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Numerical Treatment of a Time Dependent Inverse Problem in Photon Transport.

S. Pieraccini - R. Riganti - A. Belleni-Morante

Sunto. – *Si studia un problema inverso unidimensionale per un'equazione integro-differenziale del trasporto di fotoni, e si determinano le proprietà di una sorgente di fotoni ultravioletti immersa in una nube interstellare. Un procedimento iterativo che si basa sulla discretizzazione spazio-temporale del problema diretto, porta alla determinazione della intensità della sorgente e delle sue variazioni al crescere del tempo.*

Summary. – *The time-dependent intensity of a UV-photon source, located inside an interstellar cloud, is determined by formulating and solving an inverse problem for the integro-differential transport equation of photons in a one-dimensional slab. Starting from a discretizazion of the direct problem, an iterative procedure is used to compute the values of the source intensity at increasing values of time, and it is applied in some numerical simulations, whose results are presented and discussed.*

1. – Introduction.

In the framework of astrophysical applications, a great interest is devoted to inverse problems allowing to evaluate some physical and geometrical properties of UV-photon sources, that are located somewhere inside interstellar clouds.

Based on known model equations regarding both the photon transport theory [1, 8, 11] and the nature of interstellar medium [6], contributions in this field are given, among several others, in [7, 9] and more recently in [3, 5] where time-independent inverse problems were considered, related to a source $q(x)$ emitting UV-photons inside an interstellar cloud, which is represented by a slab that occupies a bounded region $-L \leq x \leq L$ of the galactic space and is composed of an homogeneous mixture of low-density gases and dust grains. These problems were studied by using a stationary version of the photon transport equation, where time is not present or is treated as a parameter [4].

On the other hand, the more general inverse problem dealing with the time evolution of some physical properties of the photon source was recently studied in [2]. Here, by assuming that the source $q(\mathbf{x}, t)$ is spatially homogeneous within a region $V_0 \subset V \subset \mathbb{R}^3$ and that a time series of "far field" measurements of the photon density is known, it was proved that it is possible to identify the time behaviour of the source by means of a suitable time-discretization procedure of the photon transport equation, leading to explicit formulae for the source intensity $q(t)$ at discrete times $t_j,\ j = 0, 1, \ldots, J$.

In this paper, such a discretization procedure is developed and a time discretization algorithm, leading to the explicit determination of the source intensity, is applied by assuming a one-dimensional model of the cloud, that will be represented by the slab $V = \{x : -L \le x \le L\}$ containing a photon source to be identified. Then a numerical iterative approach is proposed in order to obtain an approximation of the time-behaviour of the source at discrete times.

The mathematical formulation of this one-dimensional problem is given in Section 2, where the photon transport equation and the initial and boundary conditions are presented. In Section 3 the inverse problem consisting in the identification of the source $q(x,t)$ is treated, by deriving its explicit solution in terms of a sequence of values $q(x,t_j)$ depending on known values of the photon densities at discrete times t_j. In Section 4 the numerical techniques allowing the determination of an approximated solution of the above inverse problem are considered, and the results of some numerical simulations are presented and discussed.

2. – The mathematical model.

We consider the slab of width $2L$ shown in Fig. 1, "representing" an interstellar cloud where photons, produced by a source $q(x,t) > 0$, $x \in (-L_0, L_0) \subset (-L, L)$, $t \in [t_0, t_J]$, undergo capture and isotropic scattering processes with constant total and scattering cross sections σ and σ_s respectively, with $0 < \sigma_s < \sigma$. Let $\vec{v} = c\vec{u}$ be the velocity of photons (c is the speed of light) and $\mu = \vec{u} \cdot \vec{\imath} = \cos\vartheta$. Let us further assume non-reentry boundary conditions for the

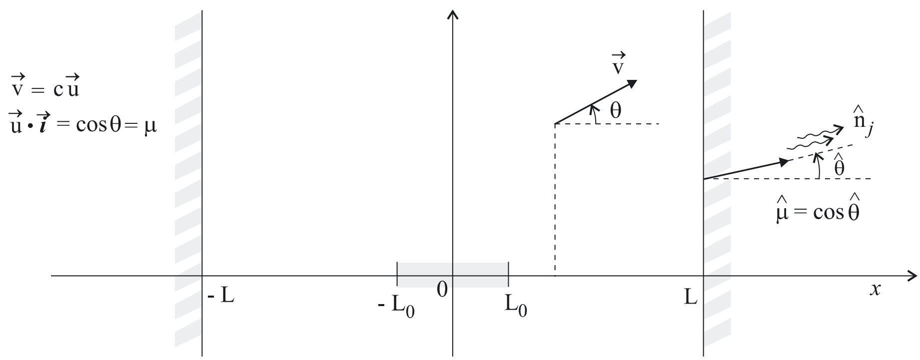

Fig. 1. – Geometry of the problem.

photons at $x = -L$ and at $x = +L$. Then the photon transport equation, the boundary condition and the initial condition have the form

$$\frac{\partial}{\partial t} N(x,\mu,t) = -c\mu \frac{\partial N}{\partial x} - c\sigma N + \frac{c\sigma_s}{2} \int_{-1}^{1} N(x,\mu',t)d\mu' + cq(x,t) \tag{1}$$

$$N(-L,\mu,t) = 0,\ \mu \in (0,1]; \quad N(L,\mu,t) = 0,\ \mu \in [-1,0) \tag{2}$$

$$N(x,\mu,0) = N_0(x,\mu) \tag{3}$$

where $N(x,\mu,t)$ is the number density of photons having a velocity component $\vec{v} \cdot \vec{\imath} = c\mu$ with $\mu \in [-1,1]$ and which at time t are at $x \in [-L,L]$. It is convenient to rescale the time independent variable by setting $t^\star = ct$ and re-write (1)-(3) in terms of the number density $M(x,\mu,t^\star) = N(x,\mu,t^\star/c) = N(x,\mu,t)$ as

$$\frac{\partial}{\partial t^\star} M(x,\mu,t^\star) = -\mu \frac{\partial M}{\partial x} - \sigma M + \frac{\sigma_s}{2} \int_{-1}^{1} M(x,\mu',t^\star)d\mu' + Q(x,t^\star) \tag{4}$$

$$M(-L,\mu,t^\star) = 0,\ \mu \in (0,1]; \quad M(L,\mu,t^\star) = 0,\ \mu \in [-1,0) \tag{5}$$

$$M(x,\mu,0) = M_0(x,\mu) = N_0(x,\mu) \tag{6}$$

where $Q(x,t^\star) = q(x,t^\star/c) = q(x,t)$.

The abstract version of system (4)-(6) in the Banach space $X = L^1([-L,L]\times [-1,1])$, endowed with the norm $\|f\| = \int_{-L}^{L} dx \int_{-1}^{+1} |f(x,\mu)|d\mu$, can be put into the form

$$\begin{cases} \dfrac{d}{dt^\star} M(t^\star) & = & (B+K)M(t^\star) + Q(t^\star), \quad t^\star > 0 \\ M(0) & = & M_0 \end{cases} \tag{7}$$

where $M(t^\star) = M(\cdot,\cdot,t^\star) : [0,+\infty) \to X,\ Q(t^\star) = Q(\cdot,t^\star) : [0,+\infty) \to X$ are now maps from $[0,+\infty)$ into the Banach space X and the operators B, K are defined as

$$(Bf)(x,\mu) = -\mu \frac{\partial f}{\partial x} - \sigma f, \quad D(B) = \left\{ f : f \in X,\ \mu \frac{\partial f}{\partial x} \in X, f \text{ satisfies the boundary condition (5)} \right\} \tag{8}$$

$$(Kf)(x,\mu) = \frac{\sigma_s}{2} \int_{-1}^{+1} f(x,\mu')d\mu', \quad D(K) = X. \tag{9}$$

The main properties of these operators have been discussed in [2] and may be summarized by the following lemma.

LEMMA 1.

(i) *$K \in \mathcal{B}(X)$, i.e. K is a bounded operator, with $\|K\| \le \sigma_s$;*

(ii) *$B \in \mathcal{G}(1, -\sigma; X)$, i.e. B is the generator of a strongly continuous semigroup $\{\exp(t^\star B), t^\star \ge 0\}$ such that $\|\exp(t^\star B)\| \le \exp(-\sigma t^\star)$ $\forall t^\star \ge 0$.*

In view of its application in the discretization procedure of (7) which will be proposed in the next Section, it is useful to calculate explicitly the inverse operator $(\lambda I - B)^{-1}$. To this aim, putting $(\lambda I - B)f = g$, from definition (8) we have that

$$\frac{\partial f}{\partial x} + \frac{\lambda + \sigma}{\mu} f = \frac{1}{\mu} g$$

or equivalently

$$\frac{\partial}{\partial x}\left[\exp\left(\frac{\lambda+\sigma}{\mu}\, x\right) f\right] = \frac{1}{\mu} \exp\left(\frac{\lambda+\sigma}{\mu}\, x\right) g \tag{10}$$

where f must satisfy the boundary conditions (5). Integration with respect to $x' \in [-L, x]$ yields

$$\exp\left(\frac{\lambda+\sigma}{\mu}\, x\right) f(x,\mu) - \exp\left(-\frac{\lambda+\sigma}{\mu}\, L\right) f(-L,\mu) = \frac{1}{\mu} \int_{-L}^{x} \exp\left(\frac{\lambda+\sigma}{\mu}\, x'\right) g(x',\mu)\, dx'.$$

Hence, if $\mu \in (0,1]$ then $f(-L,\mu) = 0$ and so we obtain

$$f(x,\mu) = \frac{1}{\mu} \int_{-L}^{x} \exp\left[-\frac{\lambda+\sigma}{\mu}\,(x - x')\right] g(x',\mu)\, dx', \qquad \mu \in (0,1]. \tag{11}$$

Likewise, by integrating (10) from x to $+L$:

$$\exp\left(\frac{\lambda+\sigma}{\mu}\, L\right) f(L,\mu) - \exp\left(\frac{\lambda+\sigma}{\mu}\, x\right) f(x,\mu) = \frac{1}{\mu} \int_{x}^{L} \exp\left(\frac{\lambda+\sigma}{\mu}\, x'\right) g(x',\mu)\, dx'$$

and taking into account that $f(L,\mu) = 0$ if $\mu \in [-1, 0)$, we have

$$f(x,\mu) = \frac{1}{|\mu|} \int_{x}^{L} \exp\left[-\frac{\lambda+\sigma}{\mu}\,(x - x')\right] g(x',\mu)\, dx', \qquad \mu \in [-1, 0). \tag{12}$$

Equations (11) and (12) hold for every $\lambda > -\sigma$, see *(ii)* of Lemma 1. In particular, if $\lambda = 1/\tau > 0$, (11) and (12) give

$$(I - \tau B)^{-1} g = \begin{cases} \dfrac{1}{\tau\mu} \displaystyle\int_{-L}^{x} \exp\left[-\frac{1+\tau\sigma}{\tau\mu}\,(x - x')\right] g(x',\mu)\, dx', & \mu \in (0,1] \\[2ex] \dfrac{1}{\tau|\mu|} \displaystyle\int_{x}^{L} \exp\left[-\frac{1+\tau\sigma}{\tau\mu}\,(x - x')\right] g(x',\mu)\, dx', & \mu \in [-1, 0). \end{cases} \tag{13}$$

3. – The time dependent inverse problem.

As anticipated in the Introduction, assume now that the values $\hat{M}_j = M(L, \hat{\mu}, t_j^\star)$ of the photon density with $x = L$ and $\hat{\mu} > 0$ are measured at the instants $t_j^\star = j\tau$, $j = 0, 1, \ldots, J$ where τ is a constant time interval. In fact, the measurements are made at some $\hat{x}$ "far from the slab", i.e. the values $M(\hat{x}, \hat{\mu}, t_j^\star + \hat{t}^\star)$ with $\hat{t}^\star = (\hat{x} - L)/\hat{\mu}$ and with $\hat{x} \gg L$ are measured. However, since $M(\hat{x}, \hat{\mu}, t_j^\star + \hat{t}^\star) = M(L, \hat{\mu}, t_j^\star)$, we may assume that the values $\hat{M}_j = M(L, \hat{\mu}, t_j^\star)$ are known.

Let us discretize (7) as follows

$$\frac{n_{j+1} - n_j}{\tau} = Bn_{j+1} + Kn_j + Q_j, \quad j = 0, 1, \ldots, J, \quad n_0 = M_0, \tag{14}$$

where $Q_j = Q(x, t_j^\star)$. Here $n_j = n_j(x, \mu)$ approximates $M_j(x, \mu) \equiv M(x, \mu, t_j^\star)$ within an error which, as proved in [2], satisfies the inequality $\|n_j - M_j\| \leq \gamma\tau, \forall j$, where γ is a suitable positive constant.

It follows from (14) that

$$(I - \tau B)n_{j+1} = n_j + \tau K n_j + \tau Q_j, \qquad j = 0, 1, \ldots, J$$

$$n_{j+1}(x, \mu) = [(I - \tau B)^{-1}(I + \tau K)n_j](x, \mu) + \tau[(I - \tau B)^{-1}Q_j](x, \mu).$$

Hence, by using (13) and the definition (9) of the operator K:

$$n_{j+1}(x, \mu) = \frac{1}{\tau\mu} \int_{-L}^{x} dx' \exp\left[-\frac{1 + \tau\sigma}{\tau\mu}(x - x')\right] \left\{ n_j(x', \mu) + \frac{\tau\sigma_s}{2} \int_{-1}^{+1} n_j(x', \mu')d\mu' + \tau Q_j(x') \right\}, \quad \mu \in (0, 1] \tag{15}$$

$$n_{j+1}(x, \mu) = \frac{1}{\tau|\mu|} \int_{x}^{L} dx' \exp\left[-\frac{1 + \tau\sigma}{\tau\mu}(x - x')\right] \left\{ n_j(x', \mu) + \frac{\tau\sigma_s}{2} \int_{-1}^{+1} n_j(x', \mu')d\mu' + \tau Q_j(x') \right\}, \quad \mu \in [-1, 0). \tag{16}$$

In order to extract from (15) and (16) the unknown values of $Q_j(x')$ in terms of the measured densities $\hat{M}_j$, some assumptions on the spatial distribution of the source are needed. If, in particular, we assume that for each $t^\star$ the source has a uniform distribution $\forall x \in [-L_0, L_0]$, so that

$$Q_j(x') = \begin{cases} Q_j = \text{a constant} > 0, & x' \in [-L_0, L_0] \\ 0, & x' \notin [-L_0, L_0] \end{cases}, \tag{17}$$

then (15) and (16) become respectively:

- if $\mu \in (0,1]$:

$$(18)\quad n_{j+1}(x,\mu) = \frac{1}{\tau\mu} \int_{-L}^{x} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]\left\{n_j(x',\mu) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x',\mu')d\mu'\right\}, \qquad -L \le x < -L_0$$

$$(19)\quad n_{j+1}(x,\mu) = \frac{1}{\tau\mu} \int_{-L}^{x} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]\left\{n_j(x',\mu) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x',\mu')d\mu'\right\} + \left\{1 - \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x+L_0)\right]\right\}\frac{\tau Q_j}{1+\sigma\tau}, \qquad -L_0 \le x \le L_0$$

$$(20)\quad n_{j+1}(x,\mu) = \frac{1}{\tau\mu} \int_{-L}^{x} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]\left\{n_j(x',\mu) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x',\mu')d\mu'\right\} + \left\{\exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-L_0)\right] - \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x+L_0)\right]\right\}\frac{\tau Q_j}{1+\sigma\tau}, \qquad L_0 < x \le L.$$

- if $\mu \in [-1,0)$:

$$(21)\quad n_{j+1}(x,\mu) = \frac{1}{\tau|\mu|} \int_{x}^{L} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]\left\{n_j(x',\mu) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x',\mu')d\mu'\right\} + \left\{\exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x+L_0)\right] - \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-L_0)\right]\right\}\frac{\tau Q_j}{1+\sigma\tau}, \qquad -L \le x < -L_0$$

$$(22)\quad n_{j+1}(x,\mu) = \frac{1}{\tau|\mu|} \int_{x}^{L} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]\left\{n_j(x',\mu) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x',\mu')d\mu'\right\} + \left\{1 - \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-L_0)\right]\right\}\frac{\tau Q_j}{1+\sigma\tau}, \qquad -L_0 \le x \le L_0$$

$$(23)\quad n_{j+1}(x,\mu) = \frac{1}{\tau|\mu|} \int_{x}^{L} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]\left\{n_j(x',\mu) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x',\mu')d\mu'\right\}, \qquad L_0 < x \le L.$$

The value of $n_{j+1}(L,\hat{\mu}) = \hat{n}_{j+1}$, which is obtained by setting $x = L,\ \mu = \hat{\mu} > 0$ into (20), approximates the "far-field" density $\hat{M}_{j+1} = M(L,\hat{\mu},t^\star_{j+1}) = M(\hat{x},\hat{\mu},t^\star_{j+1} + \hat{t})$ measured at time $t^\star_{j+1} + \hat{t}$. If $\hat{n}_{j+1}$ is known through $\hat{M}_{j+1}$, equation (20) gives explicitly the values of the source at time $t^\star_j$:

$$(24)\quad Q_j = \frac{1}{\chi}\left\{\hat{M}_{j+1} - \frac{1}{\tau\hat{\mu}}\int_{-L}^{L} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\hat{\mu}}(L-x')\right]\left[n_j(x',\hat{\mu}) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x',\mu')d\mu'\right]\right\}$$

$$j = 0, 1, \ldots, J$$

where χ is the (strictly positive) constant

$$(25)\qquad \chi = \frac{\tau}{1+\sigma\tau}\left\{\exp\left[-\frac{1+\tau\sigma}{\tau\hat{\mu}}(L-L_0)\right] - \exp\left[-\frac{1+\tau\sigma}{\tau\hat{\mu}}(L+L_0)\right]\right\},$$

and where $n_j(x,\mu)$ is known from "step j".

The set of equations (18-25) allows to determine the solution of the inverse problem, consisting in the determination of the time evolution of a source with a uniform spatial distribution as in (17). Starting with $j = 0$, the values of $n_0(x',\mu')$ at any $(x',\mu') \in [-L_0, L_0] \times [-1, 1]$, which are needed to calculate the integrals on the right hand side of (24), are given by the initial condition $M_0(x',\mu')$, see (14). Since $\hat{n}_1 = \hat{M}_1$ is known, then Q_0 can be explicitly determined. Correspondingly, inserting Q_0 into (18)-(23) with $j = 0$ gives the values of $n_1(x,\mu)$ which allow to calculate the right hand side of (24) for $j = 1$, used to determine Q_1, and so on.

Therefore, the whole set of Q_j, $j = 0, 1, \ldots, J$ can be determined, at least in principle, in terms of the photon density $n_j(x,\mu)$, to be calculated at the same time $t^\star_j$, and of the data $\hat{M}_{j+1}$ measured at time $t^\star_j + \tau$. However, it will be explained in the next Section that the numerical treatment of (24) appears to be very hard from a numerical point of view, and for this reason a different approach, consisting in a suitable iterative procedure, will be used to obtain quantitative results for the solution of the inverse problem.

REMARK 1. – If one is interested in considering a more refined spatial distribution of the source $Q_j(x)$ instead of the uniform one, see (17), then in general the knowledge of one or more additional sets of "far field" measurements is needed, in order to evaluate simultaneously other unknown parameters of the assumed distribution, such as its mean and/or its variance. Alternatively, one could identify the various $Q_j(x)$ starting from one far-field measurement, provided that the Q_j's are assumed to belong to a suitable family $\mathcal{F}$ of functions of $x \in [-L, L]$, see [2].

In particular, the solution procedure developed in this Section is still effective if model (17) is replaced by

$$Q_j(x') = \begin{cases} 2L_0 Q_j\,\varphi(x'), & x' \in [-L_0, L_0] \\ 0, & x' \notin [-L_0, L_0] \end{cases}, \tag{26}$$

where $\varphi(x')$ is a given, bounded and even function satisfying

$$\varphi(x') \geq 0; \qquad \int_0^{L_0} \varphi(x')dx' = \frac{1}{2},$$

and where the Q_j's are again the quantities to be evaluated (in this case, $\mathcal{F}$ contains only φ). In fact, by using (26) one easily obtains that the solution sequence $\{Q_j\}$ will still be given by (24) with the constant χ replaced by

$$\chi' = \frac{2L_0}{\hat{\mu}} \int_{-L_0}^{L_0} dx' \varphi(x') \exp\left[-\frac{1+\sigma\tau}{\tau\hat{\mu}}(L - x')\right],$$

and the densities n_{j+1}, $j = 0, \ldots, J$ are now determined as follows:

- if $\mu \in (0, 1]$:

$$n_{j+1}(x, \mu) = \frac{I_1(x, \mu)}{\tau\mu}, \qquad -L \leq x < -L_0$$

$$n_{j+1}(x, \mu) = \frac{1}{\tau\mu}[I_1(x, \mu) + 2L_0\tau Q_j I_3(x, \mu)], \qquad -L_0 \leq x \leq L_0$$

$$n_{j+1}(x, \mu) = \frac{1}{\tau\mu}[I_1(x, \mu) + 2L_0\tau Q_j I_4(x, \mu)], \qquad L_0 < x \leq L$$

- if $\mu \in [-1, 0)$:

$$n_{j+1}(x, \mu) = \frac{1}{\tau|\mu|}[I_2(x, \mu) + 2L_0\tau Q_j I_4(x, \mu)], \qquad -L \leq x < -L_0$$

$$n_{j+1}(x, \mu) = \frac{1}{\tau|\mu|}[I_2(x, \mu) + 2L_0\tau Q_j I_5(x, \mu)], \qquad -L_0 \leq x \leq L_0$$

$$n_{j+1}(x, \mu) = \frac{I_2(x, \mu)}{\tau|\mu|}, \qquad L_0 < x \leq L$$

where I_1, I_2 are the same integrals on the right hand side of (18)-(23), i.e.

$$I_1(x, \mu) = \int_{-L}^{x} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x - x')\right]\left\{n_j(x', \mu) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x', \mu')d\mu'\right\}$$

$$I_2(x, \mu) = \int_{x}^{L} dx' \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x - x')\right]\left\{n_j(x', \mu) + \frac{\tau\sigma_s}{2}\int_{-1}^{+1} n_j(x', \mu')d\mu'\right\}$$

and in addition the integrals

$$I_3(x,\mu) = \int_{-L_0}^{x} dx' \varphi(x') \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]$$

$$I_4(x,\mu) = \int_{-L_0}^{L_0} dx' \varphi(x') \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]$$

$$I_5(x,\mu) = \int_{x}^{L_0} dx' \varphi(x') \exp\left[-\frac{1+\tau\sigma}{\tau\mu}(x-x')\right]$$

can be calculated when the appropriate $\varphi(x')$ is chosen. □

4. – Numerical procedure and application.

Formula (24) turns out to be unsuitable from a computational point of view, since the expressions (24) and (25) tend to become extremely ill-conditioned when the time step τ is decreased.

Hence, in order to solve numerically the inverse problem, we use a different strategy, based on a iterative approach: given an initial estimate $Q_j^{(0)}$ for the value Q_j, we solve the direct problem integrating system (4)-(6) from $t_j^\star$ to $t_j^\star + \Delta T$, where ΔT is such that at time $t^\star + \Delta T$ the contribution of the source at time $t^\star$ has reached the boundary of the cloud. Let s be such that $t_j^\star + \Delta T = t_{j+s}^\star$. Then, we compare the photon number density computed on the boundary at time $t_{j+s}^\star$ with the set of measures. Let us denote by Δn_{j+s} the difference between the computed value and the measure. If $|\Delta n_{j+s}|$ is smaller then a given tolerance, we accept the estimate for the source, otherwise we properly correct the estimate by assigning a new value $Q_j^{(1)}$ and we repeat the process. When a estimate $Q_j^{(k)}$ is accepted, we proceed with the next time step, starting again the process from $t_{j+1}^\star$.

As far as the direct problem is concerned, it is solved as follows. First, we discretize the velocity field, considering N_v values μ_r in the interval $[-1,1]$. The integrals (9) are approximated by Gauss-Legendre quadrature rule, hence the values μ_r, $r=1,...,N_v$, correspond to the gaussian nodes on the interval $[-1,1]$, which we assume labeled from the largest (μ_1) to the smallest (μ_{N_v}). Let us denote by a_r the corresponding weights. Further, let us denote by $n(x,\mu_r,t^\star)$ the approximation of $M(x,\mu_r,t^\star)$. Equation (4) is then approximated by the following set of coupled equations:

$$(27)\quad \frac{\partial}{\partial t^\star} n(x,\mu_r,t^\star) = -\frac{\partial \mu_r n(x,\mu_r,t^\star)}{\partial x} - \sigma n(x,\mu_r,t^\star) + \frac{\sigma_s}{2}\sum_{p=1}^{N_v} a_p n(x,\mu_p,t^\star) + Q(x,t^\star)$$

for $r=1,...,N_v$.

Next, we consider a spatial discretization, following a method-of-lines approach. Let us introduce the grid points $x_{i+1/2} = -L + i\Delta x$, for $i = 0, ..., N_x$ and $\Delta x = 2L/N_x$. Let $x_i = (x_{i-1/2} + x_{i+1/2})/2$ denote the cell centers. By using a WENO reconstruction procedure [10], equations (27) are approximated by

$$\frac{\mathrm{d}n(x_i, \mu_r, t^\star)}{\mathrm{d}t^\star} = \frac{1}{\Delta x}\left(\tilde{f}_{i+1/2} - \tilde{f}_{i-1/2}\right) - \sigma n(x_i, \mu_r, t^\star) + \frac{\sigma_s}{2}\sum_{p=1}^{N_v} a_p n(x_i, \mu_p, t^\star) + Q(x_i, t^\star),$$

for $i = 0, ..., N_x$ and $r = 1, ..., N_v$, where the numerical flux $\tilde{f}_{i+1/2}$ is obtained by a fifth order WENO scheme.

Let $\bar{n}^r(t^\star), Q(t^\star) \in \mathbb{R}^{N_x}$ denote the vectors whose elements are $n(x_i, \mu_r, t^\star)$ and $Q(x_i, t^\star)$, respectively, and let $\bar{n}(t^\star)$ denote the matrix with entries $\bar{n}_i^r(t^\star)$. After the approximations in space and velocity previously introduced, equation (4) is approximated by a set of coupled systems of ordinary differential equations which can be written in a compact form as

$$\frac{\mathrm{d}\bar{n}^r(t^\star)}{\mathrm{d}t^\star} = g_r(t^\star, \bar{n}(t^\star); Q(t^\star)), \qquad r = 1, ..., N_v, \tag{28}$$

where

$$g_r(t^\star, \bar{n}(t^\star); Q(t^\star)) = \frac{1}{\Delta x}\left(\tilde{f}_{i+1/2} - \tilde{f}_{i-1/2}\right) - \sigma \bar{n}^r(t^\star) + \frac{\sigma_s}{2}\sum_{p=1}^{N_v} a_p \bar{n}^p(t^\star) + Q(t^\star).$$

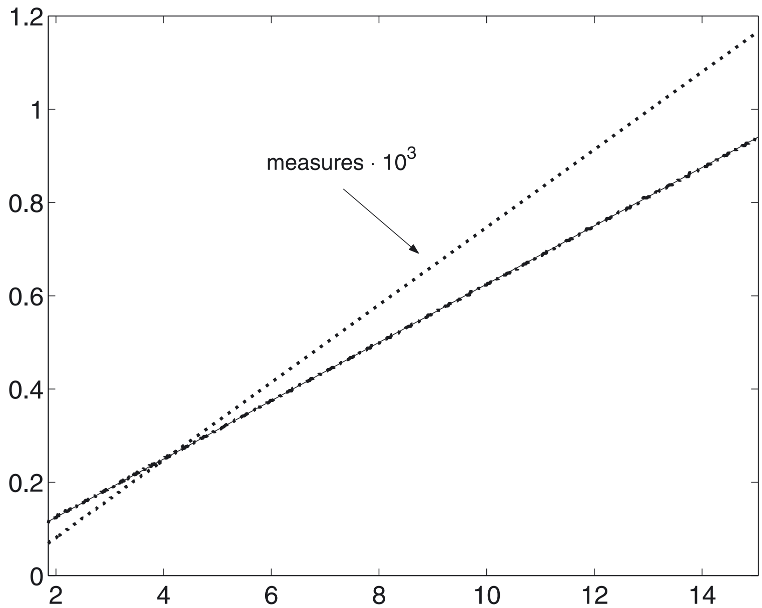

Fig. 2. – Test 1. Computed solution (dash-dot line) and exact solution (continuous line); dotted line: measured data $\hat{M}_j$ amplified by 10^3.

Finally, each system of differential equations is integrated using a proper ODE solver.

The way in which at each iteration we correct the source estimate is the following:

$$Q_j^{(k+1)} = Q_j^{(k)} - \frac{Q_j^{(k)}}{\hat{M}_{j+s}} \Delta n_{j+s}, \qquad k = 0, 1, ... \tag{29}$$

The rule is based on the following consideration: if $\Delta n_{j+s} > 0$ this means that the source at time $t_j^\star$ was over-estimated and consequently we reduce $Q_j^{(k)}$; on the other hand, if $\Delta n_{j+s} < 0$ the source was under-estimated and $Q_j^{(k)}$ is therefore increased.

The overall numerical scheme is sketched as follows.

Numerical scheme.

1. Given initial data $n_{ir}^0 = M_0(x_i, \mu_r)$ for $i = 0, 1, ..., N_x$ and $r = 1, ..., N_v$; an initial guess $Q_0^{(0)}$; a set of measures $\hat{M}_j$ for $j = 0, 1, ..., J$; a tolerance ε
2. for $j = 0, 1, .., J - s$

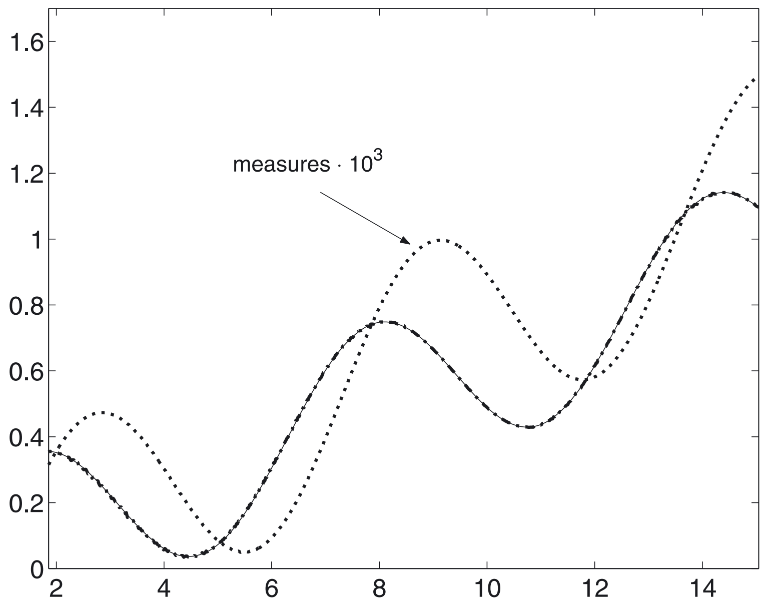

Fig. 3. – Test 2. Computed solution (dash-dot line) and exact solution (continuous line); dotted line: measured data $\hat{M}_j$ amplified by 10^3.

2.1. for $k = 0, 1, ...$
   2.1.1. Compute an approximate solution of (28) with $g_r(t^\star, \bar{n}(t); Q_j^{(k)})$ on the interval $[t_j^\star, t_{j+s}^\star]$ for $r = 1, ..., N_v$
   2.1.2. Compute $\Delta n_{j+s} = n(L, \hat{\mu}, t_{j+s}^\star) - \hat{M}_{j+s}$; if $|\Delta n_{j+s}| \leq \varepsilon \hat{M}_{j+s}$ set $Q_j = Q_j^{(k)}$, else update $Q_j^{(k)}$ according to (29).

We performed some preliminary numerical experiments in different situations. In order to have an "exact" solution for computing the errors, we first solved the direct problem with a given source on a time interval $[T_1, T_2]$; than we considered two times T_0 and T_f such that $T_1 < T_0 < T_f < T_2$, and solved the inverse problem on the time interval $[T_0, T_f]$; the initial data $n_{jr}^0 = M(x_j, \mu_r, T_0)$ and the set of measures $\hat{M}_j$ are obtained from the solution of the direct problem.

We examined three test cases. First, we considered a source given by

$$Q(t^\star) = \frac{a\, t^\star + b\, \sin(c\, t^\star)}{T_2},$$

with a, b, c given constants satisfying $Q(t^\star) \geq 0$; we assumed $a = 1,\ b = 0$ (test 1) and $a = 1,\ b = 4,\ c = 1$ (test 2). Further, we considered (test 3) a source with a more generic time behavior in which sharp peaks are superimposed to a smooth

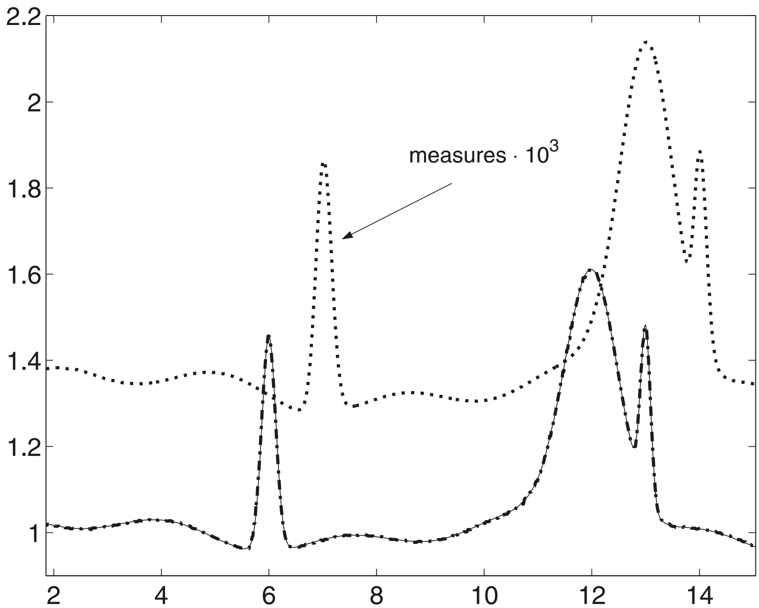

Fig. 4. – Test 3. Computed solution (dash-dot line) and exact solution (continuous line); dotted line: measured data $\hat{M}_j$ amplified by 10^3.

periodic trend, analitically given by

$$Q(t^\star) = 1 + \frac{2}{100}\sin(1.9t^\star) + \frac{3}{100}\sin(0.7t^\star) + \frac{1}{2}\mathrm{e}^{-30(t^\star-6)^2} + \frac{3}{5}\mathrm{e}^{-2(t^\star-12)^2} + \frac{2}{5}\mathrm{e}^{-60(t^\star-13)^2}.$$

In all the tests we used two initial guesses, $Q_0^{(0)} = 0.1$ (initial guess A) and $Q_0^{(0)} = 0.01$ (initial guess B); further, at each time step $j \geq 1$ we started from $Q_j^{(0)} = Q_{j-1}$. A maximum number of 150 iterations was imposed. We remark that in our experiments if such a number was reached (as happened in one of the tests here reported), failure was not declared, but the final estimate was accepted and the iteration proceeded. The given tolerance was $\varepsilon = 10^{-3}$. Further, we set $\sigma = 5$, $\sigma_s = 0.2$, $L = 1$, $L_0 = 0.1$. We assumed $\hat{\mu} = \mu_1$, $N_v = 8$ and $N_x = 100$; the time step was given by $\tau = 0.9\Delta x/\mu_1$. Finally, the system of ODEs was solved using the MATLAB function ODE45.

The results obtained on the three test cases are summarized in figures 2, 3 and 4. In figure 2 we plot (versus $t^\star$) the results obtained for Q_j for test 1 with initial guess B (dash-dot line); the computed solution is compared with the exact one (continuous line). In the same figure we plot the corresponding set of measures $\hat{M}_j$ used (dotted line), amplified by a factor of 10^3. In figures 3 and 4 we report similar data and results obtained for the second and third test cases, respectively, starting again with initial guess B.

From the figures it is clear that the qualitative behaviour of the computed solution is quite satisfactory. In order to estimate the results, we compared the computed and the exact solutions by analyzing the relative errors

$$(e_r)_j = \frac{|Q_j - Q(t_j^\star)|}{Q(t_j^\star)}, \qquad j = 0, 1, \dots, J.$$

In Table 1 we report, for each test case considered and for both initial guesses, the mean value of the relative errors and the maximum relative error computed. The table shows that the errors are low even in test 3, which apparently seems to be the most cumbersome one. The test problem in which the errors are higher is test 2, both in terms of their mean value and in terms of the maximum value. We point out that in all cases the mean relative error has the same order of magnitude of the relative tolerance used in the stopping criterion.

Table 1. – Mean error and maximum relative error for the three test cases.

	initial guess A		initial guess B	
Test #	mean error	max error	mean error	max error
1	$2.96 \cdot 10^{-3}$	$4.09 \cdot 10^{-2}$	$2.98 \cdot 10^{-3}$	$4.10 \cdot 10^{-2}$
2	$6.84 \cdot 10^{-3}$	$9.36 \cdot 10^{-2}$	$6.75 \cdot 10^{-3}$	$9.39 \cdot 10^{-2}$
3	$1.45 \cdot 10^{-3}$	$1.85 \cdot 10^{-2}$	$1.47 \cdot 10^{-3}$	$1.92 \cdot 10^{-2}$

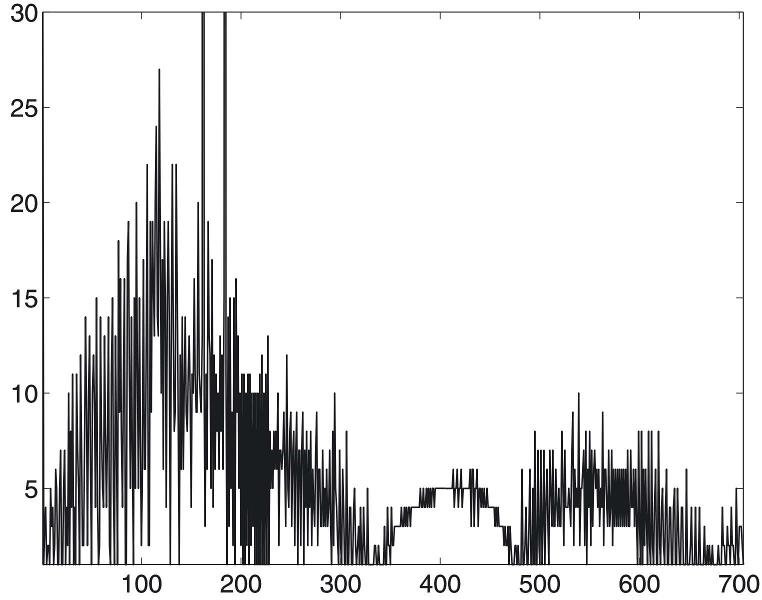

Fig. 5. – Test 2. Number of iterations versus the step counter j (initial guess A)

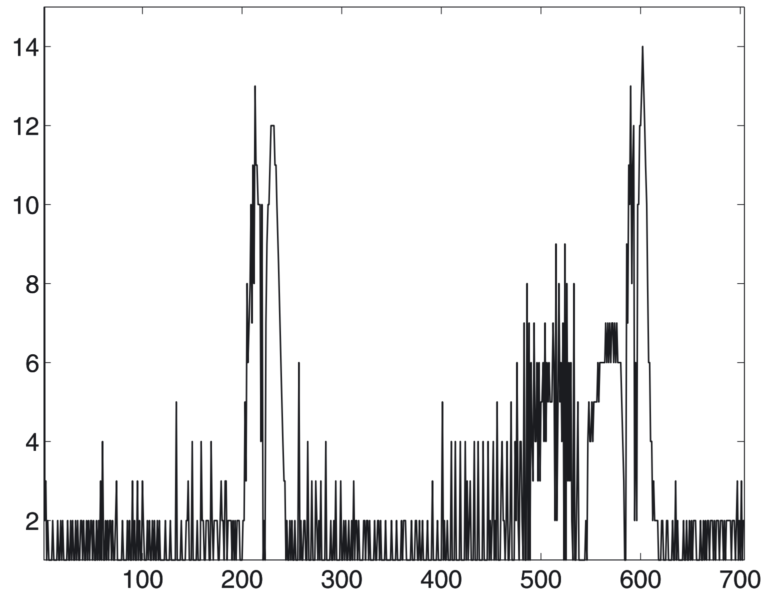

Fig. 6. – Test 3. Number of iterations versus the step counter j (initial guess A)

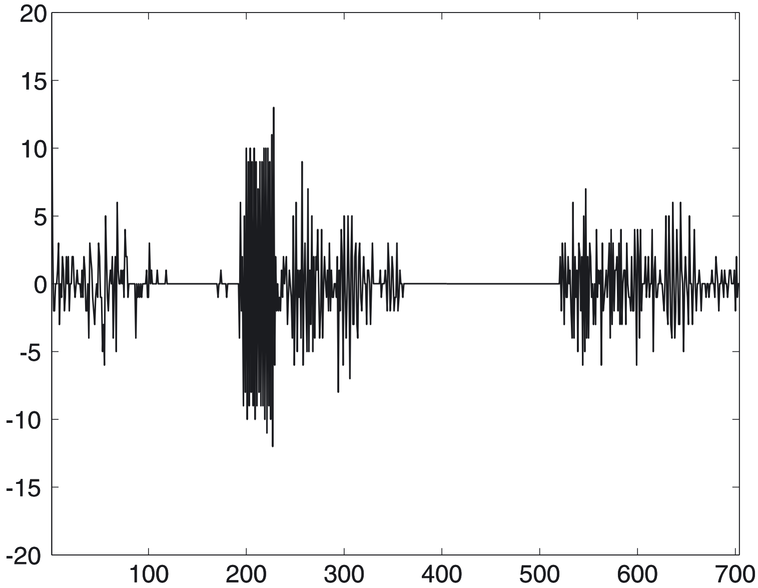

Fig. 7. – Test 2. Difference in the number of iterations with initial guesses A and B

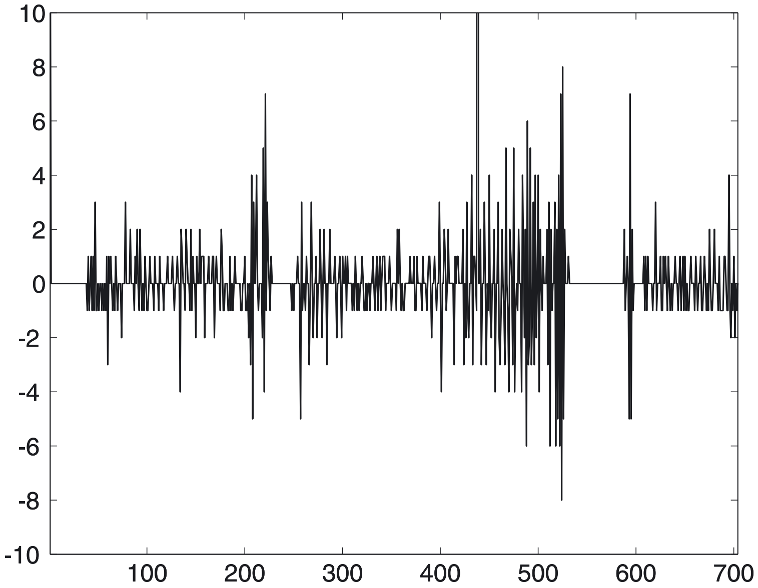

Fig. 8. – Test 3. Difference in the number of iterations with initial guesses A and B

Finally, in figures 5-8, we summarize, focusing on test 2 and 3, the computational cost in terms of the number of iterations needed at each step in order to satisfy the stopping criterion. Let us denote by $K(j)$ the number of iterations performed at step j, in such a way that $Q_j = Q_j^{K(j)}$. Figures 5 and 6 show $K(j)$ versus j for test cases 2 and 3, respectively, starting with initial guess A. In figure 5, the two peaks outside the figure boundaries correspond to $K(j) = 150$ (the maximum number of iterations was reached without satisfying the stopping criterion). Similar results are obtained by using the initial guess B. Concerning the first time step, we remark that, in test 2, using the initial guess A, 34 iterations are needed in order to satisfy the stopping criterion, while 49 iterations are needed when using the initial guess B. For test 3, at the first time step 43 iterations are needed with initial guess A and 58 with initial guess B. In the subsequent steps, the initial guess at each step is good enough that a moderate computational task is required in order to compute the next estimate Q_j of the solution, except for those regions in which the solution is steeper. In figures 7-8 we report, for both the test cases 2 and 3, the differences between the values $K(j)$ obtained by starting with the initial guess B and those obtained by starting with the initial guess A. Such results show that, as a whole, the computational cost is quite similar when using the two different initial guesses. This is clearly due to the fact that, as already mentioned, for $j \geq 2$ the starting values $Q_j^{(0)}$ are good enough and only the computational cost of the first time step strongly depends on the chosen initial guess.

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