José A. Carrillo, Marco Di Francesco, Corrado Lattanzio

Contractivity and Asymptotics in Wasserstein Metrics for Viscous Nonlinear Scalar Conservation Laws

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.2, p. 277–292.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_2_277_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 277-292

Contractivity and Asymptotics in Wasserstein Metrics for Viscous Nonlinear Scalar Conservation Laws.

J. A. Carrillo - M. Di Francesco - C. Lattanzio

Sunto. – *In questo articolo sono riportati alcuni risultati recenti riguardo il comportamento asintotico nel tempo di leggi di conservazione scalari in una dimensione spaziale e con densità di probabilità come dati iniziali. Tali risultati sono quindi applicati al caso di leggi di conservazione viscose con diffusioni nonlineari degeneri. Le proprietà di contrazione nella distanza di trasporto massimale e di uniforme espansione delle soluzioni forniscono l'esistenza di profili asintotici dipendenti dal tempo per un'ampia classe di equazioni di convenzione-diffusione con nonlinearità arbitrarie e diffusione degenere.*

Summary. – *In this work, recent results concerning the long time asymptotics of one-dimensional scalar conservation laws with probability densities as initial data are reviewed and further applied to the case of viscous conservation laws with nonlinear degenerate diffusions. The non-strict contraction of the maximal transport distance together with a uniform expansion of the solutions lead to the existence of time-dependent asymptotic profiles for a large class of convection-diffusion problems with fully general nonlinearities and with degenerate diffusion.*

1. – Introduction.

Transport metrics [35, 36] have recently received a lot of attention due to the fascinating applications in the understanding of the long time asymptotics of nonlinear diffusion equations [30, 15, 2, 31, 32, 13]. In most of these results the transport distance used is the euclidean Wasserstein distance that is proven to be contractive for the corresponding flows. Transport distances with different index were proved to be contractive for one-dimensional nonlinear diffusions in [16, 14] and used to analyse p-laplacian type equations in [1, 2]. The contraction in higher dimensions of these distances find geometrical obstacles due to focusing solutions [34].

The contraction of transport metrics for evolutionary partial differential equations of probability metrics has been shown to be an extremely powerful tool

(*) Conferenza tenuta a Torino il 3 luglio 2006 in occasione del "Joint Meeting S.I.M.A.I. - S.M.A.I. - S.M.F. - U.M.I. sotto gli auspici dell'E.M.S. Mathematics and its Applications".

in many different situations: kinetic theory [9, 6], fluid mechanics [7, 19, 29], geometric problems and Ricci curvature [32, 36, 28], conservation laws [5, 12, 8], semiconductors equations [21] and many others.

In this work, we first review the main results obtained in [12] describing the long-time asymptotics of nonlinear scalar one-dimensional conservation laws in terms of transport metrics. The main ingredients are the contraction of the maximal transport distance and a suitable scaling based on the maximal transport moment reminiscent of similar scalings used for the nonlinear diffusion case in [33, 13, 11, 17]. However, the use of the maximal transport distance posed new technical challenges to apply this strategy. As a result, long time behavior is described in terms of a time dependent family of functions playing the role of a typical asymptotic profile of the system.

The main advantage of this approach being that it gives results for quite general nonlinearities.

In this paper, we present a further application of this strategy to the case of general convection-diffusion equations in one dimension. Diffusion is assumed to be degenerate, and thus, solutions typically show finite speed of propagation. We will show that the maximal transport distance is a contractive metric for this system. The expansion of the solutions is controlled uniformly in terms of the maximal transport moment. These two facts allow us to analyse the long time behavior of convection-diffusion equations with quite general nonlinearities based on a scaling involving the maximal transport moment.

2. – Transport distances in one dimension.

All the models considered in this work preserve the total mass and enjoy a minimum principle which ensures the solution remains nonnegative if initially so. Therefore each solution can be interpreted as a curve in the space $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$. For a given $p \in [1, +\infty)$, we shall use the notation

$$\mathcal{P}_p(\mathbb{R}) = \left\{ \mu \in \mathcal{P}(\mathbb{R}) \ : \ \int_{\mathbb{R}} |x|^p d\mu < +\infty \right\}.$$

We then recall the definition of *p-Wasserstein distance* between $\mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R})$

$$d_p(\mu_1, \mu_2) := \inf \left\{ \iint_{\mathbb{R}^2} |x-y|^p d\pi(x,y), \ \ \pi \in \Gamma(\mu, \upsilon) \right\},$$

where $\Gamma(\mu_1, \mu_2)$ is the set of *transport plans* between μ_1 and μ_2, i.e. $\Gamma(\mu_1, \mu_2)$ is the set of probability measures π on $\mathbb{R}^2$ such that μ_1 and μ_2 are the marginal measures of π, see for instance [35]. When $\mu_i = u_i dx$, where $u_1, u_2 \in L^1(\mathbb{R}) \cap \cap \mathcal{P}_p(\mathbb{R})$ and dx is the one-dimensional Lebesgue measure, the p-Wasserstein between μ_1 and μ_2 can be written (dropping dx for simplicity) as

$$d_p(u_1, u_2)^p = \inf_{T: u_2 = T_\sharp u_1} \int_{-\infty}^{+\infty} |x - T(x)|^p u_1(x) dx,$$

where the constraint $u_2 = T_\sharp u_1$ (which is usually referred to as the density u_2 being the *push forward* by T of the density u_1) is expressed by the condition

$$\int_{\mathbb{R}} \varphi(x) u_2(x) dx = \int_{\mathbb{R}} \varphi(T(x)) u_1(x) dx,$$

for any $\varphi \in C_b^0(\mathbb{R})$. Due to the property

$$p \leq q \quad \Rightarrow \quad d_p(\mu_1, \mu_2) \leq d_q(\mu_1, \mu_2),$$

one can introduce the ∞-Wasserstein distance

$$d_\infty(\mu_1, \mu_2) := \lim_{p \to +\infty} d_p(\mu_1, \mu_2) = \sup_{p \geq 1} d_p(\mu_1, \mu_2).$$

In one space dimension, the Wasserstein metrics d_p, $p \in [1, +\infty]$, have a simple interpretation in terms of the pseudo-inverses of the primitive of the involved measures μ_1 and μ_2 [16, 35]. Let us denote the distribution functions of μ_i, $i = 1, 2$, by

$$F_i(x) = \mu_i((-\infty, x]), \quad x \in \mathbb{R}, \qquad i = 1, 2$$

and let us define their pseudo-inverses $F_i^{-1} : [0, 1] \to \mathbb{R}$ as follows

$$F_i^{-1}(\xi) = \inf\{x : F_i(x) > \xi\}.$$

Then, for any $p \in [1, +\infty]$,

$$d_p(\mu_1, \mu_2) = \|F_1^{-1} - F_2^{-1}\|_{L^p([0,1])}. \tag{2.1}$$

REMARK 2.1 [*Interpretations of the d_∞*]. – The above representation formula (2.1) provides an interesting interpretation of the d_∞ in terms of the supports of the involved measures. More precisely, suppose that the supports $K_i := \mathrm{supp}(\mu_i)$, $i = 1, 2$, are compact. Then, one can easily prove the estimate

$$\sup\{|\inf K_1 - \inf K_2|, |\sup K_1 - \sup K_2|\} \leq d_\infty(\mu_1, \mu_2).$$

Clearly, such an interpretation only makes sense if both measures μ_1 and μ_2 have compact support.

An alternative interpretation is possible when the supports of μ_1 and μ_2 are connected (possibly unbounded) sets of $\mathbb{R}$: we observe that for a fixed amount of mass $\xi \in (0, 1)$, the quantity $F_1^{-1}(\xi) - F_2^{-1}(\xi)$ measures the distance between the two points x_1 and x_2 in which the two measures μ_1 and μ_2 respectively "reach the same amount of mass ξ". More precisely, x_1 and x_2 are defined as the unique points such that

$$\xi = \int_{-\infty}^{x_1} d\mu_1 = \int_{-\infty}^{x_2} d\mu_2$$

and we have

$$|x_1 - x_2| = |F_1^{-1}(\xi) - F_2^{-1}(\xi)|.$$

Therefore, (2.1) implies that $d_\infty(\mu_1, \mu_2)$ controls the quantity $|x_1 - x_2|$ for all choices of the intermediate mass $\xi \in (0,1)$.

The maximal transport distance has been used to give simple proofs of growth estimates for the support of solutions in [14, 11] to nonlinear diffusion equations and for the stability of some particular solutions for fluid equations in [29]. For future use, we define the "maximal transport moment" of a compactly supported density, see also [34], as follows.

DEFINITION 2.2 [Maximal transport moment]. – *Let $v \in L^1_+(\mathbb{R})$ be compactly supported. Then, the maximal transport moment of v is defined by*

$$\mathrm{m}_\infty[v] := \sup\{|x|,\ x \in \mathrm{supp}(v)\} = d_\infty(v, \delta_0),$$

where δ_0 is the Dirac mass centered at zero.

3. – Results for the inviscid case.

In this section, we quickly review the main results on the contraction of the ∞-Wasserstein distance and their consequences on the asymptotic behavior of nonlinear scalar conservation laws given in [12]. Let us consider a general scalar conservation law

$$u_t + f(u)_x = 0, \tag{3.1}$$

with initial condition $u(x,0) = \bar{u}(x)$, where the flux function $f(u)$ is convex and $\bar{u} \in L^\infty(\mathbb{R})$, $\mathrm{supp}(\bar{u})$ compact, $\bar{u} \geq 0$ and, without loss of generality,

$$\int_{\mathbb{R}} \bar{u}(x)dx = 1.$$

Let us denote this set of initial data as $\mathcal{B}$.

The first interesting piece of information in [12] on the connection of optimal transport distances and conservation laws is that the flow of convex scalar conservation laws is a non-strict contraction with respect to d_∞. This result is based on the fact that solutions of strictly convex scalar conservation laws can be obtained through the solutions of the associated Hamilton-Jacobi equation via the Hopf-Lax formula. This allows for explicit formulas for the pseudo inverse function of solutions, and thus, for explicit estimates of the L^∞-distance between pseudo-inverses functions of two solutions. This idea is fully developed in [12], where the following result is shown:

THEOREM 3.1 ([12]). – *Given u and v solutions to* (3.1) *with initial data $\bar{u}, \bar{v} \in \mathcal{B}$ respectively. Assume that both initial data either belong to $BV(\mathbb{R})$ or that their support has a finite number of connected components. In addition, assume that the flux $f \in C^1$ in* (3.1) *is convex. Then, for any $t > 0$,*

$$d_\infty(u(t), v(t)) \leq d_\infty(\bar{u}, \bar{v}). \tag{3.2}$$

Contractivity properties of optimal transport metrics are at the basis of recent results of fine asymptotic behavior for fully nonlinear diffusion equations [13, 11, 17]. Time dependent asymptotic profiles are obtained given the asymptotic behavior of suitably scaled solutions of the equations. The main idea is that the scaling should be related to a measure of the expansion/dispersion of the solution in time. In the case of nonlinear diffusions, the euclidean Wasserstein distance and the second moment of solutions were respectively used as contractive metric and measure of expansion for solutions.

In [12], the above contractive property for solutions of scalar conservation laws was used to derive such time dependent asymptotic profile for general convex fluxes with an additional assumption allowing for control of expansion of the support. More precisely, we assume that the flux $f : [0, +\infty) \to [0, +\infty)$ is a C^1 convex function such that $f(0) = f'(0) = 0$ and such that

$$\exists\, a \in (0,1), \quad r \mapsto f(r)^{1-a} \text{ is convex on } (0, +\infty). \tag{3.3}$$

Under the requirement (3.3), it has been proven in [27] that the L^∞_x-norm of any solution $u(t)$ to (3.1) with initial data in $\mathcal{B}$ decays to zero as $t \to +\infty$. More precisely, using the results in [18], they are able to show [27, Proposition 2.1] that

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq f^{-1}\left(\frac{C(a)}{t}\|u(0)\|_{L^1(\mathbb{R})}\right) \tag{3.4}$$

for all $t > 0$. This uniform decay of the $L^\infty(\mathbb{R})$ norm implies a uniform divergence of moments of the solutions and in particular of the maximal transport moment

$$\mathrm{m}_\infty[\mu] = \sup\{|x|,\ x \in \mathrm{supp}(\mu)\} = d_\infty(\mu, \delta_0), \tag{3.5}$$

introduced in Definition 2.2. This moment will be our preferred expansion measure for the solutions of the scalar conservation law. This uniform divergence of moments together with the scaling properties of the distance between solutions and some geometric-like inequalities allow to prove the following theorem about intermediate asymptotics for scalar conservation laws:

THEOREM 3.2 ([12]). – *Let f be a flux satisfying the conditions above. Given the set*

$$\mathcal{M} = \{\mu \in \mathcal{P}(\mathbb{R}),\ \mathrm{m}_\infty[\mu] = 1\},$$

then there exist a fixed $t^* > 0$ and a one parameter family of functions $\{U_t^\infty\}_{t \geq t^*} \subset \mathcal{M} \cap \mathcal{B}$ with connected support such that, for any $u_0 \in \mathcal{M} \cap \mathcal{B}$ either having a support with a finite number of connected components or belonging to $BV(\mathbb{R})$, we have

$$d_\infty(\mathcal{T}_t[u_0], U_t^\infty) \to 0, \qquad \text{as } t \to +\infty, \tag{3.6}$$

where the map $\mathcal{T}_t$ is defined as

$$(\mathcal{T}_t[\bar{u}])(x) := \mathrm{m}_\infty[u(t)]u(\mathrm{m}_\infty[u(t)]x, t) \tag{3.7}$$

being u the solution to (3.1) *with initial datum $\bar{u}$. Moreover, for any fixed $t > t^*$, U_t^∞ is characterized as the unique fixed point of $\mathcal{T}_t : \mathcal{M} \to \mathcal{M}$.*

Hereafter we shall refer to the map $\mathcal{T}_t$ defined above as *generalized Toscani map* as introduced in [33, 17]. The objective of this work is to show that these ideas can be extended for general one-dimensional convection-diffusion equations.

In the next two sections, we show how to generalize the contraction property of the distance d_∞ for the viscous Burgers' equation and viscous conservation laws with degenerate nonlinear diffusion.

4. – Contraction for viscous Burgers' equation.

In this section we focus our attention on the viscous Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = u_{xx}, \tag{4.1}$$

with initial datum

$$u(x,0) = \bar{u} \in L^1_+(\mathbb{R}) \tag{4.2}$$

and, without loss of generality, total mass equal to one. The ideas of this section are inspired by the results in [20]. By means of the classical Hopf-Cole transformation [24]

$$\mathcal{H}(u)(x,t) := \psi(x,t) := \frac{1}{2}u(x,t)\exp\left(-\frac{1}{2}\int_{-\infty}^{x} u(\zeta,t)d\zeta\right), \tag{4.3}$$

we rewrite (4.1)-(4.2) as follows:

$$\begin{cases} \psi_t = \psi_{xx} \\ \psi(x,0) = \bar{\psi}(x) = \frac{1}{2}\bar{u}(x)\exp\left(-\frac{1}{2}\int_{-\infty}^{x} \bar{u}(\zeta)d\zeta\right) \end{cases} \tag{4.4}$$

and the initial datum $\bar{\psi}$ has total mass equal to

$$m = 1 - e^{-\frac{1}{2}}. \tag{4.5}$$

By following the same argument as in [20], we obtain the following theorem.

THEOREM 4.1. – *Let $p \in [1, +\infty]$. Let u_1 and u_2 be solutions to (4.1) with compactly supported initial data $\bar{u}_1$, $\bar{u}_2 \in L^1_+(\mathbb{R})$, both with total masses equal to one. Then the Wasserstein distance $d_p(u_1(t), u_2(t))$ satisfies the estimates*

$$d_p(u_1(t), u_2(t)) \leq e^{1/2p} d_p(\bar{u}_1, \bar{u}_2), \tag{4.6}$$

for $p \in [1, +\infty)$,

$$d_\infty(u_1(t), u_2(t)) \leq d_\infty(\bar{u}_1, \bar{u}_2), \tag{4.7}$$

for $p = +\infty$.

PROOF. – We shall make use of the one-dimensional representation formula (2.1). We recall that the two solutions $u_1(\cdot, t)$ and $u_2(\cdot, t)$ are supported on the whole real line $\mathbb{R}$ at any fixed $t > 0$, due to the effect of the linear diffusion [24]. Therefore, the two distribution functions

$$F_i(x,t) := \int_{-\infty}^{x} u_i(z,t)dz, \qquad i = 1, 2,$$

are strictly increasing and therefore invertible. Since we want to use (4.3) in order to deal with Wasserstein distances involving solutions of the heat equation, we also introduce the auxiliary distribution functions

$$G_i(x,t) := \int_{-\infty}^{x} \psi_i(z,t)dz, \qquad i = 1, 2,$$

where $\psi_i = \mathcal{H}(u_i)$ as in (4.3). For the same reasons as before, both ψ_1 and ψ_2 have $\mathbb{R}$ as support and the two distribution functions G_1 and G_2 are invertible. In order to find the relation between the F_i's and the G_i's, we introduce the function $a : [0, m] \to [0, 1]$ defined by

$$a(t) := 1 - e^{-\frac{1}{2}t}$$

and we compute

$$G_i(x,t) = \int_{-\infty}^{x} \psi_i(z,t)dz = -\int_{-\infty}^{x} \frac{\partial}{\partial z} \exp\left(-\frac{1}{2}\int_{-\infty}^{z} u_i(y,t)dy\right)dz$$
$$= 1 - \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_i(y,t)dy\right) = a(F_i(x,t)),$$

which implies

$$F_i^{-1}(\xi,t) = G_i^{-1}(a(\xi),t), \quad \xi \in (0,1),\ i = 1,2. \tag{4.8}$$

Hence, we use (4.8) and change variables in the following integral estimate

$$d_p^p(u_1(t),u_2(t)) = \int_0^1 |F_1^{-1}(\xi,t) - F_2^{-1}(\xi,t)|^p d\xi = \int_0^m |G_1^{-1}(\eta,t) - G_2^{-1}(\eta,t)|^p \frac{2}{1-\eta} d\eta$$
$$\leq 2\sqrt{e}\int_0^m |G_1^{-1}(\eta,t) - G_2^{-1}(\eta,t)|^p d\eta = 2\sqrt{e}\ d_p^p\left(\frac{\psi_1(t)}{m}, \frac{\psi_2(t)}{m}\right).$$

Since the heat equation is contractive with respect to all the p-Wasserstein distances (see e.g. [16]), we have

$$d_p^p(u_1(t),u_2(t)) \leq 2\sqrt{e}\, d_p^p\left(\frac{\psi_1(0)}{m}, \frac{\psi_2(0)}{m}\right) = 2\sqrt{e}\int_0^m |G_1^{-1}(\eta,0) - G_2^{-1}(\eta,0)|^p d\eta$$
$$= \sqrt{e}\int_0^1 |F_1^{-1}(\xi,0) - F_2^{-1}(\xi,0)|^p e^{-\frac{\xi}{2}} d\xi \leq \sqrt{e}\, d_p^p(\bar{u}_1, \bar{u}_2)$$

and the proof of (4.6) is complete. The proof of (4.7) easily follows by taking $p \nearrow +\infty$. □

REMARK 4.2. – We stated the above theorem under the assumption of $\bar{u}_1$ and $\bar{u}_2$ having compact support only to have

$$d_\infty(\bar{u}_1, \bar{u}_2) < +\infty. \tag{4.9}$$

Actually, the assumption on the supports can be replaced by (4.9), which should be interpreted according to the observations in Remark 2.1.

5. – Contraction for viscous conservation laws with slow diffusion.

In this section we examine the contractivity properties in the d_∞ distance for the nonlinear viscous conservation law

$$u_t + f(u)_x = g(u)_{xx}, \tag{5.1}$$

with nonnegative initial data in $\mathcal{B}$. The nonlinear diffusion g shall verify $g'(u) \geq 0$ and $g'(0) = 0$. The latter condition, a slow diffusion condition, guarantees that the equation (5.1) enjoy finite speed of propagation (see e.g. [25]), which implies in particular that its solutions $u(\cdot, t)$ have compact support for any $t > 0$, if the initial data $\bar{u}(\,\cdot\,)$ are compactly supported. This property is necessary to take advantage of the contractivity results already proved in the inviscid case [12] and reviewed in Section 3. Moreover, as in [12], we do not require any specific hypothesis on the nature of the nonlinear flux f and on the nonlinear diffusion g. The proof of the desired contractivity property will be obtained via the operator splitting method applied to (5.1) [22], taking advantage of the results already available for the inviscid conservation law [12] and the nonlinear diffusion equation [16]. Our main result states as follows.

THEOREM 5.1. – *Let $u_1(x,t)$ and $u_2(x,t)$ be the weak entropy solutions to* (5.1) *with nonnegative initial data $\bar{u}_1(x)$, $\bar{u}_2(x) \in \mathcal{B} \cap BV(\mathbb{R})$. Assume that f and g are locally C^2 functions, f is convex and $g'(0) = 0$.*

Then

$$d_\infty(u_1(t), u_2(t)) \leq d_\infty(\bar{u}_1, \bar{u}_2). \tag{5.2}$$

PROOF. – Following [22], we consider approximate solutions $u_{1,n}$ and $u_{2,n}$ of our solutions u_1 and u_2 defined via operator splitting method. Let us show briefly how to construct such an approximation for a general solution u to (5.1) with initial datum $\bar{u}$.

Fix a time $T > 0$ and a time step $\triangle t > 0$ such that $N \triangle t = T$, then we define an approximate solution u^n by induction. For $n = 0$, we choose as first term u^0 the initial datum $\bar{u}$. Then, if $u^n(x)$ is the approximate solution at a time $t_n = n\triangle t$, $n = 0, \ldots N-1$, we construct the successive term $u^{n+1}(x)$ as follows. Let us denote with $\mathcal{S}^1_t$ the semigroup defining the unique weak entropic solution for the Cauchy problem associated with the nonlinear conservation law

$$u_t + f(u)_x = 0.$$

Then we define

$$u^{n+1/2}(x) = \mathcal{S}^1_{\triangle t} u^n(x).$$

Similarly, we denote with $\mathcal{S}^2_t$ the semigroup defining the unique weak solution for the Cauchy problem associated with the nonlinear diffusion equation

$$u_t = g(u)_{xx}.$$

Thus, we define

$$u^{n+1}(x) = \mathcal{S}^2_{\triangle t} u^{n+1/2}(x) = (\mathcal{S}^2_{\triangle t} \circ \mathcal{S}^1_{\triangle t}) u^n(x)$$

and finally we define

$$u_n(x,t) = u^n(x),$$

for any $(x,t) \in \mathbb{R} \times (t_n, t_{n+1}]$ and $n = 0, \ldots N-1$.

Since both $\mathcal{S}^1_t$ and $\mathcal{S}^2_t$ do not increase total variation and L^∞ norm, we conclude that

$$\|u_n(\cdot,t)\|_\infty \leq \|\bar{u}\|_\infty$$

and

$$\|u_n(\cdot,t)\|_{BV(\mathbb{R})} \leq \|\bar{u}\|_{BV(\mathbb{R})}$$

for any n and $t > 0$. Hence, the Helly's compactness theorem and the continuity of u_n as a function of t with values in $L^1(\mathbb{R})$ imply there exists a subsequence u_{n_k} such that for any $t \in [0,T]$

$$u_{n_k}(\cdot,t) \to u(\cdot,t) \text{ in } L^1_{loc}(\mathbb{R}) \tag{5.3}$$

and bounded almost everywhere. Moreover, the uniqueness of entropy solutions for degenerate parabolic equations [10] implies the whole sequence u_n converges to the unique entropy solutions of (5.1) with initial datum $\bar{u} \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Moreover, since both semigroups $\mathcal{S}^1_t$ and $\mathcal{S}^2_t$ enjoy finite speed of propagation, for any fixed $t > 0$ the support of $u^n(\cdot,t)$ is bounded uniformly with respect to n and therefore the convergence in (5.3) is indeed in $L^1(\mathbb{R})$.

Let us now apply the above approximation arguments to our solutions u_1 and u_2 and let us denote with $u_{1,n}$ and $u_{2,n}$ the corresponding approximating sequences such that for any $t \in [0,T]$

$$u_{i,n}(\cdot,t) \to u_i(\cdot,t) \text{ in } L^1(\mathbb{R}),\ \ i = 1,2, \tag{5.4}$$

and bounded almost everywhere. From the results in [16, 12] we know that, for any $t > 0$, $t \in (t_n, t_{n+1}]$ for some n and

$$\begin{aligned} d_\infty(u_{1,n}(t), u_{2,n}(t)) &= d_\infty(u_1^n, u_2^n) = d_\infty(\mathcal{S}^2_{\triangle t}(\mathcal{S}^1_{\triangle t} u_1^{n-1}), \mathcal{S}^2_{\triangle t}(\mathcal{S}^1_{\triangle t} u_2^{n-1})) \\ &\leq d_\infty(\mathcal{S}^1_{\triangle t} u_1^{n-1}, \mathcal{S}^1_{\triangle t} u_2^{n-1}) \leq d_\infty(u_1^{n-1}, u_2^{n-1}) \leq \ldots \\ &\leq d_\infty(\bar{u}_1, \bar{u}_2). \end{aligned} \tag{5.5}$$

Moreover, since

$$d_\infty(u,v) = \lim_{p\to+\infty} d_p(u,v) = \sup_{p\geq 1} d_p(u,v),$$

from (5.5) we conclude

$$d_p(u_{1,n}(t), u_{2,n}(t)) \leq d_\infty(\bar{u}_1, \bar{u}_2). \tag{5.6}$$

Using the weak lower semi-continuity of the d_p's with respect to the convergence established in (5.4) (see, e.g. [35]), from (5.6) as $n \to +\infty$ we obtain

$$d_p(u_1(t), u_2(t)) \leq \liminf_{n\to+\infty} d_p(u_{1,n}(t), u_{2,n}(t)) \leq d_\infty(\bar{u}_1, \bar{u}_2). \tag{5.7}$$

Finally, passing into the limit for $p \nearrow +\infty$ in (5.7), the proof of (5.2) is complete. □

6. – Intermediate asymptotics via generalized Toscani map.

In this section we use the results proven in Section 5 to the study of the large time asymptotics of nonnegative compactly supported solutions to the degenerate diffusion-convection equation

$$u_t + f(u)_x = g(u)_{xx}, \tag{6.1}$$

where $f : [0, +\infty) \to [0, +\infty)$ and $g : [0, +\infty) \to [0, +\infty)$ are supposed to be twice continuously differentiable, f is convex and $g(0) = g'(0) = 0$. To perform this task, we use the generalized Toscani map introduced in Theorem 3.2 above (see [12, 13]) to scale the solution $u(t)$ by its own maximal transport moment (see Definition 2.2 above). For future use we also introduce the functional space

$$\mathcal{M}_{BV} := \{v \in \mathcal{B} \cap BV(\mathbb{R}) : \{\mathrm{m}_\infty[v] = 1\}$$

which is dense on $\mathcal{M}$ with the distance d_∞ as shown in [12, Theorem 5.8].

We now introduce the generalized Toscani map of the equation (6.1). Let $\bar{u} \in \mathcal{M}_{BV}$. Let $t \geq 0$ and let $u(t)$ be the solution to (6.1) at time t with initial datum $\bar{u}$. We define

$$\mathcal{T}_t[\bar{u}](x) := \mathrm{m}_\infty[u(t)]\, u(\mathrm{m}_\infty[u(t)]\, x, t). \tag{6.2}$$

It is easily checked that $\mathrm{m}_\infty[\mathcal{T}_t[\bar{u}]] = 1$ for all $t \geq 0$ and for any $\bar{u} \in \mathcal{M}_{BV}$, and thus, the map $\mathcal{T}_t : \mathcal{M}_{BV} \longrightarrow \mathcal{M}_{BV}$ is well-defined. Let us now state the main result of this section.

THEOREM 6.1. – *Assume that f and g are locally C^2 functions, f is convex and $g(0) = g'(0) = 0$. Then, there exist a fixed $t^* > 0$ and a one parameter family of* *functions* $\{U_t^\infty\}_{t\geq t^*} \subset \mathcal{M} \cap \mathcal{B}$ *such that, for any* $\bar{u} \in \mathcal{M}_{BV}$ *we have*

$$d_\infty(\mathcal{T}_t[\bar{u}](x), U_t^\infty) \to 0, \qquad \text{as } t \to +\infty, \tag{6.3}$$

where $\mathcal{T}_t[\bar{u}](x)$ *is defined by* (6.2). *Moreover, for any fixed* $t > t^*$, U_t^∞ *is characterized as the unique fixed point of the generalized Toscani map* $\mathcal{T}_t : \mathcal{M} \to \mathcal{M}$.

PROOF. – We shall only discuss the general outline of the proof, which follows the same strategy as in the proof of [12, Theorem 3.4] (see also [13, Theorem 2]). The main steps of the proof are:

STEP 0. – *Uniform Expansion of Solutions:* The first important ingredient of this argument is to show that our chosen measure of expansion of the solution, the maximal transport moment $\mathrm{m}_\infty[u(t)]$, diverges uniformly in the set of initial data in $\mathcal{M}_{BV}$ as $t \to +\infty$. This fact is guaranteed by the temporal decay estimates of the L^∞ norm for convection diffusion equations proven in [26, Lemma 2.7] (which is based on a previous result in [4]). Under the conditions of this theorem, they show an $L^1 - L^\infty$ decay estimate that depends only on the L^1 norm of the initial data, namely

$$\|I(u(t))\|_{L^\infty(\mathbb{R})} \leq \|\bar{u}\|^2_{L^1(\mathbb{R})} t^{-1}, \qquad t > 0, \tag{6.4}$$

where

$$I(r) = \int_0^r (g(r) - g(\rho))d\rho, \qquad r \geq 0.$$

This information and the arguments performed in [12, Lemma 3.1] and [13, Lemma 2.1] imply that we have a uniform expansion of the solution, in the sense that

$$\lim_{t\to\infty} \mathrm{m}_\infty[u(t)] = +\infty \qquad \text{uniformly in the set } \mathcal{M}_{BV}. \tag{6.5}$$

STEP 1. – *Contraction Estimate in* d_∞*:* We first establishes a contraction result for the map $\mathcal{T}_t$ (for large enough t) in the set $\mathcal{M}_{BV}$ metrized by the Wasserstein d_∞. Given two elements $\bar{u}_1, \bar{u}_2 \in \mathcal{M}_{BV}$, let u_1 and u_2 be the entropy solutions having $\bar{u}_1$ and $\bar{u}_2$ as initial data respectively. We define for any $t > 0$

$$\widetilde{\mathrm{m}}_\infty(t) := \min\{\mathrm{m}_\infty[u_1(t)], \mathrm{m}_\infty[u_2(t)]\}$$

and we introduce the mass-preserving scaling

$$\widetilde{u}_i(x,t) := a_i(t)u_i(a_i(t)x,t), \qquad a_i(t) := \frac{\mathrm{m}_\infty[u_i(t)]}{\widetilde{\mathrm{m}}_\infty(t)}, \qquad i = 1,2.$$

After the above scaling procedure, one of the two between u_1 and u_2 (namely, the one with less maximal transport moment) remains unchanged, while the other one is rescaled in such a way that $\widetilde{u}_1$ and $\widetilde{u}_2$ have the same maximal transport moment. Moreover, for further use we observe

$$\mathcal{T}_t[\bar{u}_i(x)] = \widetilde{\mathrm{m}}_\infty(t)\,\widetilde{u}_i(\widetilde{\mathrm{m}}_\infty(t)x, t), \quad i = 1, 2. \tag{6.6}$$

At this moment, a technical lemma based on a geometric inequality comes into play: consider any two probability densities u, v on $\mathbb{R}$ such that $u \neq v$ on a set of positive Lebesgue measure and $\mathrm{m}_\infty[u] = \mathrm{m}_\infty[v]$. For $a \geq 1$ let $v_a(x) := a^{-d} v(a^{-1}x)$. Then,

$$d_\infty(u, v) \leq 2\, d_\infty(u, v_a) \tag{6.7}$$

for any $a \geq 1$. We refer to [12, Lemma 3.2] for its complete proof.

Since $\mathrm{m}_\infty[u_i(t)] \geq \widetilde{\mathrm{m}}_\infty(t)$ for $i = 1, 2$, we can apply the result in (6.7) which implies

$$d_\infty(\widetilde{u}_1(t), \widetilde{u}_2(t)) \leq 2\, d_\infty(u_1(t), u_2(t)). \tag{6.8}$$

A trivial scaling property of the Wasserstein distances and the identity (6.6) yield

$$d_\infty(\mathcal{T}_t[\bar{u}_1], \mathcal{T}_t[\bar{u}_2]) \leq \widetilde{\mathrm{m}}_\infty(t)^{-1} d_\infty(\widetilde{u}_1(t), \widetilde{u}_2(t)),$$

which, together with (6.8) implies

$$d_\infty(\mathcal{T}_t[\bar{u}_1], \mathcal{T}_t[\bar{u}_2]) \leq 2\, \widetilde{\mathrm{m}}_\infty(t)^{-1} d_\infty(u_1(t), u_2(t)).$$

Finally, thanks to (6.5) and to the contraction result of Theorem 5.1 we have, for a sufficiently large t^*,

$$d_\infty(\mathcal{T}_t[\bar{u}_1], \mathcal{T}_t[\bar{u}_2]) \leq \beta(t)\, d_\infty(\overline{u}_1, \overline{u}_2) \tag{6.9}$$

for a suitable function $[t^*, +\infty) \ni t \to \beta(t) \in (0, +\infty)$ such that $\beta(t) < 1$ for all $t \geq t^*$ and such that $\beta(t) \to 0$ as $t \to +\infty$.

STEP 2. – *Extension and Fixed Points:* Using that the set $\mathcal{M}_{BV}$ is dense on $\mathcal{M}$ with the distance d_∞ as proven in [12, Theorem 5.8], it is a simple matter to extend by continuity to $\mathcal{M}$ the generalized Toscani map. We can then apply Banach's fixed point Theorem for $t > t^*$, which yields the existence of the desired family of fixed points $\{U_t^\infty\} \in \mathcal{M}$. The convergence statement in (6.3) follows easily from the contraction estimate by choosing one of the initial data to be the fixed point itself.

STEP 3. – *"Regularity" of Asymptotic Profile:* Finally, one can easily prove that the family of fixed points is a subset of $\mathcal{M} \cap \mathcal{B}$ due to standard regularizing effect results for convection diffusion equations with bounded measures as initial data, see for instance the quite general result in [3]. □

REMARK 6.2. – We remark that an explicit decay rate with respect to time can be determined in the formula (6.3), depending on the rate of divergence of the maximal transport moment, which in turn depends on the L^∞ decay rate of $u(t)$. In this context, we observe that the decay rate in the estimate (6.4) only depends on the nonlinear diffusion function g. Such a decay rate can be improved in those cases in which the convection part dominates the diffusive part for large times. For instance, when $f(u) = u^q$ and $g(u) = u^m$ with $1 < q < m + 1$ the following estimate is valid (see [26, Lemma 2.10] or [23, Lemma 2.1])

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq C(\|\bar{u}\|_{L^1(\mathbb{R})})t^{-1/q}, \qquad t > 0.$$

Acknowledgements. JAC acknowledges the support from DGI-MEC (Spain) FEDER-project MTM2005-08024, 2005SGR00611 and the Acc. Integ. program HI2006-0111. MDF acknowledges financial support and kind hospitality from the Centre de Recerca Matemàtica (Barcelona, Spain).

REFERENCES

[1] M. AGUEH, *Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory,* Adv. Differential Equations, **10** (2005), 309-360.

[2] L. AMBROSIO - N. GIGLI - G. SAVARÉ, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[3] D. ANDREUCCI, *Degenerate parabolic equations with initial data measures*, Trans. Amer. Math. Soc., **349** (10) (1997), 3911-3923.

[4] P. BÉNILAN - J. E. BOUILLET, *On a parabolic equation with slow and fast diffusions*, Nonlinear Anal., **26** (4) (1996), 813-822.

[5] F. BOLLEY - Y. BRENIER - G. LOEPER, *Contractive metrics for scalar conservation laws*, J. Hyperbolic Differ. Equ., **2** (2005), 91-107.

[6] F. BOLLEY - J. A. CARRILLO, *Tanaka Theorem for Inelastic Maxwell Models*, To appear in Comm. Math. Phys.

[7] Y. BRENIER, *Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations*, Comm. Pure Appl. Math., **52** (1999), 411-452, 1999.

[8] Y. BRENIER, *L^2 formulation of multidimensional scalar conservation laws*, to appear in Arch. Ration. Mech. Anal.

[9] E. CARLEN - W. GANGBO, *Constrained steepest descent in the 2-Wasserstein metric*, Ann. of Math., **157** (2003), 807-846.

[10] J. CARRILLO, *Entropy solutions for nonlinear degenerate problems*, Arch. Ration. Mech. Anal., **147** (1999), 269-361.

[11] J. A. CARRILLO - M. DI FRANCESCO - M. P. GUALDANI, *Semidiscretization and long-time asymptotics of nonlinear diffusion equations*, Comm. Math. Sci., **1** (2007), 21-53.

[12] J. A. CARRILLO - M. DI FRANCESCO - C. LATTANZIO, *Contractivity of Wasserstein metrics and asymptotic profiles for scalar conservation laws*, J. Differential Equations, **231** (2006), 425-458.

[13] J. A. CARRILLO - M. DI FRANCESCO - G. TOSCANI, *Intermediate asymptotics beyond homogeneity and self-similarity: long time behavior for $u_t = \Delta\phi(u)$*, Arch. Rational Mech. Anal., **180** (2006), 127-149.

[14] J. A. CARRILLO - M. P. GUALDANI - G. TOSCAN, *Finite speed of propagation in porous media by mass transportation methods*, C. R. Acad. Sci. Paris, **338** (2004), 815-818.

[15] J. A. CARRILLO, R. J. MCCANN- C. VILLANI, *Contractions in the 2-Wasserstein length space and thermalization of granular media*, Arch. Rational Mech. Anal., **179** (2006), 217-264.

[16] J. A. CARRILLO - G. TOSCANI, *Wasserstein metric and large-time asymptotics of nonlinear diffusion equations*, In *New trends in mathematical physics*, World Sci. Publ., Hackensack, NJ, 2004, 234-244.

[17] J. A. CARRILLO - J. L. VÁZQUEZ, *Asymptotic Complexity in Filtration Equations*, To appear in J. Evol. Equ.

[18] M. CRANDALL - M. PIERRE, *Regularizing effects for $u_t + A\varphi(u) = 0$ in L^1*, J. Funct. Anal., **45** (1982), 194-212.

[19] M. CULLEN W. GANGBO, *A variational approach for the 2-dimensional semi-geostrophic shallow water equations*, Arch. Ration. Mech. Anal., **156** (2001) 241-273.

[20] M. DI FRANCESCO - P. A. MARKOWICH, *Entropy dissipation and Wasserstein metric methods for the viscous Burgers' equation: convergence to diffusive waves*, In *Partial Differential Equations and Inverse Problems,* **362** of Contemp. Math., Amer. Math. Soc., Providence, RI, 2004, 145-165.

[21] M. DI FRANCESCO - M. WUNSCH, *Large time behavior in Wasserstein spaces and relative entropy for bipolar drift-diffusion-Poisson models*, to appear in Monat. Math.

[22] S. EVJE - K. H. KARLSEN, *Viscous splitting approximation of mixed hyperbolic-parabolic convection-diffusion equations*, Numer. Math., **83** (1) (1999), 107-137.

[23] M. ESCOBEDO - J. L. VÁZQUEZ - E. ZUAZUA, *Asymptotic behaviour and source-type solutions for a diffusion-convection equation*, Arch. Rational Mech. Anal., **124** (1) (1993), 43-65.

[24] E. HOPF, *The partial differential equation $u_t + uu_x = \mu u_{xx}$*, Comm. Pure Appl. Math., **3** (1950), 201-230.

[25] A. S. KALASHNIKOV, *Some problems of the qualitative theory of second-order non-linear degenerate parabolic equations*, Uspekhi Mat. Nauk, **42** (2(254)) (1987), 135-176, 287.

[26] P. LAURENÇOT - F. SIMONDON, *Long-time behaviour for porous medium equations with convection*, Proc. Roy. Soc. Edinburgh Sect. A, **128** (2) (1998), 315-336.

[27] T. P. LIU - M. PIERRE, *Source-solutions and asymptotic behavior in conservation laws*, J. Differential Equations, **51** (1984), 419-441.

[28] J. LOTT - C. VILLANI, *Ricci curvature for metric-measure spaces via optimal transport*, To appear in Ann. of Math.

[29] R. J. MCCANN, *Stable rotating binary stars and fluid in a tube*, Houston J. Math., **32** (2006), 603-632.

[30] F. OTTO, *The geometry of dissipative evolution equation: the porous medium equation*, Comm. Partial Differential Equations, **26** (2001), 101-174.

[31] F. OTTO - M. WESTDICKENBERG, *Eulerian calculus for the contraction in the wasserstein distance*, SIAM J. Math. Anal., **37** (2005), 1227-1255.

[32] K. T. STURM, *Convex functionals of probability measures and nonlinear diffusions on manifolds*, J. Math. Pures Appl., **84** (2005) 149-168.

[33] G. TOSCANI, *A central limit theorem for solutions of the porous medium equation*, J. Evol. Equ., **5** (2005), 185-203.

[34] J. L. VÁZQUEZ, *The Porous Medium Equation. New contractivity results,* Progress in Nonlinear Differential Equations and Their Applications, **63** (205) (Volume in honor of H. Brezis, Proceedings of Gaeta Congress, June 2004), 433-451.

[35] C. Villani, *Topics in optimal transportation*, 58 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2003.
[36] C. Villani, *Optimal transport, old and new*, Lecture Notes for the 2005 Saint-Flour summer school, to appear in Springer 2007.

José A. Carrillo: ICREA (Institució Catalana de Recerca i Estudis Avançats)
and Departament de Matemàtiques, Universitat Autònoma de Barcelona
e-mail: carrillo@mat.uab.es

Marco Di Francesco: Sezione di Matematica per l'Ingegneria,
Dipartimento di Matematica Pura ed Applicata, Università di L'Aquila
e-mail difrance@univaq.it

Corrado Lattanzio: Sezione di Matematica per l'Ingegneria,
Dipartimento di Matematica Pura ed Applicata, Università di L'Aquila
e-mail: corrado@univaq.it

Pervenuta in Redazione
il 29 marzo 2007