
BOLLETTINO UNIONE MATEMATICA ITALIANA

LÁSZLÓ ZSILINSZKY

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. 10-B
(2007), n.3, p. 1071–1079.

Unione Matematica Italiana

http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_1071_0

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On Baireness of the Wijsman Hyperspace.

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Sunto. – *Si fornisce una caratterizzazione della proprietà di essere di Baire per la topologia di Wijsman nell'ipotesi in cui lo spazio di base sia quasi localmente separabile e metrizzabile. Si esibisce inoltre un esempio di spazio metrico di prima categoria la cui topologia di Wijsman risulta essere di Baire.*

Summary. – *Baireness of the Wijsman hyperspace topology is characterized for a metrizable base space with a countable-in-itself π -base; further a separable 1st category metric space is constructed with a Baire Wijsman hyperspace.*

1. – Introduction.

There has been a considerable effort in exploring various completeness properties of the Wijsman hyperspace topology w_d , i.e. the weak topology on the nonempty closed subsets of the metric space (X, d) generated by the distance functionals viewed as functions of set argument [17]. It was first shown by *Effros* [10], that a Polish space admits a metric for which the Wijsman topology is Polish; later, *Beer* showed [2],[3], that given a separable *complete* metric base space, the corresponding Wijsman hyperspace is Polish. Finally, *Costantini* demonstrated in [6], that Polish base spaces always generate Polish Wijsman topologies (a short proof, using the so-called strong Choquet game, was found by the author in [19]). As a related result, note that the Wijsman hyperspace is analytic iff X is analytic [1].

Beer asked, whether complete metrizability of X alone (without separability) is equivalent to some completeness property of the Wijsman hyperspace. *Costantini* [7] showed that a natural candidate, Čech-completeness, is not the right property; on the other side, complete metrizability of X guarantees Baireness [18], even strong a -favorability [19], of the Wijsman hyperspace regardless of the underlying metric on X . It is also known, that less than complete metrizability of X – e.g. having a dense completely metrizable subspace [20] or being a separable Baire space [18], respectively – guarantees Baireness of the Wijsman topology; however, w_d may be non-hereditarily Baire, even if X is se-

parable, hereditarily Baire and has a dense completely metrizable subspace [20], or X is completely metrizable [9], respectively. It is the purpose of this paper to continue in this research by characterizing Baireness of the Wijsman hyperspace for almost locally separable metrizable spaces.

A space is *almost locally separable*, provided the set of points of local separability is dense. In a metrizable space, this is equivalent to having a *countable-in-itself* π -base, i.e. a π -base, each element of which contains only countably many elements of the π -base [21] (cf. locally countable pseudo-base of Oxtoby [14]). A topological space is a *Baire space*, provided countable collections of dense open subsets have a dense intersection or, equivalently, if nonempty open sets are of 2nd Baire category [12].

Let $CL(X)$ stand for the space of nonempty closed subsets of a topological space (X, τ) (the so-called *hyperspace* of X), and for $M \subseteq X$ define

$$M^- = \{A \in CL(X) : A \cap M \neq \emptyset\},$$

$$M^+ = \{A \in CL(X) : A \subseteq M\}.$$

We will write M^c, \overline{M} for the complement and closure, respectively, of M in X . If (X, τ) is a metrizable topological space with a compatible metric d , denote by $S(x, \varepsilon)$ (resp. $B(x, \varepsilon)$) the open (resp. closed) ball of radius $\varepsilon > 0$ about $x \in X$, and put $S(M, \varepsilon) = \bigcup_{m \in M} S(m, \varepsilon)$ for the open ε -hull of $M \subseteq X$. Denote by $B(X)$ the collection of finite unions of closed balls.

The *Wijsman topology* w_d on $CL(X)$ is the weak topology generated by the distance functionals $d(x, A) = \inf\{d(x, a) : a \in A\}$ ($x \in X, A \in CL(X)$) viewed as functionals of set argument. It is easy to show that subbase elements of w_d are of the form U^- and $\{A \in CL(X) : d(x, A) > \varepsilon\}$, where $U \in \tau, x \in X$ and $\varepsilon > 0$. The Wijsman topology is a fundamental tool in the construction of the lattice of hyperspace topologies, since many of these arise as suprema and infima, respectively of appropriate Wijsman topologies [5],[8].

The *ball proximal topology* bp_d has subbase elements of the form U^- and $\{A \in CL(X) : \inf\{d(a, b) : (a, b) \in A \times B\} > \varepsilon\}$, where $U \in \tau, B \in \mathcal{A}, \varepsilon > 0$; it coincides with the Wijsman topology when X is a normed space (for a characterization of this coincidence see [11]). Moreover, bp_d is Baire if and only if w_d is [18]. As we will see, there is an even simpler hypertopology on $CL(X)$ with this “Baire connection”, the so-called ball topology.

The *ball topology* b_d has subbase elements of the form U^- and $(B^c)^+$, where $U \in \tau$ and $B \in B(X)$. It is not hard to show that the collection

$$\mathcal{B} = \{(B^c)^+ \cap \bigcap_{i \leq n} S(x_i, r)^- : B \in B(X), r > 0, n \in \omega,$$

the $S(x_i, r)$'s are pairwise disjoint and miss $B\}$

is a base for b_d . The ball topology is a hit-and-miss topology, like the well-known Vietoris or Fell topologies [3]; b_d may be non-regular [11], so it is certainly not always equal to the (completely regular) Wijsman topology, however, we have:

THEOREM 1.1. – *The following are equivalent:*

- (i) $(CL(X), w_d)$ is a Baire space,
- (ii) $(CL(X), b_d)$ is a Baire space.

PROOF. – If $\mathcal{U}_w = \bigcap_{i \leq n} U_i^- \cap \bigcap_{j \leq m} \{A \in CL(X) : d(x_j, A) > \varepsilon_j\} \in w_d$ is nonempty, $r = \min\{\frac{d(x_j, A) - \varepsilon_j}{2} : j \leq m\}$, and

$$\mathcal{U}_b = \bigcap_{i \leq n} U_i^- \cap \bigcap_{j \leq m} ((B(x_j, \varepsilon_j + r))^c)^+ \in b_d,$$

then $\emptyset \neq \mathcal{U}_b \subseteq \mathcal{U}_w$.

Conversely, if $\mathcal{U}_b = \bigcap_{i \leq n} U_i^- \cap \bigcap_{j \leq m} ((B(x_j, \varepsilon_j))^c)^+ \in b_d$ is nonempty, and $a_i \in U_i$ so that $d(x_j, a_i) > \varepsilon_j$ for all i, j , then $r = \min_{i,j} d(x_j, a_i) > \varepsilon_j$ for all j , and for

$$\mathcal{U}_w = \bigcap_{i \leq n} U_i^- \cap \bigcap_{j \leq m} \left\{ A \in CL(X) : d(x_j, A) > \frac{\varepsilon_j + r}{2} \right\} \in w_d$$

we have $\{a_0, \dots, a_n\} \in \mathcal{U}_w \subseteq \mathcal{U}_b$. The theorem now follows by [16], Proposition 3.4. □

2. – Main Results.

We need some auxiliary material first: natural numbers will be viewed as sets of predecessors, $\Delta \subseteq CL(X)$ will be closed under finite unions, bold symbols will denote notions related to the product space $\mathbf{X} = X^\omega$ endowed with the so-called *pinched-cube topology* [15] $\tau = \tau(\Delta)$ having the base

$$\mathcal{B} = \left\{ \left(\prod_{i \leq n} U_i \right) \times (B^c)^{\omega \setminus (n+1)} : B \in \Delta, n \in \omega \text{ and } U_i \in \tau, U_i \subseteq B^c \text{ for all } i \leq n \right\}.$$

If $J \subseteq \omega$, the projection map $\pi_J : (\mathbf{X}, \tau) \rightarrow X^J$ is continuous and open. If $C \subseteq \mathbf{X}$, $J \subseteq \omega$ and $x \in X^J$, denote $C[x] = C \cap \pi_J^{-1}(x)$; further, if \mathcal{C} is a collection of subsets of \mathbf{X} , put $\mathcal{C}[x] = \{C \in \mathcal{C} : C[x] \neq \emptyset\}$.

The proof of the next theorem is a modification of analogous results about the Tychonoff and box products of Baire spaces from [21], we sketch the proof for completeness:

THEOREM 2.1. – *If (X, τ) is a topological Baire space with a countable-in-itself π -base, then (\mathbf{X}, τ) is a Baire space.*

PROOF. – Let \mathcal{P} be a countable-in-itself π -base of X . Let $\{G_n\}_n$ be a decreasing sequence of dense open subsets of (X, τ) and fix a nonempty τ -open V . Choose $V_0 \in \mathcal{B}$ with $V_0 \subseteq V \cap G_0$, $\pi_1(V_0) \in \mathcal{P}$ and put $\mathcal{B}_0 = \{V_0\}$. By induction, we can define $\mathcal{B}_i \subseteq \mathcal{B}$ for each $i \geq 1$ so that $\mathcal{B}_i = \bigcup_{B \in \mathcal{B}_{i-1}} \mathcal{B}_i(B)$, where for all $B \in \mathcal{B}_{i-1}$, $\mathcal{B}_i(B)$ is a maximal collection such that

- (1) $A \subseteq B \cap G_i$ for each $A \in \mathcal{B}_i(B)$,
- (2) $\pi_{i+1}(A) \in \{\prod_{k \leq i} P_k : (P_0, \dots, P_i) \in \mathcal{P}^{i+1}\}$ for each $A \in \mathcal{B}_i(B)$,
- (3) $\{\pi_i(A) : A \in \mathcal{B}_i(B)\}$ is pairwise disjoint.

Notice, that each $\mathcal{B}_i(B)$ is countable, since, by (2), $\pi_i(B)$ has ccc for each $i \geq 1$ and $B \in \mathcal{B}_{i-1}$. For $i \geq 1$ denote

$$U_i = \bigcup_{B \in \mathcal{B}_i} \{U \in \mathcal{B} : \pi_{\omega \setminus i}(U) = \pi_{\omega \setminus i}(B) \text{ and } \{U \subseteq B\}.$$

For each $B \in \mathcal{B}_1(V_0)$ define

$$Y_B^{(0)} = \{x \in X : B[x] \neq \emptyset \text{ and } \forall U \in U_2(U \subseteq B \Rightarrow U[x] = \emptyset)\}.$$

CLAIM 2.1.1. $Y_B^{(0)}$ is nowhere dense in X for each $B \in \mathcal{B}_1(V_0)$.

(Assume by contradiction that for some $B \in \mathcal{B}_1(V_0)$, $Y_B^{(0)}$ is dense in some $U \in \tau$. Then $B \cap \pi_1^{-1}(U) \neq \emptyset$, and by maximality of $\mathcal{B}_2(B)$, there exists $E \in \mathcal{B}_2(B)$ such that $U = E \cap \pi_1^{-1}(U) \neq \emptyset$. Then $U \in U_2$, $U \subseteq B$, and $\pi_1(U)$ is a nonempty open subset of U ; thus, it intersects with $Y_B^{(0)}$, say, in x . To get a contradiction, note that $x \in \pi_1(U)$ means $U[x] \neq \emptyset$; on the other hand, $x \in Y_B^{(0)}$ implies $U[x] = \emptyset$.)

Define $W_0 = \pi_1(\bigcup \mathcal{B}_1(V_0))$. Since X is a Baire space, by Claim 2.1.1, we can choose some

$$x_0 \in W_0 \setminus \bigcup \{Y_B^{(0)} : B \in \mathcal{B}_1(V_0)\}.$$

Then, by (3), there is a unique $V_1 \in \mathcal{B}_1(V_0)[x_0]$; further, for each $B \in \mathcal{B}_1(V_0)[x_0]$ there is a $U \in U_2$ with $U \subseteq B$ and $U[x_0] \neq \emptyset$.

By induction, assume that for all $i \leq n$ ($n \in \omega$), $V_i \in \mathcal{B}_i$, and

$$x_i \in W_i = \pi_{(i+1) \setminus i}(\bigcup \mathcal{B}_{i+1}(V_i)[x_0, \dots, x_{i-1}])$$

have been defined (when $i = 0$, $\mathcal{B}_1(V_0)[x_0, x_{-1}]$ is meant to be $\mathcal{B}_1(V_0)$) so that

- (4) $\forall 1 \leq i \leq n (V_i \in \mathcal{B}_i(V_{i-1})[x_0, \dots, x_{i-1}])$,
- (5) $\forall i \leq n \forall B \in \mathcal{B}_{i+1}(V_i)[x_0, \dots, x_i] \exists U \in U_{i+2} (U \subseteq B \text{ and } U[x_0, \dots, x_i] \neq \emptyset)$.

Since $x_n \in W_n$, and, by (3), $\{\pi_{(n+1) \setminus n}(A) : A \in \mathcal{B}_{n+1}(V_n)[x_0, \dots, x_{n-1}]\}$ is pairwise disjoint, there is a unique $V_{n+1} \in \mathcal{B}_{n+1}(V_n)[x_0, \dots, x_{n-1}]$ with $V_{n+1}[x_0, \dots, x_n] \neq \emptyset$; thus, (4) holds for $1 \leq i \leq n + 1$.

Moreover, (5) implies that there is some $U \in \mathcal{U}_{n+2}$ with $U[x_0, \dots, x_n] \neq \emptyset$ and $U \subseteq V_{n+1}$. We can also find $B \in \mathcal{B}_{n+2}$ with $U \subseteq B$ and $\pi_{\omega \setminus (n+2)}(U) = \pi_{\omega \setminus (n+2)}(B)$. By (1), $B \subseteq G_{n+2}$, and since B intersects V_{n+1} , it follows by (3), that $B \subseteq V_{n+1}$; thus, $B \in \mathcal{B}_{n+2}(V_{n+1})[x_0, \dots, x_n]$, so

$$W_{n+1} = \pi_{(n+2) \setminus (n+1)}(\bigcup \mathcal{B}_{n+2}(V_{n+1})[x_0, \dots, x_n])$$

is a nonempty open subset of X . For each $B \in \mathcal{B}_{n+2}(V_{n+1})[x_0, \dots, x_n]$ define

$$Y_B^{(n+1)} = \{x \in X : B[x_0, \dots, x_n, x] \neq \emptyset \text{ and } \forall U \in \mathcal{U}_{n+3}(U \subseteq B \Rightarrow U[x_0, \dots, x_n, x] = \emptyset)\}.$$

CLAIM 2.1.2. $Y_B^{(n+1)}$ is nowhere dense in X for each $B \in \mathcal{B}_{n+2}(V_{n+1})[x_0, \dots, x_n]$.

(Indeed, if $Y_B^{(n+1)}$ is dense in some $U \in \tau$, then $B \cap \pi_{(n+2) \setminus (n+1)}^{-1}(U) \neq \emptyset$, and by maximality of $\mathcal{B}_{n+3}(B)$, there exists $E \in \mathcal{B}_{n+3}(B)$ such that $U = E \cap \pi_{(n+2) \setminus (n+1)}^{-1}(U) \neq \emptyset$. Then $U \in \mathcal{U}_{n+3}$, $U \subseteq B$, and $\pi_{(n+2) \setminus (n+1)}(U)$ is a nonempty open subset of U ; thus, it intersects with $Y_B^{(n+1)}$, say, in x . Finally, $x \in \pi_{(n+2) \setminus (n+1)}(U)$ means that $U[x] \neq \emptyset$; on the other hand, $x \in Y_B^{(n+1)}$ implies $U[x] = \emptyset$.)

Now, X is a Baire space so, by Claim 2.1.2, we can find

$$x_{n+1} \in W_{n+1} \setminus \bigcup \{Y_B^{(n+1)} : B \in \mathcal{B}_{n+2}(V_{n+1})[x_0, \dots, x_n]\},$$

so (5) is satisfied for $i \leq n + 1$. By induction, we have constructed $\mathbf{x} = (x_n)_{n \in \omega} \in X$, and a sequence $\{V_n \in \mathcal{B} : n \in \omega\}$ with $V_{n+1} \in \mathcal{B}_{n+1}(V_n)[x_0, \dots, x_n]$ for all $n \in \omega$. Then $\mathbf{x} \in V_n \subseteq V \cap G_n$ for each $n \in \omega$, so $V \cap \bigcap_n G_n \neq \emptyset$, and (X, τ) is a Baire space. □

THEOREM 2.2. – Let X be metrizable with a compatible metric d . If $(X, \tau(B(X)))$ is a Baire space, then $(CL(X), b_d)$ is a Baire space.

PROOF. – The set

$$S(X) = \{A \in CL(X) : A \text{ separable}\}$$

is dense in $(CL(X), b_d)$, since even the set of finite subsets of X is; thus, we only need to prove that $(S(X), b_d \upharpoonright_{S(X)})$ is a Baire space.

Define $\varphi : (X, \tau) \rightarrow (S(X), b_d \upharpoonright_{S(X)})$ via

$$\varphi(\mathbf{x}) = \overline{\{x_k : k \in \omega\}}, \text{ where } \mathbf{x} = (x_k)_k \in X.$$

We will be done, if we show that φ is continuous and feebly open (i.e. the interior of $\varphi(V)$ is nonempty for each nonempty $V \in \tau$), since Baire spaces are invariant of these mappings (see [12] or [13]).

To see continuity, take a basic $\mathcal{Z}' = (B^c)^+ \cap \bigcap_{i \leq m} V_i^- \in \mathcal{B}$, where $B = \bigcup_{j \leq p} B(z_j, \varepsilon_j) \in B(X)$. If $\mathbf{x} = (x_k)_k \in \varphi^{-1}(\mathcal{Z}')$, then $\varphi(\mathbf{x}) \in \mathcal{Z}'$, so there exists a $k_i \in \omega$ with $x_{k_i} \in V_i$ for each $i \leq m$. We can find some $\delta_j > \varepsilon_j$ so that $d(z_j, f(\mathbf{x})) > \delta_j$ for each $j \leq p$, as well as neighborhoods U_i of x_{k_i} (for all $i \leq m$) that are subsets of $U = (\bigcup B(z_j, \delta_j))^c$. Then $\mathbf{U} = (\prod_{i \leq m} U_i) \times U^{\omega \setminus (m+1)} \in \mathcal{B}$, and $\mathbf{x} \in \mathbf{U} \subseteq \varphi^{-1}(\mathcal{Z}')$.

To justify feeble openness of φ , take a nonempty $\mathbf{V} = (\prod_{i \leq m} V_i) \times (B^c)^{\omega \setminus (m+1)} \in \mathcal{B}$, where $B = \bigcup_{j \leq p} B(z_j, \varepsilon_j) \in B(X)$. Denote $\mathcal{Z}' = (B^c)^+ \cap \bigcap_{i \leq m} V_i^- \in \mathcal{B}$, and take an $A \in \mathcal{Z}' \cap S(X)$.

If $C = \{c_k : k \in \omega\}$ is a countable dense subset of A , where $c_i \in C \cap V_i$ for each $i \leq m$, then $\mathbf{c} = (c_k)_k \in \mathbf{V}$, so $A = \varphi(\mathbf{c}) \in \varphi(\mathbf{V})$. Consequently, $\emptyset \neq \mathcal{Z}' \cap S(X) \subseteq \varphi(\mathbf{V})$. □

THEOREM 2.3. – *If X is almost locally separable, then the following are equivalent:*

- (i) $(CL(X), w_d)$ is Baire for each compatible metric d on X ,
- (ii) $(CL(X), bp_d)$ is Baire for every compatible metric d on X ,
- (iii) $(CL(X), b_d)$ is Baire for every compatible metric d on X ,
- (iv) X is a Baire space.

PROOF. – (i) \Leftrightarrow (ii) \Leftrightarrow (iii) follows from Theorem 1.1, and (iv) \Rightarrow (iii) from Theorems 2.1 and 2.2

(iii) \Rightarrow (iv) If X is not a Baire space, it has a nonempty open 1st category subset U , and so there exists a separable closed 1st category subset C of U with a nonempty interior $intC$. If d_0 is a compatible totally bounded metric on C , then by a theorem of Bing [4], it can be extended to a compatible metric d on X . Let C_n be an increasing sequence of closed nowhere dense sets such that $C = \bigcup_{n \in \omega} C_n$. By total boundedness of d on C , there exists a finite set $F_{k,n} \subseteq C_n$ for each $n \in \omega, k \geq 1$, such that $C_n \subseteq S(F_{k,n}, \frac{1}{k})$. Define

$$\mathcal{S}_n = \bigcup_{k \geq 1} \bigcup_{x \in F_{k,n}} \left(B\left(x, \frac{1}{k}\right)^c \right)^+.$$

Then \mathcal{S}_n is clearly b_d -open. Moreover, it is also dense in $(CL(X), b_d)$: indeed, let $\mathcal{Z}' = (B^c)^+ \cap \bigcap_{i \leq m} V_i^- \in \mathcal{B}$. For each $i \leq m$ there exist $k_i \geq 1$ and $v_i \in V_i \setminus S(C_n, \frac{1}{k_i})$, since otherwise, $V_i \subseteq \bigcap_{k \geq 1} S(C_n, \frac{1}{k}) = C_n$ for some $i \leq m$, which would contradict nowhere density of C_n . If $k = \max\{k_i : i \leq m\}$ and $A = \{v_0, \dots, v_m\}$, then $A \cap S(F_{k,n}, \frac{1}{k}) = \emptyset$, so $A \in \mathcal{Z}' \cap \mathcal{S}_n$. To conclude, notice that $\bigcap_{n \in \omega} \mathcal{S}_n$ is disjoint to $(intC)^-$. □

Since the Wijsman hyperspace is metrizable iff X is separable [3], we have

COROLLARY 2.4. – *The following are equivalent:*

- (i) $(CL(X), w_d)$ is a metrizable Baire space for each compatible metric d on X ,
- (ii) X is a separable Baire space.

In light of Theorem 2.3, it is natural to ask whether Baireness of just a single Wijsman topology w_d (bp_d, b_d , respectively) implies Baireness of X . The following example (given by *R. Pol*) shows, that this is not the case:

EXAMPLE 2.5. – There exists a separable 1st category metric space with a Baire Wijsman (ball proximal, ball, resp.) hyperspace.

PROOF. – Consider ω^ω with the Baire metric

$$e(x, y) = 1/\min\{n + 1 : x(n) \neq y(n)\}$$

and its 1st category subset $\omega^{<\omega}$ of sequences eventually equal to zero. Then the product $X = \omega^{<\omega} \times \omega^\omega$ is a separable, 1st category space endowed with the metric $d((x_0, x_1), (y_0, y_1)) = \max\{e(x_0, y_0), e(x_1, y_1)\}$.

We claim that $(CL(X), b_d)$ is a Baire space: let $p : X \rightarrow \omega^{<\omega}$ be the projection onto the first axis. Let $\mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \dots$ be dense open sets in $(CL(X), b_d)$, and $\mathcal{U} \in \mathcal{B}$. Inductively, we will define $\mathcal{U}_i \in \mathcal{B}$ with $\mathcal{U}_i \subseteq \mathcal{S}_i$, and finite sets $F_i \in \mathcal{U}_i$ such that for each $u \in F_i$ there is $u^* \in F_{i+1}$ with $p(u) = p(u^*)$ and $d(u, u^*) < \frac{1}{i}$.

Let $\mathcal{U}_1 \in \mathcal{B}$ be a nonempty subset of \mathcal{S}_1 , and choose a finite set $F_1 \in \mathcal{U}_1$. Suppose that \mathcal{U}_i and F_i have been defined for some $i \geq 1$. If

$$m_i \geq \max\{n : p(u)(n) \neq 0 \text{ for some } u \in F_i\},$$

and $u' \in S(u, \frac{1}{m_i})$ for some $u \in F_i$, then $p(u) = p(u')$. Since \mathcal{S}_{i+1} is dense and $F_i \in \mathcal{U}_i \cap \bigcap_{u \in F_i} S(u, \frac{1}{m_i})$, we can find $\mathcal{U}_{i+1} \in \mathcal{B}$ with

$$\mathcal{U}_{i+1} \subseteq \mathcal{S}_{i+1} \cap \mathcal{U}_i \cap \bigcap_{u \in F_i} S\left(u, \frac{1}{m_i}\right),$$

and choose a finite $F_{i+1} \in \mathcal{U}_{i+1}$. Assume that

$$\mathcal{U}_i = (B_i^c)^+ \cap \bigcap_{u \in F_i} V_u^- \in \mathcal{B}.$$

For any $u \in F_i$, the sequence u, u^*, u^{**}, \dots is Cauchy in $\{p(u)\} \times \omega^\omega$, so it converges to some $u^\infty \in S(u, \frac{1}{m_i}) \subseteq V_u$. By the definition of \mathcal{B} , the set

$\{u^\infty : u \in \bigcup_{i \geq 1} F_i\}$ misses the clopen B_i for each $i \geq 1$, so

$$\emptyset \neq \overline{\{u^\infty : u \in \bigcup_{i \geq 1} F_i\}} \in \bigcap_{i \geq 1} \mathcal{U}_i \subseteq \mathcal{U} \cap \bigcap_{i \geq 1} \mathcal{G}_i;$$

thus, $(CL(X), b_d)$ is a Baire space. \square

Acknowledgements. The author wishes to thank *Roman Pol* for communicating Example 2.5 to him, as well as the referee for the useful comments concerning Theorem 2.1 that greatly improved the paper.

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Department of Mathematics and Computer Science
The University of North Carolina at Pembroke
Pembroke, NC 28372, USA
e-mail: laszlo@uncp.edu

Pervenuta in Redazione

il 6 novembre 2006 e in forma rivista il 14 aprile 2008

