Cédric Villani


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# Hypocoercive Diffusion Operators

CÉDRIC VILLANI (*)

**Sunto.** – *In molti problemi provenienti dalla fisica matematica, l'associazione di un operatore di diffusione degenere con un operatore conservativo può portare a dissipazione in tutte le variabili e a convergenza verso l'equilibrio. Si può tracciare un'analogia con il fenomeno ben studiato di ipoellitticità nella teoria della regolarità, ed effettivamente entrambi i fenomeni sono stati studiati insieme. Ora una teoria distinta di "ipocoercività" sta iniziando ad emergere con alcuni risultati già sorprendenti e numerosi problemi aperti pieni di sfida. Questo testo (una versione abbreviata di quello che ho preparato per il Congresso Internazionale dei Matematici) ne analizza alcuni.*

**Summary.** – *In many problems coming from mathematical physics, the association of a degenerate diffusion operator with a conservative operator may lead to dissipation in all variables and convergence to equilibrium. One can draw an analogy with the well-studied phenomenon of hypoellipticity in regularity theory, and actually both phenomena have been studied together. Now a distinctive theory of "hypocoercivity" is starting to emerge, with already some striking results, and several challenging open problems. This text (an abbreviated version of the one which I prepared for the International Congress of Mathematicians) will review some of them.*

## Introduction

During the past decade, considerable progress has been achieved in the qualitative study of diffusion equations in large time, be it for linear or nonlinear models. Quantitative functional methods have become especially popular. Here are some of the keywords in the field: spectral gap (Poincaré) inequalities, logarithmic Sobolev inequalities, analysis of entropy production, gradient flows, rescalings. Most of the time, estimates on the rate of convergence are established in the end by means of some Gronwall-type inequality $dE/dt \leq -\Phi(E)$, where $E$ is a Lyapunov functional for the system. Among a large literature, I shall only quote some of my own works: entropy production estimates for the spatially homogeneous Boltzmann equation, in collaboration with Giuseppe Toscani [25],

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[26], [28]; and for certain nonlinear diffusion equations with a convex mean-field interaction, in collaboration with José Antonio Carrillo and Robert McCann [2].

While these subjects are still very active, in this text I shall focus on a newer direction of research which has emerged only a few years ago, and can be loosely described as "the role of the non-dissipative part in the dissipation process".

Indeed, it happens not so rarely that the dissipative properties of an equation are strongly influenced by some of the conservative terms in this equation. In the context of diffusion equations, the interaction between dissipative and conservative terms is also well-known, since it is at the basis of the phenomenon of *hypoellipticity*. To make the discussion a bit more precise, let me recall a particularly simple theorem of hypoelliptic regularization, which is a direct consequence of Lars Hörmander's celebrated regularity theorem [20]. Let $A_1, \ldots, A_k$ and $B$ be $C^\infty$ vector fields on $\mathbb{R}^N$, identified with derivation operators, and let $L = -\sum A_j^2 + B$. If the rank of $(A_1, \ldots, A_k)$ is strictly less than $N$, then the operator $L$ is not elliptic, and there is no a priori reason why the semigroup $e^{-tL}$ would be regularizing in all variables. But if $-\sum [A_j, B]^2 - \sum A_j^2$ is elliptic, where $[A_j, B]$ is the Lie bracket between $A_j$ and $B$, then $e^{-tL}$ is regularizing in all variables, and the operator $L$ is said to be hypoelliptic. (This is not the classical definition of hypoellipticity, but it will do for the purpose of this presentation.) We see here how the "nondissipative" first-order operator $B$ interacts with the "dissipative part" of $L$, or more precisely the derivation operators $A_j$, to produce the missing directions of regularization. Possibly the most important instance of application is to the operator $L = -\Delta_v + v \cdot \nabla_x$, where $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$; in that case $A_j = \partial/\partial v_j$, $B = v \cdot \nabla_x$, $[A_j, B] = \partial/\partial x_j$. The corresponding evolution equation $\partial_t f + Lf = 0$ is degenerate, but still presents some of the typical features of a parabolic equation.

Hypoelliptic regularity has been the object of hundreds of works for the past four decades. But what was understood only very recently is that quite similar phenomena arise in the study of rates of convergence to equilibrium. To describe this, I shall use the word "hypocoercivity", which was suggested to me by Thierry Gallay. A typical hypocoercivity theorem will give sufficient conditions on an operator $L$ so that $e^{-tL}$ will converge to equilibrium at a certain rate, even though $L$ is not "coercive", in the sense that the kernel of its dissipative part is much larger than the set of equilibria.

Hypoellipticity and hypocoercivity are often found together, and have been actually studied together, by refined hypoelliptic techniques [6], [7], [15], [16], [19], and sometimes by probabilistic methods [8], [22], [23]. However, these two phenomena are distinct: Each of them can occur without the other; and the structures which underlie them are not exactly the same. This motivates the development of a separate theory of hypocoercivity. In the sequel, I shall present some of the first results in this direction. For a much more detailed presentation, the reader can refer to [31].

## 1. – Motivations

In this section I shall describe some concrete examples which motivate the study of hypocoercivity. All of them come from mathematical physics, and none of them is academic. Of course the list is far from exhaustive.

*The kinetic Fokker-Planck equation*

In stochastic analysis, Fokker-Planck equations are often encountered as equations satisfied by the time-dependent laws of solutions of first-order stochastic differential equations. In "real life" however, equations of motion are not first-order, but second-order. Consider for instance a particle in $\mathbb{R}^n$, following Newton's equations with a potential force $-\nabla V$, a white noise random forcing, and a linear friction with coefficient $\theta = 1$: Then its position $X_t$ at time $t$ satisfies the second-order stochastic differential equation

$$\frac{d^2X_t}{dt^2} = -\nabla V(X_t) + \sqrt{2}\,\frac{dB_t}{dt} - \frac{dX_t}{dt},$$

where $B_t$ is a standard Brownian motion. (Of course, the coefficient $\sqrt{2}$ is just a convenient normalization, and the writing is formal in the sense that $B_t$ is not differentiable.) To write the associated partial differential equation, define $f_t(x, v)$ as the density of the law of $(X_t, \dot{X}_t)$ in $\mathbb{R}^n \times \mathbb{R}^n$. Then $f$ is a solution of

$$(1) \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (fv),$$

where $\Delta_v$ and $\nabla_v \cdot$ respectively stand for the Laplace and divergence operators in velocity space. Equation (1) is *kinetic* in the sense that it involves not only the position, but also the velocity variable; it is one of the fundamental equations in gas dynamics. It admits many nonlinear variants, among which the Vlasov-Poisson-Fokker-Planck equation, which is accepted as one of the fundamental equations of stellar dynamics.

When $V$ is quadratic, the fundamental solution of (1) is explicit and Gaussian. Its examination shows that there is relaxation to a Gaussian equilibrium (in $x$ and $v$ variables) as $t \to \infty$, and this convergence is exponentially fast, with an explicit rate. Here we see a perfect illustration of the hypocoercivity phenomenon: The differential operator on the left-hand side of (1) is conservative (it describes the trajectories of a classical dynamical system in $\mathbb{R}^n \times \mathbb{R}^n$ with Hamiltonian $V(x) + |v|^2/2$), and the right-hand side alone is diffusive degenerate (it only acts on the velocity variable $v$, so cannot cause any relaxation to equilibrium with respect to the $x$ dependence); however, their combination leads to an exponential convergence to equilibrium.

For more general potentials, there is still a global equilibrium:

$$f_\infty(x,v) = \frac{e^{-\left[V(x)+\frac{|v|^2}{2}\right]}}{Z},$$

where $Z$ is a normalizing constant. Then it is an obviously natural question whether exponential convergence to $f_\infty$ holds true under adequate assumptions on the potential $V$, which go beyond the "trivial" quadratic case. Shockingly enough, the first such results were obtained only around 2002, by Frédéric Hérau and Francis Nier [19]. They used a quite sophisticated approach taking roots in Joseph Kohn's approach to hypoellipticity. Since then, their method has been very much simplified, as I shall describe later.

It should be noted that in most works on the subject, one studies the Fokker-Planck equation after making the change of unknown $h = f/f_\infty$, so the resulting equation reads

$$\frac{\partial h}{\partial t} + v\cdot\nabla_x h - \nabla V(x)\cdot\nabla_v h = \Delta_v h - v\cdot\nabla_v h. \tag{2}$$

This has the advantage that the operator appearing on the right-hand side is self-adjoint in the Hilbert space $L^2(f_\infty\, dx\, dv)$, so (2) might lend itself to a spectral treatment. Of course, formally, equations (1) and (2) are equivalent; but in practise they are studied in different functional spaces. For simplicity, I shall not say more on this important issue.

*Oscillator chains*

Even though Fourier's law of conduction of heat is one of the oldest partial differential equations, it is still extremely far from a rigorous theoretical understanding. Many models of statistical physics have been proposed to describe heat conduction. Here is one of them, described in [8]. Each atom in a solid body is labelled (for the sake of this discussion, we may assume that the dimension is 1, so atoms are labelled $0, 1, \ldots, N$), and the unknowns are the displacements $X^0, \ldots, X^N$ of the atoms with respect to their respective equilibrium positions. Each atom is bound to its equilibrium position with a "pinning potential" $V$, and it also interacts with its two neighbors by an interaction potential $W$, assumed to be symmetric ($W(z) = W(-z)$). So the equation for $X^k$ is just

$$\frac{d^2X_t^k}{dt^2} = -\nabla V(X_t^k) - \nabla W(X_t^k - X_t^{k-1}) - \nabla W(X_t^k - X_t^{k+1}). \tag{3}$$

Of course these equations do not apply to the atoms that are at the extreme left ($k = 0$) and the extreme right ($k = N$) in the chain, since they have only one neighbor. But these extremal atoms are also *shaken* by some external bath, with a temperature of agitation $T^{(\ell)}$ on the left, and $T^{(r)}$ on the right. The corresponding equations for, say, $k = 0$, can be written

$$(4) \qquad \begin{cases} \dfrac{d^2 X_t^0}{dt^2} = -\nabla V(X_t^0) - \nabla W(X_t^0 - X_t^1) + \ell, \\ \dfrac{d\ell}{dt} = \lambda^{(\ell)} \sqrt{2T^{(\ell)}}\, \dfrac{dB_t^{(\ell)}}{dt} - \ell + (\lambda^{(\ell)})^2 \ell\, \dfrac{dX_t^0}{dt}. \end{cases}$$

Here $\lambda^{(\ell)}$ is a coefficient describing the strength of the coupling between the particle and the heat bath.

Again, the law of this system is described by a linear partial differential equation in the variables $\ell, r, X^0, \dots, X^N, \dot{X}^0, \dots, \dot{X}^N$. It is very similar to the kinetic Fokker-Planck equation, except that it is much more degenerate, since the diffusion only acts on the variables $\ell$ and $r$.

There are now two difficult problems which naturally arise: (i) Show that the solution $f_t(\ell, r, x^0, \dots, x^N, v^0, \dots, v^N)$ approaches some stationary distribution as $t \to \infty$; (ii) Study the properties of this stationary distribution, and in particular the associated energy flux. (In this case, it is better to say "stationary distribution" rather than "equilibrium", precisely because the temperatures are not necessarily equal.) In particular, if $T^{(\ell)} > T^{(r)}$, in the asymptotic regime $N \to \infty$, is it true that energy flows from the left to the right, and what is the relation between the average flux and the difference of temperatures!?

When $T^{(\ell)} = T^{(r)}$, the equilibrium distribution is easy to write down explicitly, and problem (ii) is trivially solved. But as soon as these temperatures are different, the stationary solution is not explicit - except in the case when $V$ and $W$ are quadratic, but then the results are physically irrelevant!! It is conjectured that some anharmonicity is *necessary* to get the Fourier law (ironically enough, the heat equation, although one of the most basic *linear* models in science, needs some dose of microscopic nonlinearity to be explained). Then problem (ii) becomes incredibly difficult.

Even when the two temperatures are equal, problem (i) appears to be quite difficult. It is actually a typical hypocoercive situation: The diffusion on $\ell$ and $r$ should lead in the end to a relaxation to equilibrium in all variables.

Exponential convergence to the stationary distribution has been proved recently by several authors [8], [7], even for the case when $T^{(\ell)} \neq T^{(r)}$, under various assumptions on the potentials; but the dependence of the estimates upon the number of atoms is just terrible.

### *The Boltzmann equation*

The Boltzmann equation is one of the basic partial differential equations in statistical mechanics. It is a kinetic model for the evolution of a rarefied gas of particles interacting via binary collisions. Historically, it has preceded the Fokker-Planck equation; but the analytical problems that it raises are considerably more acute. A mathematically-oriented presentation of the Boltzmann equation can be found in my long review paper [27]. The classical Boltzmann equation in $n$ dimensions of space can be written

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \quad x \in \boldsymbol{\Omega}_x, \ v \in \mathbb{R}^n, \tag{5}$$

where $\boldsymbol{\Omega}_x$ is a bounded connected open spatial domain, and $Q$ is the Boltzmann collision operator, defined by

$$Q(f,f) = \int_{\mathbb{R}^n} \int_{S^{n-1}} B(v - v_*, \sigma) \big[ f(x,v') f(x,v'_*) - f(x,v) f(x,v_*) \big] \, d\sigma \, dv_*.$$

Here $\sigma$ is a variable unit vector in $\mathbb{R}^n$, $B(v - v_*, \sigma)$ is a collision kernel depending on the particular form of the interaction (for instance $B(v - v_*, \sigma) = |v - v_*|$), and the transform $(v, v_*) \to (v', v'_*)$ is computed by the rules of elastic collision:

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

This equation should of course be supplemented with boundary conditions. To simplify things, one can assume that $\boldsymbol{\Omega}_x$ is just the $n$-dimensional torus $\mathbb{T}^n$ (periodic boundary conditions); another common choice is specular reflexion in a bounded open set.

In spite of hundreds of papers, the mathematical theory of the Boltzmann equation is far from complete; in particular there is still no theory of classical solutions in the large. However, strong regularity results have been obtained in a close-to-equilibrium regime. A complete theory can also be put together as long as there is a pointwise control of certain hydrodynamic fields (density in physical space, mean velocity, temperature, pressure tensor).

It was Boltzmann's beautiful observation that the $H$ functional (negative of the entropy),

$$H(f) = \int f \log f \, dx \, dv$$

is nonincreasing with time along solutions of the Boltzmann equation. Then there is a unique large-time equilibrium, which takes the form of a Maxwellian (Gaussian) distribution:

$$f_\infty(x,v) = \rho \, \frac{e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{n/2}},$$

where $\rho \geq 0$ (total mass), $u \in \mathbb{R}^n$ (total mean momentum) and $T \geq 0$ (mean temperature) are constants. This equilibrium is obtained by maximizing the entropy given the conservation laws.

The problem of convergence to equilibrium for the Boltzmann equation is famous for historical reasons (it triggered a hot controversy in the nineteenth century) and also for theoretical reasons (as a manifestation of irreversibility in the statistical description of a reversible mechanical system; and as the justification of the law of maximum entropy on a basic model). See my lecture notes [29] for an overview of this question.

Of course the complexity of the Boltzmann equation, and its nonlinearity are major difficulties in the study. But behind that, we can recognize once again a hypocoercive situation: The dissipation (collision) operator $Q$ on the right-hand side of (5) is very degenerate since is only acts on the velocity dependence, and it is only its association with the conservative transport operator $v \cdot \nabla_x$ on the left-hand side which can lead to convergence to equilibrium.

*Stability of Oseen's vortices*

The last example in this gallery comes from hydrodynamics and was brought to my attention by Thierry Gallay. It is a well-documented fact in turbulence theory that the vorticity of a two-dimensional incompressible flow tends to coalesce and form large vortices. Thierry Gallay and Eugene Wayne [11] have studied this phenomenon rigorously for a two-dimensional incompressible viscous fluid in the whole space: If $\omega = \omega_t(x)$ is the vorticity, the equation is just

$$\frac{\partial \omega}{\partial t} + \mathrm{BS}[\omega] \cdot \nabla \omega = \Delta \omega,$$

where $\mathrm{BS}[\omega]$ is the velocity field obtained from the vorticity $\omega$ via the Biot-Savart law:

$$\mathrm{BS}[\omega](x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y)\, dy,$$

and $v^\perp$ is obtained from $v$ by rotation of angle $\pi/2$.

If $\omega_0 \in L^1(\mathbb{R}^2)$, then $\omega_t$ converges to 0 as $t \to \infty$, due to viscous dissipation. But a refined analysis shows that $\omega_t$ is asymptotically close to an explicit self-similar Gaussian solution, which physically corresponds to a unique large vortex, called Oseen's vortex. In fact, in suitably rescaled variables, the vorticity does converge to a stationary Gaussian distribution.

The linear stability analysis of this phenomenon reduces to the spectral analysis of the operator $S + aB$ in $L^2(\mathbb{R}^2)$, where

$$\text{(6)} \qquad \begin{cases} S\omega = -\Delta \omega + \dfrac{|x|^2}{16}\, \omega - \dfrac{\omega}{2}, \\ B\omega = \mathrm{BS}[G] \cdot \nabla \omega + 2\mathrm{BS}[G^{1/2}\omega] \cdot \nabla G^{1/2}. \end{cases}$$

Here $G$ is a Gaussian distribution: $G(x) = e^{-|x|^2/4}/(4\pi)$; and $a$ is the value of the "circulation Reynolds number", which in the present set of conventions is just $\int \omega_0$.

The spectral study of $S + aB$ turns out to be quite tricky. In the hope of getting a better understanding, one can decompose $\omega$ in Fourier series: $\omega = \sum_{n\in\mathbb{Z}} \omega_n(r)e^{in\theta}$, where $(r,\theta)$ are standard polar coordinates in $\mathbb{R}^2$. For each $n$, the operators $S$ and $B$ can be restricted to the vector space generated by $e^{in\theta}$, and can be seen as just operators on a function $\omega(r)$:

$$\begin{cases} (S_n\omega)(r) = -\partial_r^2\omega - \left(\dfrac{r}{2} + \dfrac{1}{r}\right)\partial_r\omega - \left(1 - \dfrac{n^2}{r^2}\right)\omega, \\ (B_n\omega)(r) = i\,n\,(\varphi\omega - g\Omega_n). \end{cases}$$

Here $g(r) = e^{-r^2/4}/4\pi$, $\varphi(r) = (1 - e^{-r^2/4})/2\pi r^2$, and $\Omega_n(r)$ solves the differential equation

$$-(r\Omega')' + \frac{n^2}{r}\Omega = \frac{r}{2}\omega.$$

The regime $|a| \to \infty$ is of physical interest and has already been the object of numerical investigations by physicists. There are two families of eigenvalues which are imposed by symmetry reasons; but apart from that, it seems that all eigenvalues converge to infinity as $|a| \to \infty$, and for some of them the precise asymptotic rate of divergence $O(|a|^{1/2})$ has been established by numerical evidence. If that is correct, this means that the "perturbation" of $S$ by $aB$ is strong enough to send most eigenvalues to infinity as $|a| \to \infty$. This is particularly striking when one realizes that $S$ is symmetric in $L^2(\mathbb{R}^2)$, while $B$ is antisymmetric. Obviously, this is again a manifestation of a hypocoercive phenomenon.

Let us simplify things just a bit by throwing away the nonlocal term $g\Omega_n$ in the expression of $B_n$. After a few manipulations, the problem reduces to the following

MODEL PROBLEM 1.1. – Identify sufficient conditions on $f : \mathbb{R} \to \mathbb{R}$, so that the real parts of the eigenvalues of

$$L_a : \omega \longmapsto (-\partial_x^2\omega + x^2\omega - \omega) + iaf\omega$$

in $L^2(\mathbb{R})$ go to infinity as $|a| \to \infty$, and estimate this rate.

So far this problem has been solved only partially, by Isabelle Gallagher and Thierry Gallay; I shall describe their results later on.

## 2. – A dynamical approach

Together with Laurent Desvillettes [3], [5], I have developed a method to study quite general hypocoercive situations. The method ultimately relies on the analysis of a *system* of coupled differential inequalities of first and second order (instead of just one first-order differential inequality as in Gronwall's lemma). The method was devised with the aim of proving convergence to equilibrium for *uniformly smooth* solutions of the Boltzmann equation. The main result reads as follows:

THEOREM 2.1. – *Let $(f_t)_{t\geq 0}$ be a smooth solution of the Boltzmann equation (5), such that all the derivatives of $f$ are uniformly bounded, and all the moments of $f$ are bounded, uniformly in time. Further assume that $f$ satisfies a pointwise lower bound of the form $f_t(x,v) \geq K_0 e^{-A_0|v|^{q_0}}$. Then, under adequate boundary conditions, $f_t$ converges to global equilibrium as $t \to \infty$, at least as fast as $O(t^{-\kappa})$ for all $\kappa > 0$.*

Some information about the implementation of this program in the context of the Boltzmann equation can be found in the original research paper [5], or, in a lighter form, in the lecture notes [30], [29]. Before being used on the Boltzmann equation, the dynamical approach had been tried on the Fokker-Planck equation [3] and on some other linear models [1], [9]. It is quite robust and adapted to equations with very little structure.

One of its appealing features is that it seems to provide a good physical intuition of what is going on: The system approaches hydrodynamical state under the influence of collisions, then it is driven out of hydrodynamical state by the influence of the transport, etc. Numerical simulations have corroborated this qualitative analysis surprisingly well. In the diagram below, computed numerically by Francis Filbet, one sees very clearly that the solution of the Boltzmann equation oscillates between states where it is close to hydrodynamical, and states where it is close to homogeneous. In particular, contrary to a widespread belief, the approach to hydrodynamical regime is not faster than the approach to global equilibrium. (All of this is valid only on scales of time on which the Knudsen number is of order 1.)

From the point of view of physics, the discovery of these oscillations may be one of the most noticeable outcomes of the program of hypocoercivity applied to the Boltzmann equation. They are not easy to observe, and have even been used as a "benchmark" to test the accuracy of certain numerical schemes (see e.g. [10]).

However, the dynamical method suffers from the complexity of its practical application, and its heavy computational cost. In the next section, I shall describe another method which may be less appealing from the physical point of view, and requires a bit more structure, but has the advantage to be much lighter.

Also, I emphasize that Theorem 2.1 requires strong regularity and decay estimates on the solutions. As discussed in [5], all these estimates can be proven in the *close-to-equilibrium* regime, but remain a major open problem for solutions in the large. Even in a close-to-equilibrium regime, decay estimates based

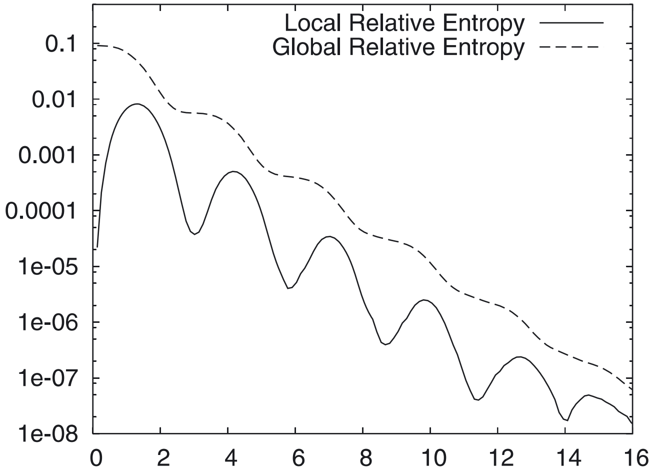


Fig. 1. – The upper curve is the $H$ functional as a function of time, in semi-log plot; the lower curve is the purely kinetic part of the $H$ functional. When the two curves are far away, the distribution is almost in hydrodynamical state; when they are very close, it is almost homogeneous.

on a linearization method are quite hard to obtain, and were not available at the time when [5] was published; since then this gap has been filled in a series of important works by Yan Guo and Robert Strain [13], [14].

## 3. – A functional approach

In the previous method the resolution of the main degeneracy problem was done via the time-differentiation of certain (relatively) simple functionals. Now the idea is to put as much as possible of the difficulty in a careful choice of the functional; and more precisely to add "correction terms" which are negligible in size, but contribute in an important way to the time-derivative of the functional. This will be more easily explained on the example of the Fokker-Planck equation, in the form (2).

Suppose you want to get a Gronwall inequality for some well-chosen functional, applied to the Fokker-Planck equation. First try the $L^2$ norm. With the notation $\mu(dx\,dv) = f_\infty(x,v)\,dx\,dv$, and omitting once again the dependence upon time, one has

$$\frac{d}{dt}\int h^2\,d\mu = -\int |\nabla_v h|^2\,d\mu.$$

Since the derivatives in the right-hand side involve only the velocity variables, there is no way to dominate the integral in the left-hand side by the right-hand side (choose $h = h(x)$, then the right-hand side vanishes).

So go to a higher order norm, involving gradients of $h$. After a bit of work, under suitable assumptions on $V$, one can find constants $a, c, K > 0$ such that

$$(7)\quad \frac{d}{dt}\left(\int h^2\,d\mu + a\int |\nabla_x h|^2\,d\mu + c\int |\nabla_v h|^2\,d\mu\right) \le -K\left(\int |\nabla_v h|^2\,d\mu + \int |\nabla_v\nabla_x h|^2\,d\mu + \int |\nabla_v\nabla_v h|^2\,d\mu\right).$$

Again, the right-hand side is not sufficient to control the expression in brackets on the left-hand side. So still nothing!

But now correct the functional on the left-hand side by adding an innocent-looking term $2b\int \nabla_x h\cdot\nabla_v h\,d\mu$. If $b < \sqrt{ac}$, this term does not play any noticeable role in the value of the functional, since

$$\left|2b\int \nabla_x h\cdot\nabla_v h\,d\mu\right| \le (1-\delta)\left[a\int |\nabla_x h|^2\,d\mu + c\int |\nabla_v h|^2\,d\mu\right]$$

for some positive constant $\delta$. However, if $a$, $b$ and $c$ are properly chosen, then we have a differential inequality which is much better than (7):

$$(8)\quad \frac{d}{dt}\left(\int h^2\,d\mu + a\int |\nabla_x h|^2\,d\mu + 2b\int \nabla_x h\cdot\nabla_v h\,d\mu + c\int |\nabla_v h|^2\,d\mu\right) \le -K\left(\int |\nabla_x h|^2\,d\mu + \int |\nabla_v h|^2\,d\mu + \int |\nabla_v\nabla_x h|^2\,d\mu + \int |\nabla_v\nabla_v h|^2\,d\mu\right).$$

Now it very easy to close this differential inequality: It suffices that $\mu$ satisfies a Poincaré inequality (in the $x$ and $v$ variables).

*The algebraic core*

I started to work on this approach while struggling to understand the results of Frédéric Hérau and Francis Nier [19], without resorting to the technical hypoelliptic machinery used in their work. After deciding that there should be an elementary approach based on integration by parts and chain rule, I was still flooded by the complex calculations. Then I decided that there should be an even simpler approach with no analysis at all. After going to an abstract formulation of the problem, I found out that there was indeed an extremely simple "algebraic core" which can be presented as follows. Take two operators $A$ and $B$ on a Hilbert space (in the present case $A$ would be the vector-valued differential operator $\nabla_v$, while $B$ would be $v\cdot\nabla_x - \nabla V(x)\cdot\nabla_v$), with $B^* = -B$. Then, at least formally, the time-derivative of

$$\langle Ah, [A,B]h\rangle$$

along the influence of $B$ can be written as

$$\langle ABh, [A,B]h \rangle + \langle Ah, [A,B]Bh \rangle.$$

Pretend that $B$ commutes with $[A,B]$; then the previous expression is

$$\langle ABh, [A,B]h \rangle + \langle Ah, B[A,B]h \rangle,$$

and since $B^* = -B$, this can be rewritten as

$$\langle ABh, [A,B]h \rangle - \langle BAh, [A,B]h \rangle = \langle [A,B]h, [A,B]h \rangle. \tag{9}$$

In the example of the Fokker-Planck equation, $[A,B] = \nabla_x$, so $\langle Ah, [A,B]h \rangle = \int \nabla_v h \cdot \nabla_x h \, d\mu$, and the right-hand side of (9) is the desired term in $\|\nabla_x h\|^2$.

The advantage to input terms with "mixed derivatives" such as $\int \nabla_x h \cdot \nabla_v h$ had been actually noted before in studies of global in time propagation of the smoothness for kinetic equations, most notably by Denis Talay [24] and Yan Guo [12]. The simple algebraic core presented above explains why this trick also applies to problems of convergence to equilibrium.

The rest of this section will be devoted to a presentation of some results which I have obtained by pushing further this approach [31]. There are other works in this direction by Clément Mouhot and Lukas Neumann [21], and Frédéric Hérau [18], [17]; all of them based on quite similar tools.

*The basic theorem*

Two important features of the next theorem are that

– it applies to a general abstract framework: $\mathcal{H}$ is a Hilbert space (think of $\mathcal{H}$ as $L^2(\mu)$, where $\mu$ is the equilibrium measure); and $\mathcal{V}$ another Hilbert space (think of $\mathcal{V}$ as $\mathbb{R}^n$, the space of velocities); then $A$ is an unbounded operator $\mathcal{H} \to \mathcal{H} \otimes \mathcal{V}$, and $B$ is an unbounded operator $B \to B$ with $B^* = -B$;

– it considers a linear operator $L$ which is in (abstract) *Hörmander form*, that is $L = A^*A + B$ for some operators $A, B$ as above.

Assume that the semigroup $e^{-tL}$ is well defined and that there is no problem to differentiate the square norms, etc. Systematic tensorization with the identity operator will be used to make sense of notation such as $[A,B] = AB - (B \otimes I)A$. The scalar product in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle$, the norm in $\mathcal{H}$ by $\| \cdot \|$, and an operator $S$ will be said to be bounded respectively to a family of operators $T_1, \ldots, T_k$ if there is a constant $C$ such that $\|Sy\| \le C(\|T_1 y\| + \ldots + \|T_k y\|)$. The symbol $\Re$ stands for real part.

THEOREM 3.1. – *With the above notation, write* $[A,B] = C$, *and assume that*

(i) $[A,C] = 0$, $[A^*,C] = 0$;

(ii) $[A, A^*]$ *is bounded relatively to* $I$ *and* $A$;
(iii) $[B, C]$ *is bounded relatively to* $A$, $A^2$, $C$ *and* $AC$.

*Further assume that*

(H) $A^*A + C^*C$ *is coercive.*

*Then for a suitable choice of constants* $a, b, c$ *one has the differential inequality*

$$\frac{d}{dt}\mathcal{F}(e^{-tL}h) \leq -K\,\mathcal{F}(e^{-tL}h),$$

*where*

$$\mathcal{F}(h) = \|h\|^2 + a\,\|Ah\|^2 + 2b\,\Re\,\langle Ah, Ch\rangle + c\,\|Ch\|^2,$$

*and* $K$ *is a positive constant which only depends on the constants appearing implicitly in assumptions* (ii), (iii) *and* (H).

REMARK 3.2. – The assumption (H) is obviously an analog in this context of Lars Hörmander's bracket condition.

This theorem applies to the Fokker-Planck equation (2) under simple assumptions on the potential $V$, and yields exponential convergence to equilibrium for initial data $h_0$ satisfying $\|\nabla_x h_0\|^2 + \|\nabla_v h_0\|^2 < +\infty$. The latter restriction can finally be removed by an independent study of hypoelliptic regularity [16], [31]. (This is not a standard hypoelliptic estimate since it is global; there would be much to say about it, but this would take us too far.) In the end, one obtains the following theorem, which generalizes and improves the results of [19], [16]. Recall that a measure $\upsilon$ is said to satisfy a Poincaré inequality if one has a functional inequality of the form $\|\nabla h\|_{L^2(\upsilon)} \geq P\|h - \langle h\rangle\|_{L^2(\upsilon)}$, $P > 0$, where $\langle h\rangle$ is the average value of $h$ with respect to $\upsilon$.

THEOREM 3.3. – *Let* $V \in C^2(\mathbb{R}^n)$ *with* $\inf V > -\infty$, *such that*

(a) $|\nabla^2 V| \leq C(1 + |\nabla V|)$;
(b) *the reference measure* $\upsilon(dx) = e^{-V(x)}\,dx$ *satisfies a Poincaré inequality with constant* $P$.

*Let* $\mu(dx\,dv) = e^{-(V(x)+|v|^2/2)}\,dx\,dv/Z$, *where* $Z$ *is a normalizing constant. Then there are constants* $\lambda > 0$ *and* $C'$, *explicitly computable in terms of* $C$ *and* $P$, *such that solutions of the Fokker-Planck equation* (2) *satisfies*

$$\|h_t - \langle h_0\rangle\|_{L^2(\mu)} \leq C' e^{-\lambda t}\,\|h_0 - \langle h_0\rangle\|_{L^2(\mu)}.$$

Theorem 3.1, or more precisely its proof, was also used by Isabelle Gallagher and Thierry Gallay to provide a first solution to the Model Problem 1.1, as follows. Set $\mathcal{H} = L^2(\mathbb{R};\mathbb{C})$, $A = \partial_x\omega + x\omega$, $B\omega = (iaf)\omega$. Then $C\omega = iaf'\omega$, so the operator $A^*A + C^*C$ is of Schrödinger type:

$$(A^*A + C^*C)\,\omega = (\,-\,\partial_x^2\omega + x^2\omega - \omega) + a^2 f'^2\omega,$$

and the spectrum of $A^*A + C^*C$ can be studied via standard semi-classical techniques. For instance, if $f'(x)^2 = x^2/(1+x^2)^k$, $k \in \mathbb{N}$, then the spectral gap of $A^*A + C^*C$ is bounded below like $O(|a|^{2\upsilon})$, with $\upsilon = \min(1, 2/k)$. Then a careful examination of the proof of Theorem 3.1 yields a lower bound like $O(|a|^{\upsilon})$ on the real part of the spectrum of $A^*A + B$.

*Multiple commutators*

As in Lars Hörmander's hypoellipticity theorem, multiple commutators are also allowed in hypocoercivity results. But as an important difference, it seems that one only needs to consider commutators with the antisymmetric part. Here is such a theorem:

THEOREM 3.4. – *With the same notation as before, assume the existence of (possibly unbounded) operators $C_0, C_1, \ldots, C_{N_c+1}$, $R_1, \ldots, R_{N_c+1}$, and $Z_1, \ldots, Z_{N_c+1}$ such that*

$$C_0 = A, \quad [C_j, B] = Z_{j+1}C_{j+1} + R_{j+1} \ \ (0 \le j \le N_c), \quad C_{N_c+1} = 0,$$

and, for all $k \in \{0, \ldots, N_c\}$,

(i) $[A, C_k]$ *is bounded relatively to* $\{C_j\}_{0\le j\le k}$ *and* $\{C_jA\}_{0\le j\le k-1}$;
(ii) $[A^*, C_k]$ *is bounded relatively to $I$ and* $\{C_j\}_{0\le j\le k}$;
(iii) $R_k$ *is bounded relatively to* $\{C_j\}_{0\le j\le k-1}$ *and* $\{C_jA\}_{0\le j\le k-1}$;
(iv) $Z_j$ *is bounded relatively to $I$, and $I$ is bounded relatively to* $Z_j$;
(H) $\sum_{j=0}^{N_c} C_j^*C_j$ *is coercive.*

*Then one can choose constants $a_k$ and $b_k$ in such a way that the functional*

$$\mathcal{F}(h) = \|h\|^2 + \sum_{k=0}^{N_c}\bigl(a_k\|C_kh\|^2 + 2b_k\,\Re\,\langle C_kh, C_{k+1}h\rangle\bigr)$$

*satisfies the differential inequality*

$$\frac{d}{dt}\mathcal{F}(e^{-tL}h) \le -K\mathcal{F}(e^{-tL}h)$$

*for some constant $K$ which can be computed explicitly in terms of the constants appearing implicitly in* (i)-(iv) *and* (H).

This result generalizes Theorem 3.1 in several ways: Multiple commutators are allowed; a remainder $R_{j+1}$, and a multiplier $Z_j$ are allowed in the identity defining $C_{j+1}$ in terms of $C_j$; and the various directions $C_k$ are not assumed to commute.

As a simple application of Theorem 3.4, it is possible to prove exponential convergence to equilibrium for the oscillator chain described by equations (3)-(4), under the assumption that $V$ and $W$ are uniformly convex and have a bounded Hessian, *and that the temperatures on the left and on the right are equal*, that is $T^{(\ell)} = T^{(r)}$. Interestingly enough, bounded Hessians are not covered by the results of Jean-Pierre Eckmann and Martin Hairer [7], who impose a superquadratic growth at infinity. Conversely, it is not clear whether Theorem 3.4 can be used to recover the results in [7]. Still some work is required to clarify the situation about these assumptions on the potentials. I shall also come back in the end of these notes to the very unsatisfactory restriction $T^{(\ell)} = T^{(r)}$.

*Beyond the Hörmander form*

All the examples treated so far were dealing with linear operators in the form $L = A^*A + B$, where $B$ is antisymmetric. In theory, any operator can of course be cast in this form, but this might be a terrible thing to do in practise; for instance, if the symmetric part of $L$ is an integral operator then $A$ would look horrendous. So it is desirable to prove results under alternative structure assumptions.

It would be illusory to hope for a gain based on commutators like $[L, B]$. Instead, one can introduce an adequate auxiliary operator $A$ into the estimates, in such a way that (i) $A^*A + [A, B]^*[A, B]$ is coercive, and (ii) $A$ "almost commutes" with $L$. Recently, Clément Mouhot and Lukas Neumann [21] have derived such a hypocoercivity theorem in the particular framework of kinetic equations. I shall not say more about this promising approach, and just refer to [21] for more information. One of the outcomes of their work is a simplification of Yan Guo's theory of collisional kinetic equations close to equilibrium.

*Nonlinear equations*

To conclude this section, I shall show how to recover fully nonlinear hypocoercivity estimates by a variant of the approach developed above. For general nonlinear operators, it is probably hopeless to try to get anywhere unless one assumes some strong assumptions of smoothness and decay at infinity, to make sure that all norms involved are "almost comparable" (that is, they are comparable if one allows them to be raised to powers that are arbitrarily close to 1). So I will assume that $(f_t)_{t\geq 0}$ satisfies uniform bounds in a scale of weighted Sobolev spaces $(X^s)_{s\in\mathbb{R}}$ of arbitrarily high smoothness and decay, that are in interpolation. (For instance, $X^s$ might be defined as the space of functions $f$ such that $(I - \Delta_v - \Delta_x)^{s/2} f(x,v)(1 + |x|^2 + |v|^2)^{s/2}$ lies in $L^2$.) Then all the nonlinear operators involved will be assumed to be Lipschitz when restricted on balls of $X^s$, with values in some higher order space $X^{s+k}$. In practise, this means that our nonlinearities are not worse than polynomial, with coefficients that do not increase faster than polynomial. Then I shall denote the functional derivative of a functional $\mathcal{F}$ at function $f$ by just $\mathcal{F}'_f$. I shall further assume that there is a unique equilibrium $f_\infty$, and a Lyapunov functional $\mathcal{E}$ satisfying

$$\mathcal{E}(f_t) - \mathcal{E}(f_\infty) \geq K \, \|f_t - f_\infty\|_s^{2(1+\varepsilon)}$$

for some suitable $s = s(\varepsilon)$, $K = K(\varepsilon)$, where $\varepsilon$ is arbitrarily small. In words, this means that $\mathcal{E}$ essentially controls the square of the distance to equilibrium.

THEOREM 3.5. – *With the above notation, let*

$$L = B - \mathcal{C}$$

*be a nonlinear differential operator, such that $B$ preserves the Lyapunov functional $\mathcal{E}$ (that is, $\mathcal{E}'_f \cdot Bf = 0$), let $(f_t)_{t\geq 0}$ solve $\partial_t f + Lf = 0$, and let $(\Pi_j)_{1\leq j\leq J}$ be nonlinear operators satisfying*

$$\Pi_j \circ \Pi_k = \Pi_{\max(j,k)}, \tag{10}$$

*such that, for all $t \geq 1$,*

(i) $\mathcal{C} \circ \Pi_1 = 0$; $-\mathcal{E}'_{f_t} \cdot (\mathcal{C} f_t) \geq K_\varepsilon [\mathcal{E}(f) - \mathcal{E}(\Pi_1 f)]^{1+\varepsilon}$;
(ii) $K_\varepsilon \|\Pi_1 f_t - f_\infty\|^{2+\varepsilon} \leq \mathcal{E}(\Pi_1 f) - \mathcal{E}(f_\infty) \leq C_\varepsilon \|\Pi_1 f_t - f_\infty\|^{2-\varepsilon}$;
(iii) $\Pi_J f = f_\infty$; $Bf_\infty = 0$;

(H) $\|(\mathrm{Id} - \Pi_j)'_{\Pi_j f} \cdot (B\Pi_j f)\|^2 \geq K_\varepsilon \|(\Pi_j - \Pi_{j+1}) f\|^{2+\varepsilon}$ for all $j \in \{1, \ldots, J-1\}$.

*Then* $\|f_t - f_\infty\| = O(t^{-\infty})$.

This theorem may seem particularly abstract and confusing, so I should give some explanations. First, $B$ plays the role of the antisymmetric part, but this shows only in the assumption that it does not contribute to the decay of $\mathcal{E}$; on the contrary, $\mathcal{C}$ should be thought of as the symmetric, or collisional part, and it does make the Lyapunov functional decay.

Next, the operators $\Pi_j$ act as a family of "nested projections". The first one, $\Pi_1$, sends $f$ to the kernel of the "collision operator" $\mathcal{C}$; then the second one sends $f$ to a smaller subspace, and then each $\Pi_j$ takes values in a smaller subspace until finally one reaches $f_\infty$. The "concrete" examples are the maps $f \to M^f_{\rho u T}$, $f \to M^f_{\rho u \langle T\rangle}$, $f \to M^f_{\rho 0 1}$, $f \to f_\infty$ which we considered in Section 2 (so for the Boltzmann equation we need four such nonlinear projections).

Finally, the key hypocoercivity condition is (H): It ensures basically that the effect of the "antisymmetric part" $B$ is strong enough to get us out of the image of $\Pi_j$, unless we are in the image of $\Pi_{j+1}$.

Theorem 3.5 leads to a simplified proof of Theorem 2.1, which does not involve any second-order differential inequality, but just variants of Gronwall's lemma. Once again, the key point is to add a correction to the Lyapunov functional $\mathcal{E}$ into another functional $\mathcal{F}$. The correction is small enough that the value of $\mathcal{F}$ is very close to the value of $\mathcal{E}$; but its structure is such that $\mathcal{F}$ satisfies (almost) a Gronwall-type estimate. More explicitly,

$$\mathcal{F}(f) = \big[\mathcal{E}(f) - \mathcal{E}(f_\infty)\big] + \sum_{j=1}^{J-1} a_j \big\langle (\mathrm{Id} - \Pi_j) f, (\mathrm{Id} - \Pi_j)'_f \cdot (Bf) \big\rangle, \tag{11}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in, say, $X^0$, and $\varepsilon > 0$, $a_j > 0$ $(1 \le j \le J-1)$ are small numbers depending on the smoothness of $f_t$, on $\delta$, and on estimates on the distance of $f$ to $f_\infty$ (in general $1 \gg a_1 \gg \ldots \gg a_{J-1}$).

If the reader thinks that I am being too abstract and formal here, I invite him or her to write down explicitly what (11) is for the Boltzmann equation: Take $\mathcal{E}(f) = \int f \log f$, $f_\infty = M(v)$, $\Pi_1 f = M^f_{\rho u T}$, $\Pi_2 f = M^f_{\rho u \langle T \rangle}$, $\Pi_3 f = M^f_{\rho 01}$, $\Pi_4 f = f_\infty$, and $Bf = v \cdot \nabla_x f$; then the expression of $\mathcal{F}(f)$ would fill up basically a whole page. Expression (11) is not only quite general, it is also the best way to conduct calculations.

Let me conclude this section with another theorem that can be derived from Theorem 3.5: convergence to equilibrium for the nonlinear Vlasov-Fokker-Planck interaction with moderate interaction and small coupling.

THEOREM 3.6. – *Let $W \in C^\infty(\mathbb{T}^n)$ be an even smooth function with* $\sup W - \inf W$ *small enough, and let $f_0$ be a probability density on $\mathbb{T}^n_x \times \mathbb{R}^n_v$, with all moments finite. Then there is a unique solution $(f_t)_{t \ge 0}$ to the partial differential equation*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x (W * \rho) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (fv), \qquad \rho_t(x) = \int f_t(x,v)\, dv;$$

*and it does converge to a uniquely determined equilibrium distribution $f_\infty$, with*

$$\|f_t - f_\infty\|_{L^1} = O(t^{-\infty}).$$

This theorem also follows directly from Theorem 3.5, now by choosing just\break $\Pi_1 f = \rho M$, $\Pi_2 f = f_\infty$, where the equilibrium $f_\infty$ is the unique minimizer of the energy functional $H(f) + (1/2) \int \rho(x) \rho(y) W(x-y)\, dx\, dy$. The assumption on $W$ being smooth and small enough guarantees the uniqueness of the minimizer (it implies that we stay away from phase transitions) and allows to develop a very strong regularity theory for the equation.

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UMPA (UMR CNRS 5669),
ENS Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France
e-mail: umpa.ens-lyon.fr/˜cvillani/

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