of $\mathbb{P}^4$ at A_0 is given by $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$ (section 1). The total transform Φ^* of $\Phi_m \cap U_0$ with respect to $\mathcal{B}_{t_4}$ is given by

$$\Phi^* = \mathcal{B}_{z_3}^*(\Phi_m \cap U_0) : \sum_{i+j+k+h+l=m} b_{ijkhl} (x_3 z_3)^j (y_3 z_3)^k z_3^h (z_3 t_3)^l = 0,$$

The double surface S_0 infinitely near A_0 in affine coordinates (x_3, y_3, z_3, t_3) is given by $\begin{cases} y_3 = 0 \\ z_3 = 0 \end{cases}$ and the blow-up π_2 along S_0 is given locally by the formulas:

$$\mathcal{B}_{y_{31}} : \begin{cases} x_3 = x_{31} \\ y_3 = y_{31} \\ z_3 = y_{31}z_{31} \\ t_3 = t_{31} \end{cases}; \quad \mathcal{B}_{z_{32}} : \begin{cases} x_3 = x_{32} \\ y_3 = y_{32}z_{32} \\ z_3 = z_{32} \\ t_3 = t_{32} \end{cases}.$$

The total transform Φ^{**} of $\Phi^* = \mathcal{B}^*_{z_3}(\Phi_m \cap U_0)$ with respect to $\mathcal{B}_{z_{32}}$ is given by

$$\Phi^{**} = \mathcal{B}^*_{z_{32}}(\Phi^*) : \sum_{i+j+k+h+l=m} b_{ijkhl}(x_{32}z_{32})^j (y_{32}z_{32}^2)^k z_{32}^h (z_{32}t_{32})^l = 0,$$

Since Φ_m is a global m-canonical adjoint to V'', by definition in ($^\diamondsuit$), section 17, we have $D''_m \geq 0$. This implies that

$$\frac{\Phi^{**}}{(z_{32}^m)} = \frac{1}{(z_{32}^m)} \left(\sum_{i+j+k+h+l=m} b_{ijkhl} x_{32}^j y_{32}^k z_{32}^{j+2k+h+l} t_{32}^l = 0 \right) \geq 0.$$

Here, as well as in the proof of Lemma 1, we use the fact that $\mathcal{B}_{z_{32}} \circ \mathcal{B}_{z_3}$ coincides with the desingularization $\sigma_{|_{Z_5}}$ on the affine open set $V''_{z_{32}}$, in fact $V''_{z_{32}}$ is nonsingular (see the tree of blow-ups, section 16).

Clearly, the latter inequality is equivalent to

$$j + 2k + h + l - m \geq 0, \text{ i.e. } i \leq k.$$

Note that we obtained the latter result in the affine open set U_0. Next, without any further computations, simply using the rotations of indices and variables (section 14), we obtain similar results in the other affine open sets U_1, U_2, U_3 and U_4:

in U_1, we obtain $j \leq h$;

in U_2, we obtain $k \leq l$;

in U_3, we obtain $h \leq i$;

in U_4, we obtain $l \leq j$.

Composing all the inequalities, we deduce that $i \leq k \leq l \leq j \leq h \leq i$, that is $i = j = k = h = l$.

This proves Lemma 3.

LEMMA 4. – *Let us consider an m-canonical adjoint to V''* (not necessarily global)

$$\Phi_m : F_m(X_0, X_1, X_2, X_3, X_4) = \sum_{i+j+k+h+l=m} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = 0,$$

where $b_{ijkhl} \in k$. *Φ_m can be replaced with*

$$\Psi_m : F_m - B_0 f_6'' = \sum_{i_0+j_0+k_0+h_0+l_0=m} c_{i_0 j_0 k_0 h_0 l_0} X_0^{i_0} X_1^{j_0} X_2^{k_0} X_3^{h_0} X_4^{l_0} = 0,$$

such that $i_0 \le k_0$, i.e. $j_0 + 2k_0 + h_0 + l_0 \ge m$, for all monomials.

Before proving Lemma 4, we must point out that the inequality $i_0 \le k_0$ is equivalent to

$$\frac{\Psi^{**}}{(z_{32}^m)} = \frac{1}{(z_{32}^m)} \left(\sum_{i_0+j_0+k_0+h_0+l_0=m} c_{i_0 j_0 k_0 h_0 l_0} x_{32}^{j_0} y_{32}^{k_0} z_{32}^{j_0+2k_0+h_0+l_0} t_{32}^{l_0} = 0 \right) \ge 0.$$

Here, we used the same notations as in the proof of Lemma 3, where $c_{i_0 j_0 k_0 h_0 l_0} \in k$, B_0 is a suitable form and $f_6'' = 0$ is the equation of V''.

Proof of Lemma 4. Since $\Phi_m : F_m = 0$ is an m-canonical adjoint to V'', by definition (in ($^\diamond$), section 17) we have $D''_{m|_{Z_5}} \ge 0$.

Let us consider the first two blow-ups $\mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}^4$ for the resolution of the singularities of V''. If V_1'' is the strict transform of V'' with respect to π_1, and V_2'' is the strict transform of V_1'' with respect to π_2, then we obtain the sequence

$$V_2'' \xrightarrow{\pi_2'} V_1'' \xrightarrow{\pi_1'} V''$$

of morphisms, where π_i' is the restriction of π_i to V_i'', $i = 1, 2$.

Now, let us consider the affine open set U_0 and $V'' \cap U_0$. With the notations in the proof of Lemma 3 and the affine open set of affine coordinates $(x_{32}, y_{32}, z_{32}, t_{32})$, we have that V_2'' has the equation given by the polynomial

$$\frac{1}{z_{32}^5} f_6''(1, x_{32}z_{32}, y_{32}z_{32}^2, z_{32}, z_{32}t_{32}) = a_{30201} y_{32}^2 t_{32} + a_{13020} x_{32}^3 + a_{01302} x_{32} y_{32}^3 z_{32}^4 t_{32}^2 + a_{20130} y_{32} + a_{02013} x_{32}^2 z_{32} t_{32}^3 .$$

Next, in the inequality $D''_{m|_{Z_5}} \ge 0$, we consider only the first two blow-ups. So, in the affine open set of coordinates $(x_{32}, y_{32}, z_{32}, t_{32})$, we obtain

$$\left(\frac{\Phi^{**}}{(z_{32}^m)} \right)_{|_{V_2'' \cap \dots}} \ge 0.$$

Here again, we use the fact that $\mathcal{B}_{z_{32}} \circ \mathcal{B}_{z_3}$ coincides with the desingularization $\sigma_{|_{Z_5}}$ on the affine open set $V''_{z_{32}}$, in fact $V''_{z_{32}}$ is nonsingular (see the tree of blow-ups, section 16).

In the language of polynomials, the latter inequality is equivalent to writing the equality of polynomials

$$(\diamond\diamond)\qquad \sum_{i+j+k+h+l=m} b_{ijkhl}x_{32}^j y_{32}^k z_{32}^{j+2k+h+l} t_{32}^l + B(x_{32},y_{32},z_{32},t_{32})(a_{30201}y_{32}^2 t_{32}$$
$$+ a_{13020}x_{32}^3 + a_{01302}x_{32}y_{32}^3 z_{32}^4 t_{32}^2 + a_{20130}y_{32} + a_{02013}x_{32}^2 z_{32} t_{32}^3) = z_{32}^m(\ldots),$$

where $B(x_{32},y_{32},z_{32},t_{32})$ is a suitable polynomial.

In the particular case, we have $j+2k+h+l \geq m$ for all monomials, the Lemma 4 is true with $B_0 = B = 0$. So, we assume $j+2k+h+l < m$ for some monomials (now $B \neq 0$) and we distinguish the cases $r \leq j+2k+h+l < m$, with $r \geq 0$. We can conveniently rewrite $\sum_{i+j+k+h+l=m} b_{ijkhl}x_{32}^j y_{32}^k z_{32}^{j+2k+h+l} t_{32}^l$ distinguishing these cases

$$\sum_{i+j+k+h+l=m} b_{ijkhl}x_{32}^j y_{32}^k z_{32}^{j+2k+h+l} t_{32}^l = \sum_{j+2k+h+l=r} b_{ijkhl}x_{32}^j y_{32}^k z_{32}^r t_{32}^l$$
$$+ \sum_{j+2k+h+l=r+1} b_{ijkhl}x_{32}^j y_{32}^k z_{32}^{r+1} t_{32}^l + \cdots + \sum_{j+2k+h+l=m-1} b_{ijkhl}x_{32}^j y_{32}^k z_{32}^{m-1} t_{32}^l$$
$$+ \sum_{j+2k+h+l=m} b_{ijkhl}x_{32}^j y_{32}^k z_{32}^m t_{32}^l + \sum_{j+2k+h+l=m+1} b_{ijkhl}x_{32}^j y_{32}^k z_{32}^{m+1} t_{32}^l + \cdots.$$

Substituting in $(\diamond\diamond)$, we obtain $B(x_{32},y_{32},z_{32},t_{32}) = z_{32}^r C(x_{32},y_{32},z_{32},t_{32})$. Again, substituting the latter equality in $(\diamond\diamond)$, then simplifying z_{32}^r and putting $z_{32} = 0$, we obtain the equality of polynomials

$$(\diamond\diamond\diamond)\qquad \sum_{j+2k+h+l=r} b_{ijkhl}x_{32}^j y_{32}^k t_{32}^l$$
$$= C(x_{32},y_{32},0,t_{32})(a_{30201}y_{32}^2 t_{32} + a_{13020}x_{32}^3 + a_{20130}y_{32}).$$

Multiplying the left- and right-hand sides of $(\diamond\diamond\diamond)$ by z_{32}^r, and taking $\mathcal{B}_{z_3}$ and$\mathcal{B}_{z_{32}}$ into account, we obtain

$$\sum_{j+2k+h+l=r} b_{ijkhl}x^j y^k z^h t^l = z^{r-5} C\left(\frac{x}{z},\frac{y}{z^2},0,\frac{t}{z}\right)(a_{30201}y^2 t + a_{13020}x^3 z^2 + a_{20130}yz^3).$$

We write $C\left(\frac{x}{z},\frac{y}{z^2},0,\frac{t}{z}\right) = \frac{D(x,y,z,t)}{z^\rho}$. Thus,

$$\sum_{j+2k+h+l=r} b_{ijkhl}x^j y^k z^h t^l = z^{r-5-\rho} D(x,y,z,t)(a_{30201}y^2 t + a_{13020}x^3 z^2 + a_{20130}yz^3).$$

Replacing $D(x,y,z,t)$, if necessary, we can assume $r-5-\rho \geq 0$. In fact, if $r-5-\rho < 0$, then we deduce $z^{-r+3+\rho}\left(\sum\ldots\right) = D(x,y,z,t)(\ldots)$. Since polynomial rings are factorial, we obtain $D(x,y,z,t) = z^{-r+3+\rho}E(x,y,z,t)$ and $\sum\ldots = E(x,y,z,t)(\ldots)$, as desired.

Now, we suitably homogenize the latter equality to obtain an addendum of the form F_m, so we can write

$$(\Diamond^v) \quad \sum_{j+2k+h+l=r} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = G(X_0, X_1, X_2, X_3, X_4)(a_{30201} X_0^2 X_2^2 X_4 + a_{13020} X_1^3 X_3^2 + a_{20130} X_0 X_2 X_3^3),$$

where $i + j + k + h + l = m$.

From the hypothesis $r \le m - 1$, we deduce $k \le i - 1$ and particularly $i \ge 1$. The latter inequality means that X_0 can be put in evidence in $\sum_{j+2k+h+l=r} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l$. So, from $(\Diamond^v)$, we obtain the equality of polynomials $G(X_0, X_1, X_2, X_3, X_4) = X_0 H(X_0, X_1, X_2, X_3, X_4)$, and we can thus rewrite $(\Diamond^v)$ as follows.

$$\sum_{j+2k+h+l=r} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = H(X_0, X_1, X_2, X_3, X_4)(a_{30201} X_0^3 X_2^2 X_4 + a_{13020} X_0 X_1^3 X_3^2 + a_{20130} X_0^2 X_2 X_3^3).$$

But now $a_{30201} X_0^3 X_2^2 X_4 + a_{13020} X_0 X_1^3 X_3^2 + a_{20130} X_0^2 X_2 X_3^3$ is an addendum of the form $f_6''(X_0, X_1, X_2, X_3, X_4)$ defining V''. This enables us to replace the m-canonical adjoint $\Phi_m : F_m(X_0, X_1, X_2, X_3, X_4) = 0$ with the new one Φ'_m given by

$$F_m(X_0, X_1, X_2, X_3, X_4) - H(X_0, X_1, X_2, X_3, X_4) f_6''(X_0, X_1, X_2, X_3, X_4) = 0,$$

where the form $F_m - H f_6''$ is now given by

$$\sum_{j+2k+h+l=r+1} c_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l + \cdots ,$$

i.e. the form $F_m - H f_6''$ starts with $j + 2k + h + l = r + 1$ instead of $j + 2k + h + l = r$.

By iterating this process, we can replace the m-canonical adjoint $\Phi_m : F_m = 0$ with $\Psi_m : F_m - B_0 f_6''$, so that

$$F_m - B_0 f_6'' = \sum_{j+2k+h+l=m} c_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l + \cdots .$$

This proves Lemma 4.

The rotations of the indices and variables concerning the affine open set U_0 can be repeated for each of the other affine open sets U_1, U_2, U_3 and U_4 (in the same way as in Lemma 4), i.e. if we choose one affine open set U_s, then a result like the one in Lemma 4 holds in U_s too. That is to say, we have

COROLLARY. – *If we consider an m-canonical adjoint to V''*

$$\Phi_m : F_m = \sum_{i+j+k+h+l=m} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = 0,$$

then we can replace Φ_m with

$$\Phi_m^{(s)} : F_m - B_s f_6'' = \sum_{i_s+j_s+k_s+h_s+l_s=m} c_{i_s j_s k_s h_s l_s} X_0^{i_s} X_1^{j_s} X_2^{k_s} X_3^{h_s} X_4^{l_s} = 0,$$

such that either for $s=0$, the inequality $i_0 \leq k_0$ holds for all monomials (cf. Lemma 4), *or*

for $s=1$, the inequality $j_1 \leq h_1$ holds for all monomials, or
for $s=2$, the inequality $k_2 \leq l_2$ holds for all monomials, or
for $s=3$, the inequality $h_3 \leq i_1$ holds for all monomials, or
for $s=4$, the inequality $l_4 \leq j_4$ holds for all monomials.

LEMMA 5. – *If we consider a non-global m-canonical adjoint to V''*

$$\Phi_m : F_m(X_0, X_1, X_2, X_3, X_4) = \sum_{i+j+k+h+l=m} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = 0,$$

then a form $B = B(X_0, X_1, X_2, X_3, X_4)$ exists such that $\Phi_m^\star : F_m - Bf_6'' = 0$ is a global m-canonical adjoint to V''. In other words, the following equality holds

$$\Phi_{m|_{V''}} = \Phi^\star_{m|_{V''}}.$$

Before considering the proof, note that Lemma 5 holds essentially because of the particular rotations of indices and variables. In other words, if we consider other permutations of indices and variables, and we leave the same five imposed singularities, then Lemma 5 may be false (cf. [S_2]).

PROOF OF LEMMA 5. If a form of the type $(\cdots)X_0X_1^3X_3^2$ appears in F_m as an addendum, then we can replace $X_0X_1^3X_3^2$ with

$$\frac{-f_6''}{a_{13020}} + X_0X_1^3X_3^2 =$$

$$-\frac{a_{30201}}{a_{13020}} X_0^3 X_2^2 X_4 - \frac{a_{01302}}{a_{13020}} X_1 X_2^3 X_4^2 - \frac{a_{20130}}{a_{13020}} X_0^2 X_2 X_3^3 - \frac{a_{02013}}{a_{13020}} X_1^2 X_3 X_4^3.$$

This is the same as replacing the form F_m with $F_m - (...)f_6''$ and this can be done for the reasons given in the proof of Lemma 1 (section 5). Clearly, we can repeat this replacement several times, obtaining a new form $F'_m = F_m - (...)f_6''$, which contains no addendum of the type $(\cdots)X_0X_1^3X_3^2$. So, from now on, we can consider $\Phi'_m : F'_m = 0$, instead of $\Phi_m : F_m = 0$, because if we prove the lemma for Φ'_m, then it is also proved for Φ_m. If $F'_m = 0$, then $F_m = (...)f_6''$ and Φ_m is a global adjoint (cutting the zero divisor on Z_5), and in this case Lemma 5 holds true. So we can assume that $F'_m \neq 0$ and F'_m contains no addendum of the type $(\ldots)X_0X_1^3X_3^2$.

Let us write $F'_m = \sum_{i'+j'+k'+h'+l'=m} b'_{i'j'k'h'l'} X_0^{i'}X_1^{j'}X_2^{k'}X_3^{h'}X_4^{l'}$. We claim that $i' \leq k'$. To prove this claim, we assume by contradiction that $i' > k'$. From Lemma 4, Φ'_m can be replaced with

$$\Psi_m : F'_m - B_0 f''_6 = \sum_{i_0+j_0+k_0+h_0+l_0=m} c_{i_0j_0k_0h_0l_0} X_0^{i_0}X_1^{j_0}X_2^{k_0}X_3^{h_0}X_4^{l_0} = 0,$$

so that $i_0 \leq k_0$. From the proof of Lemma 4, this can be done if and only if we subtract $B_0(a_{30201}X_0^3X_2^2X_4 + a_{13020}X_0X_1^3X_3^2 + a_{20130}X_0^2X_2X_3^3)$, $B_0 \neq 0$, from F'_m. But this means that the form F'_m has the addendum $B_0a_{13020}X_0X_1^3X_3^2$, and this contradiction proves our claim.

Next, we claim that $j' \leq h'$ also holds. Otherwise, we would have to subtract the form $B_1(a_{13020}X_0X_1^3X_3^2 + a_{01302}X_1X_2^3X_4^2 + a_{02013}X_1^2X_3X_4^3)$, $B_1 \neq 0$, from F'_m (cf. Corollary). But this is impossible because F'_m does not contain the addendum $(\ldots)X_0X_1^3X_3^2$.

Similarly, $h' \leq i'$ also holds. Otherwise, we would have to subtract the form $B_3(a_{13020}X_0X_1^3X_3^2 + a_{20130}X_0^2X_2X_3^3 + a_{02013}X_1^2X_3X_4^3)$, $B_3 \neq 0$, from F'_m, which is again impossible.

In short, in F'_m we have $i' \leq k', j' \leq h', h' \leq i'$. If we also have $k' \leq l'$ and $l' \leq j'$ in F'_m, then Lemma 5 is demonstrated, because the five inequalities for the same F'_m tell us that $\Phi'_m : F'_m = 0$ is a global m-canonical adjoint to V''.

So, let us assume that the two inequalities $k' \leq l'$ and $l' \leq j'$ do not hold, for some monomials. In this case, we show that there is a contradiction, or we construct F''_m such that $F''_m = F'_m - B_2f''_6$ and F''_m defines a global m-canonical adjoint, proving Lemma 5.

First we consider $l' > j'$. In this case, as in the proof of Lemma 4 (using the rotations of indices and variables), we see that F'_m has an addendum of the type $(\ldots)X_0^3X_2^2X_4$. We replace $X_0^3X_2^2X_4$ with the form $\dfrac{-f''_6}{a_{30201}} + X_0^3X_2^2X_4$ in F'_m (several times), which is equivalent to replacing the form F'_m with $F''_m = F'_m - (\ldots)f''_6$. As before, we can see that in the form

$$F''_m = \sum_{i''+j''+k''+h''+l''=m} b''_{i''j''k''h''l''} X_0^{i''}X_1^{j''}X_2^{k''}X_3^{h''}X_4^{l''}$$

the inequality $l'' \leq j''$ holds (for all monomials). In addition, the inequality $k'' \leq l''$ also holds.

Likewise (or as in the proof of Lemma 4), $l'' \leq j''$ can be obtained if and only if we subtract the addendum $B_2(a_{30201}X_0^3X_2^2X_4 + a_{01302}X_1X_2^3X_4^2 + a_{20130}X_0^2X_2X_3^3)$ from F'_m. This has two consequences:

1) in F'_m there is the addendum $B_2(a_{30201}X_0^3X_2^2X_4 + a_{01302}X_1X_2^3X_4^2 + a_{20130}X_0^2X_2X_3^3)$;

2) in F''_m there is the addendum $-B_2(\delta_1 a_{13020}X_0X_1^3X_3^2 + \delta_2 a_{02013}X_1^2X_3X_4^3)$, where $\delta_i = \begin{cases} 0 \\ 1 \end{cases}, i = 1, 2$, and $-B_2(\delta_1 a_{13020}X_0X_1^3X_3^2 + \delta_2 a_{02013}X_1^2X_3X_4^3) \neq 0$.

We note that, for example, if $\delta_2 = 0$, then the addendum $B_2 a_{02013}X_1^2X_3X_4^3$ appears in F'_m.

Now, let us consider the case where $\delta_2 = 1$.

Since i' is the power of the variable X_0 and k' is the power of X_2 in F'_m, we deduce from $i' \leq k'$ and from 1) that $B_2 = \sum X_0^r X_2^{r+s}(\ldots)$, where $r \geq 0$ and $s \geq 1$. Similarly, k'' is the power of X_2 and l'' is the power of X_4 in F''_m, so from $k'' \leq l''$ and from 2), we deduce that $B_2 = \sum X_0^r X_2^{r+s} X_4^{r+s+v}(\ldots)$, where $v \geq 0$.

In addition, the inequality $j' \leq h'$ holds in F'_m. Again from 1), we obtain $B_2 = \sum X_0^r X_1^t X_2^{r+s} X_3^{t+u} X_4^{r+s+v}(\ldots)$, where $t \geq 0$ and $u \geq 1$. From $l'' \leq j''$ in F''_m, we obtain that in B_2 the inequality $r + s + v < t$ must hold.

Finally, from $h' \leq i'$ in F'_m, in B_2 the inequality $t + u < r$ must hold too.

From $r + s + v < t$ and $t + u < r$, we deduce that $t - s - v > r > t + u$, but this is a contradiction. The contradiction proves that the case of $\delta_2 = 1$ cannot occur.

Next, let us consider the case where $\delta_2 = 0$.

In this case, all the above inequalities hold except $r + s + v < t$, which must be replaced by $r + s + v \leq t + 3$. Here, we also consider the inequality $l' > j'$ in F'_m, which was assumed at the beginning. Said inequality implies that $r + s + v > t + 1$, so $r + s + v = \begin{cases} t+2 \\ t+3 \end{cases}$. From the inequalities $t + u < r$ and $r + s + v \leq t + 3$, we deduce that $t + u + s < r + s + v = \begin{cases} t+2 \\ t+3 \end{cases}$. The case of $t + u + s < t + 2$ does not occur because $u \geq 1$ and $s \geq 1$. There remains the case of $t + u + s < t + 3$. This inequality implies that $s = 1, u = 1, v = 0$ and $r = t + 2$. Thus, the form B_2 is of the type $B_2 = \sum X_0^{t+2} X_1^t X_2^{t+3} X_3^{t+1} X_4^{t+3}$ and

$$-B_2 a_{13020}X_0X_1^3X_3^2 = -a_{13020}(\sum X_0^{t+3}X_1^{t+3}X_2^{t+3}X_3^{t+3}X_4^{t+3}),$$

where $t \geq 0$. But this is a global $5(t + 3)$-canonical adjoint to V'' and the statement in Lemma 5 is true because $F''_m = -B_2 a_{13020}X_0X_1^3X_3^2 + G_m$ and $F'_m = B_2(a_{30201}X_0^3X_2^2X_4 + a_{01302}X_1X_2^3X_4^2 + a_{20130}X_0^2X_2X_3^3 + a_{02013}X_1^2X_3X_4^3) + G_m$, where G_m defines global m-canonical adjoints; in fact, the monomials of G_m, as addenda of F'_m satisfy $i' \leq k', j' \leq h', h' \leq i'$ and, as addenda of F''_m satisfy $l'' \leq j'', k'' \leq l''$, with $j' = j'', k' = k'', l' = l''$.

So, in the case of the inequality $l' > j'$, Lemma 5 is proved.

We have examined the inequality $l' > j'$ and it remains for us to consider the inequality $k' > l'$. Here, applying the same proof as for $l' > j'$, we prove our thesis. So Lemma 5 is completely proved.

PROOF OF PROPOSITION 3. The proof is immediate because, from Lemma 5, to compute the plurigenera of Z_5, we can assume that the m-canonical adjoints to V'' are global; the statement in Proposition 3 therefore follows from Lemma 1. □

19. – The irregularities of $\mathbf{Z_5}$.

With the same proof as in the case of Z_3, cf. section 6, the irregularities of Z_5 are $q_1 = q_2 = 0$.

THIS COMPLETES THE CONSTRUCTION OF THE THIRD THREEFOLD Z_5.

Appendix

With a construction similar to that of Z_4, but imposing only three of the four singularities imposed on $V' \subset \mathbb{P}^4$ at the four vertices A_0, A_1, A_3, A_4, we obtain a new hypersurface V'^* such that a desingularization X of V'^* is a threefold of general type. This threefold X has the birational invariants $q_1 = q_2 = 0$, $p_g = 1$, $P_2 = 2$ and its m-canonical transformation is birational if and only if $m \geq 11$.

For instance, let us choose the vertex A_4 and put no singularities at A_4, while imposing on V'^* the same singularities as on V' in section 8 at the other vertices. Lemma 1 (section 5) and Lemma 2 (section 12) clearly hold, but the proof of Proposition 2 (also in section 12) has to be modified, removing the conditions given by the singularity at A_4. So, the remaining conditions are the inequalities

$$\begin{cases} j + h + 2l + 4 \geq m \\ j' + h' + 2l' \geq m \end{cases}, \quad \begin{cases} i + 2h + l + 4 \geq m \\ i' + 2h' + l' \geq m \end{cases}, \quad \begin{cases} 2i + j + l + 4 \geq m \\ 2i' + j' + l' \geq m \end{cases}$$

regarding the following equation of the m-canonical adjoints to V'^* (loc. cit.)

$$\Phi_m : B'''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4) = 0.$$

The union of these inequalities now gives $\begin{cases} l \geq i + 1 \geq h + 2 \geq j + 3 \\ l' \geq i' \geq h' \geq j' \end{cases}$, where $i + j + h + l = m - 5$ and $i' + j' + h' + l' = m$.

In this case, $B'''(X_0, X_1, X_3, X_4)$ can differ from zero: the first value of m for which $B'''(X_0, X_1, X_3, X_4) \neq 0$ is 11, according to the values: $l = 3$, $i = 2$, $h = 1$ and $j = 0$. Moreover, the irregularities of X are $q_1 = q_2 = 0$ and the first plurigenera of X are given by

$p_g = 1$, because $\Phi_1 : \lambda_1 X_4 = 0$ ($l' = 1, i' = h' = j' = 0$),

$P_2 = 2$, because $\Phi_2 : X_4(\lambda_1 X_4 + \lambda_2 X_0) = 0 \begin{cases} l' = 2, i' = h' = j' = 0 \\ l' = i' = 1, h' = j' = 0 \end{cases}$,

$P_3 = 3$, because $\Phi_3 : X_4(\lambda_1 X_4^2 + \lambda_2 X_0 X_4 + \lambda_3 X_0 X_3) = 0$,

$P_4 = 5$, because $\Phi_4 : X_4(\lambda_1 X_4^3 + \lambda_2 X_0 X_4^2 + \lambda_3 X_0^2 X_4 + \lambda_4 X_0 X_3 X_4 + \lambda_5 X_0 X_1 X_3) = 0$,

where $\lambda_i \in k$.

$P_5 = 6, P_6 = 9, P_7 = 11, P_8 = 14, P_9 = 17.$

Now, considering the linear system of 4-canonical adjoints to V'^*, it is not difficult to see that the 4-canonical transformation $\varphi_{|4K_X|}$, where K_X is a canonical divisor on X, is generically a rational transformation $2:1$. Roughly speaking, in an open set of X, $\varphi_{|4K_X|}$ can be identified with $\varphi_{|_{V'^*}}$, where φ is the rational tranformation defined by the linear system of 4-canonical adjoints (cf. for instance [S_1], section 16); the equation for V'^* is of the type $(...)X_2^2 + (...) = 0$; the equations for the 4-canonical adjoints do not contain the variable X_2; so two distinct points, that are mapped to one point, are of the type $(a, b, x_2, c, d), (a, b, -x_2, c, d)$. Since $p_g > 0$, it follows that $\varphi_{|mK_X|}$ is either generically $2:1$ or birational for $m > 4$. Note that $\varphi_{|mK_X|}$ is not generically $n:1$ for $m < 4$.

Next, the first value of m for which $B'''(X_0, X_1, X_3, X_4)$ is $\neq 0$ is 11 (see above); the m-canonical adjoint $\Phi_m : B'''(X_0, X_1, X_3, X_4)X_2 = 0$ "separates" the two points $(a, b, x_2, c, d), (a, b, -x_2, c, d)$ in the rational transformation $\varphi_{|11K_X|}$, thanks to the presence of the variable X_2 to the power 1. We thus deduce that $\varphi_{|11K_X|}$ is a birational tranformation. Again from $p_g > 0$, it follows that $\varphi_{|mK_X|}$ is a birational transformation for $m \geq 11$.

Therefore, we have proved that

the m-canonical transformation (improperly called a 'map') *of the threefold X is generically* $2:1$ *if and only if* $10 \geq m \geq 4$ *and it is birational if and only if* $m \geq 11$.

We note that X is birationally distinct from the threefolds appearing in the lists of [Re], pp. 358-359 and [F], pp. 151-154, because X has different plurigenera from those of the threefolds in said lists.

Based on a result given by M. Chen [C], we know that a threefold, with the bigenus $P_2 \geq 2$, has the m-canonical transformation that is birational for $m \geq 16$. As a consequence of this and of the above result, the optimal limitation for the birationality of the m-canonical transformation for threefolds with $P_2 = 2$ is now between 11 and 16.

We also constructed a nonsingular threefold Y of general type in [S_2] where $\varphi_{|mK_Y|}$ birational if and only if $m \geq 11$, but in that case Y has $p_g = 0$ and $P_2 = 1$.

Added in proofs. Meng Chen and Kang Zuo have proved that a nonsingular algebraic threefold of general type with $p_g = 1$ and $P_2 = 2$ has the m-canonical map (transformation) $\varphi_{|mK|}$ which is birational for $m \geq 11$ (cf. Theorem 4.4 in M. Chen - K. Zuo, *Complex projective threefolds with non-negative canonical Euler-Poincaré characteristic*, preprint, arXiv:math/0609545v2 [math.AG] 23 Oct. 2007). Our Example in the Appendix proves that such a limitation is optimal. Another example, having $p_g = 1$ and $P_2 = 2$ with $\varphi_{|11K|}$ birational and $\varphi_{|10K|}$ non-birational, like our example, was also presented by Iano-Fletcher (loc. cit. Example 4.8).

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Pervenuta in Redazione
il 9 febbraio 2007