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Hypoellipticity and Local Solvability of Anisotropic PDEs with Gevrey Nonlinearity.

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Sunto. – *In questo articolo viene proposto un approccio unificato, che si basa sulle tecniche dell'analisi microlocale, per caratterizzare sia l'ipoellitticità sia la risolubilità locale, in C^∞ e nelle classi di Gevrey G^λ, di operatori alle derivate parziali anisotropi, in dimensione $n \geq 3$, i quali, vengono perturbati con non linearità di tipo Gevrey. Per ottenere questi risultati sono state imposte alcune condizioni sul segno dei termini di ordine inferiore della parte lineare dell'operatore, vedere Teorema* 1.1 *e Teorema* 1.3.

Summary. – *We propose a unified approach, based on methods from microlocal analysis, for characterizing the hypoellipticity and the local solvability in C^∞ and Gevrey G^λ classes of semilinear anisotropic partial differential operators with Gevrey nonlinear perturbations, in dimension $n \geq 3$. The conditions for our results are imposed on the sign of the lower order terms of the linear part of the operator, see Theorem* 1.1 *and Theorem* 1.3 *below.*

1. – Introduction.

We consider a class of semilinear anisotropic equations with multiple characteristics in n variables $x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$, $n \geq 3$, belonging to the set $\Omega := \{|x| < \delta\}$, δ small, of the form:

$$P(x', x_n, D_{x'}, D_{x_n})u + G(x', x_n; \partial_{x'}^{\gamma'} \partial_{x_n}^{j} u)|_{\left|\frac{\gamma'}{\rho'}\right| + j < k^*} = \mu f(x), \tag{1.1}$$

where the linear part is given by:

$$P(x', x_n, D_{x'}, D_{x_n}) = D_{x_n}^m - \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x) D_{x'}^{\alpha'} + \sum_{k^* \leq \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) D_{x'}^{\beta'} D_{x_n}^{j}, \tag{1.2}$$

with $m, j \in \mathbb{Z}_+$, $\alpha', \beta', \gamma' \in \mathbb{Z}_+^{n-1}$, $0 < k^* < m$, μ small constant, $\left|\frac{\alpha'}{\rho'}\right| := \sum_{i=1}^{n-1} \alpha'_i \frac{1}{\rho_i}$, $\rho' = (\rho_1, \ldots, \rho_{n-1})$ with $0 < \rho_i \leq 1$ for $i = 1, \ldots, n-1$; we shall also say that

$\left|\frac{\beta'}{\rho'}\right| + j$ is the anisotropic order of $D^{\beta'}_{x'} D^{j}_{x_n}$, so the nonlinearity involves derivatives of anisotropic order less than k^*. We give for (1.1) and (1.2) results of hypoellipticity and local solvability at the origin. For related results in the case $n = 2$ see De Donno-Oliaro [5], in which C^∞ and Gevrey local solvability have been obtained by considering respectively C^∞ nonlinearity G and analytic nonlinearity G in (1.1); hypoellipticity in C^∞ and Gevrey classes was also proved for C^∞ nonlinearity G.

In this paper in Theorem 1.1 we extend to the multidimensional case $n \geq 3$ the results in [5], while the main result is contained in Theorem 1.3, where a new approach allows us to admit Gevrey nonlinearity G in the study of Gevrey local solvability of the equation (1.1), see Section 3.3. About this frame see also [29], in which the composition in Gevrey classes is treated, and see [1] that requires a bit more than the Gevrey regularity on the perturbation G but it deals with exponential Gevrey norms which are suitable for the machinary of p.d.o. calculus. We observe also that nonlinear composition estimates for Gevrey functions for different scales of Banach-spaces (using norms based on infinite sums) have been proved in [2], [12] [22].

The arguments in our proofs are based mainly on microlocal tools, allowing relevant simplifications in the study: pseudo-differential operators, wave front sets and $S^m_{\rho,\delta}$ techniques.

We list some papers devoted to this kind of problems, using the same techniques: Hounie-Santiago [14] and Gramchev-Popivanov [10] on the local solvability of semilinear partial differential equations with simple characteristics, Gramchev-Rodino [12] about Gevrey solvability for semilinear equations with multiple characteristics, Garello [7] regarding the inhomogeneous elliptic case, see also Šananin [31] on the C^∞ local solvability of linear equations of quasi principal type and Lorenz [21] regarding linear anisotropic operators with characteristics of constant multiplicity.

We start by considering here C^∞ nonlinearity G, namely, writing $G(x; v)$ as $F(x; \Re v, \Im v)$, we assume $F \in C^\infty(\Omega \times \mathbb{R}^N)$. We also assume C^∞ coefficients in (1.2) and in the following we always suppose that

$$\Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(0) \xi'^{\alpha'} \neq 0 \text{ for } \xi' \neq 0, \tag{1.3}$$

$$G(x', x_n; 0) = 0. \tag{1.4}$$

We recall that the nonzero hypothesis on $\Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'} \xi'^{\alpha'}$ is a nondegeneracy condition with invariant meaning, usually required in the study of the local solvability and hypoellipticity of the linear operator (1.2) in C^∞ and G^σ, $\sigma > \frac{m}{m-1}$, see for example Liess-Rodino [20], De Donno-Rodino [6], concerning Gevrey hypoellipticity for 2 variables PDEs with high multiplicity. Let us also observe that if $\Im \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(0)\xi'^{\alpha'} \neq 0$ then the operator is quasi-elliptic; the results of hypoellipticity and local solvability are well known in this case. Starting point for our discussion is the result of Marcolongo-Oliaro [23], where the G^{σ} local solvability is proved under hypotheses on the quasi principal symbol; in the present paper we admit a larger class of data $f(x)$ with respect to the case studied in [23], but we add hypotheses on the lower order terms. In the following it will be convenient to use the Sobolev anisotropic space $H^s_{\rho'}$, where as before $\rho' = (\rho_1, \ldots, \rho_{n-1})$, defined by

$$\|f\|_{H^s_{\rho'}} := \left(\int \left(1 + \sum_{j=1}^{n-1} |\xi_j|^{2\rho_j} + |\xi_n|^2 \right)^s |(\mathcal{F}_{x\to\xi} f)(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty$$

$\mathcal{F}_{x\to\xi} f$ being the Fourier transform of $f(x)$. For $s > \frac{1+\sum_{i=1}^{n-1} \frac{1}{\rho_i}}{2}$, the space $H^s_{\rho'}$ is an algebra, cf. the inhomogeneous Schauder estimates in Garello [7, Proposition 2.5]. Moreover we define the anisotropic characteristic manifold

$$\Sigma := \{(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus 0) : \xi_n^m - \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(x)\xi'^{\alpha'} = 0\}. \tag{1.5}$$

We may regard the next results as an extension of De Donno-Oliaro [5] in which local solvability and hypoellipticity are proved in the case of 2-variables equations with analytic nonlinearity. Let us state the main results, starting from C^∞ frame.

THEOREM 1.1. – *Let us fix k^* in* (1.1), *$m - \frac{1}{2} < k^* < m$, in such a way that there exists at least one n-uple $(\beta'^*, j^*) \in \mathbb{Z}^n_+$ such that $\left|\frac{\beta'^*}{\rho'}\right| + j^* = k^*$. We suppose $a_{\beta' j}(x)$, $b_{\alpha'}(x) \in C^\infty(\Omega)$, and assume that for $(x,\xi) \in \Sigma$ the following conditions hold:*

i) $\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^* j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*} \neq 0$ *for* $\xi' \neq 0,\ \xi_n \neq 0$;

ii) $\Im \sum_{k^*<\left|\frac{\beta'}{\rho'}\right|+j<m} a_{\beta' j}(x)\xi'^{\beta'}\xi_n^{j} \cdot \Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^* j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*} \geq 0$;

iii) $\Im \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(x)\xi'^{\alpha'} \cdot \Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^* j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*} \leq 0.$

Assume moreover that (1.3) *holds and the nonlinear function $G \in C^\infty$ satisfies* (1.4). *Then the operator P in* (1.2) *is C^∞-hypoelliptic and hypoelliptic* *with gain of k^* derivatives, i.e.: if $f \in H^s_{\rho'}$ then all the solutions u of $Pu = f$, belong to $H^{s+k^*}_{\rho'}$.*

Moreover (1.1) *is C^∞ solvable in $\Omega = \{|x| < \delta\}$ for δ sufficiently small. More precisely, if $f \in H^s_{\rho'}$ with compact support in Ω for $s > \frac{1}{2}\left(1 + \sum_{i=1}^{n-1} \frac{1}{\rho_i}\right)$, then the equation* (1.1) *admits a local solution $u \in H^{s+k^*}_{\rho'}$.*

As examples of operators satisfying Theorem 1.1 we consider in $\mathbb{R}^3$:

$$D^{2bp+1}_{x_3} - (1 - i|x|^{2l})(D^{2a}_{x_1} + D^{2b}_{x_2})^p + i(D^{2a}_{x_1} + D^{2b}_{x_2})^{p-1} D^{2b}_{x_3}, \tag{1.6}$$

where $p, a, b \in \mathbb{N}$, $p \geq 3$, $1 \leq a \leq b$; we have $\rho_1 = \frac{2ap}{2bp+1}$, $\rho_2 = \frac{2bp}{2bp+1}$, $k^* = 2bp + 1 - \frac{1}{p}$, $(\beta'^*, j^*) = (2a(p-1-k), 2bk, 2b)$ for $k = 0, \ldots, p-1$. As example of operator of even order

$$D^{2bp}_{x_3} - (1 - i|x|^{2l})(D^{2a}_{x_1} + D^{2b}_{x_2})^{p-1} + i(D^{2a}_{x_1} + D^{2b}_{x_2})^{p-2} D^{2b}_{x_3}, \tag{1.7}$$

where $p, a, b \in \mathbb{N}$, $p \geq 4b + 2$, $1 \leq a \leq b$; we have $\rho_1 = \frac{a(p-1)}{bp}$, $\rho_2 = \frac{p-1}{p}$, $k^* = 2bp - \frac{2b}{p-1}$, $(\beta'^*, j^*) = (2a(p-2-k), 2bk, 2b)$ for $k = 0, \ldots, p-2$. We may add in (1.6)-(1.7) arbitrary nonlinear C^∞ perturbation of lower anisotropic order satisfying the hypotheses of Theorem 1.1, and we obtain that (1.6)-(1.7) are C^∞ locally solvable and C^∞ hypoelliptic.

We want to study now the case when the hypothesis *i)* in Theorem 1.1 is not satisfied: the basic idea is to refer to the Gevrey classes and transform the operator P in (1.2) into another operator that satisfies it. To this aim, we introduce the Gevrey-Sobolev anisotropic spaces $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta))$, defined as the set of all L^2 functions for which

$$\|f\|_{\mathbb{H}^{s,\psi}_{\tau,\sigma',r}} := \|e^{\tau\psi(x_n, D')} f\|_{H^s_{\rho'}} < +\infty, \tag{1.8}$$

where $\sigma' := \left(\frac{1}{\rho_1}, \ldots, \frac{1}{\rho_{n-1}}\right)$ is the Gevrey order, $s > 0$ the Sobolev index, and we take $r \in (0,1)$, $\tau > 0$; $\psi = \psi(x_n, \xi')$ is a weight function of order (r, ρ'), i.e.: a non-negative function satisfying the following estimates:

$$\sup_{x_n \in (-\delta, \delta)} |D^j_{x_n} D^{\beta'}_{\xi'} \psi(x_n, \xi')| \leq C_{j\beta'} \langle \xi' \rangle_{\rho'}^{r - \left|\frac{\beta'}{\rho'}\right|}, \tag{1.9}$$

where $\langle \xi' \rangle_{\rho'} := \sum_{i=1}^{n-1} (1 + |\xi_i|^{\rho_i})$. The "Gevrey-Sobolev" spaces $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta))$ have been introduced in the isotropic case in Gramchev-Rodino [12], and they are studied, in this anisotropic form, in Marcolongo-Oliaro [23]. We refer to the quoted papers for a detailed exposition.

As standard, the Gevrey anisotropic space $G^{\lambda}(\Omega)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, is defined by the estimates:

$$\sup_K |\partial_x^\gamma f(x)| \le C_K^{|\gamma|+1}(\gamma_1!)^{\lambda_1} \cdots (\gamma_n!)^{\lambda_n}, \text{ for every } K \subset\subset \Omega, \tag{1.10}$$

where $\lambda_i \ge 1$ for $i = 1, \ldots, n$. Let us observe that $G^\lambda \subset \mathrm{H}^{s,\psi}_{\tau,\sigma',r}$ for $\lambda_i < \frac{1}{\rho_i r}$, $i = 1, \ldots, n-1$.

REMARK 1.2. – *Taking analytic coefficients in Theorem* 1.1 *we obtain* G^λ-*hypoellipticity of the operator* P *in* (1.2) *for* $\lambda_i \ge \frac{1/\rho_i}{k^* - (m-1)}$, *cf. De Donno-Rodino* [6] *regarding the isotropic case.*

Before stating the result of solvability in Gevrey classes we observe that for a suitable $\varepsilon_{k^*} > 0$ we can write G in (1.1) as $G(x', x_n; \partial_{x'}^{\gamma'} \partial_{x_n}^j u)|_{\left|\frac{\gamma'}{\rho'}\right| + j \le k^* - \varepsilon_{k^*}}$. About estimates regarding the composition in Gevrey classes see for example Bourdaud-Reissig-Sickel [1].

THEOREM 1.3. – *Let us fix* $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n)$ *with* $1 < \lambda_i < \frac{1}{\rho_i r}$ *for* $i = 1, \ldots, n-1$, $\lambda_n > 1$, $r \in (\frac{1}{2}, 1)$. *In the equation* (1.1), (1.2) *let the datum* f *and the coefficients of* P *be in* $G_0^\lambda(\Omega)$, *and* $k^* - \varepsilon_{k^*} < m - 1 + r$. *Assume that for* $(x, \xi) \in \Sigma$, *beside* (1.3), (1.4), *the following conditions hold:*

– $\Im \sum_{k^* \le \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^{j+m-1} \ge 0 \; (\le 0)$;

– $\Im \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x) \xi'^{\alpha'} \xi_n^{m-1} \le 0 \; (\ge 0)$.

We rewrite $G(x', x_n; \partial_{x'}^{\gamma'} \partial_{x_n}^j u)$ *as* $F(x', x_n; \Re(\partial_{x'}^{\gamma'} \partial_{x_n}^j u), \Im(\partial_{x'}^{\gamma'} \partial_{x_n}^j u))$, *and assume that* F *satisfies the following conditions:*

– $F(x; y_0) \in G^\lambda(\Omega)$ *for every* $y_0 \in \mathbb{R}^N$;

– $F(x_0; y) \in \bigcup_{r' < \frac{1}{r\rho_{\max}}} G^{r'}(\mathbb{R}^N)$, *for every* $x_0 \in \Omega$, *where* $\rho_{\max} := \max_{i=1,\ldots,n-1} \rho_i$.

Then the semilinear equation (1.1) *admits a classical solution in* Ω. *More precisely, for a suitably fixed* ψ *and taking* s *large, if* $f \in \mathrm{H}^{s,\psi}_{\tau,\sigma',r}$, $\sigma' = \left(\frac{1}{\rho_1}, \ldots, \frac{1}{\rho_{n-1}}\right)$, *is compactly supported in* Ω, *we obtain a local solution* $u \in \mathrm{H}^{s+m-(1-r),\psi}_{\tau,\sigma',r}$.

Multiplying $(D_{x_1}^{2a} + D_{x_2}^{2b})^{p-1} D_{x_3}^{2b}$ and $(D_{x_1}^{2a} + D_{x_2}^{2b})^{p-2} D_{x_3}^{2b}$ by $|x|^{2h}$ in (1.6) and (1.7) respectively (and possibly adding a Gevrey nonlinear perturbation involving lower order derivatives and satisfying the conditions requested in Theorem 1.3) we obtain examples of operators satisfying Theorem 1.3: more precisely, easy computation show that (1.6) is $G^{(\lambda_1,\lambda_2\lambda_3)}$ locally solvable for $\lambda_1 < \frac{(2bp+1)m_{ab}}{2a(p-1)m_{ab}-a}$, $\lambda_2 < \frac{(2bp+1)m_{ab}}{2b(p-1)m_{ab}-b}$, $\lambda_3 > 1$, where m_{ab} is the lowest common multiple between a and b; the operator (1.7) is $G^{(\lambda_1,\lambda_2\lambda_3)}$ locally solvable for $\lambda_1 < \frac{bp}{ra(p-1)}$, $\lambda_2 < \frac{p}{r(p-1)}$, $\lambda_3 > 1$, where

$$r = \begin{cases} 1/2 & \text{if } a = 1,\ p = 4b + 2 \\ \dfrac{ap - a - 2ab - 1}{a(p-1)} & \text{otherwise} \end{cases}$$

in the next section 2 we give a preliminary result on $S^m_{\rho,\delta}$ estimates; Theorem 1.1 is proved in Section 2, Theorem 1.3 in Section 3.

2. – Hypoellipticity for a class of differential polynomials.

In this Section we begin to prove $S^m_{\rho,\delta}$ estimates study for a pseudo-differential model in n variables, $n \geq 3$ (for related results in the case $n = 2$ see De Donno-Oliaro [5]). We recall that an operator P is said to be hypoelliptic at (a neighorhood Ω of) a point x_0 when $sing\,supp\,Pu = sing\,supp\,u$ for all $u \in \mathcal{E}'(\Omega)$. We take $m \in \mathbb{Z}_+$, $m \geq 4$ and the anisotropic weight $\rho = (\rho_1, ..., \rho_{n-1}, 1)$, $0 < \rho_i \leq 1$, $i = 1, ..., n-1$. Let the function in $\Omega \times \mathbb{R}^n$

$$(2.1) \qquad p(x,\xi) = \xi_n^m - \sum_{\left|\frac{a'}{\rho'}\right| = m} b_{a'}(x)\xi'^{a'} + \sum_{k^* \leq \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x)\xi'^{\beta'}\xi_n^j + \sigma(x,\xi)\ ,$$

be the symbol of the pseudo-differential operator P, where $(\xi', \xi_n) = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$, $b_{a'}, a_{\beta' j} : \Omega \to \mathbb{C}$, are in $C^\infty(\Omega)$, $a' = (a'_1, ..., a'_{n-1}) \in \mathbb{Z}_+^{n-1}$, $\beta' = (\beta'_1, ..., \beta'_{n-1}) \in \mathbb{Z}_+^{n-1}$; $j \in \mathbb{Z}_+$. We define the following sets for $k \in \mathbb{Q}_+$, $0 < k < m$:

$$I_k = \left\{ (\beta', j) \in \mathbb{Z}_+^n : \left|\frac{\beta'}{\rho'}\right| + j = k \right\}$$

and let k^* such that $(m - \frac{1}{2}) < k^* < m$. We use the notation k^- for all $k < k^*$ and k^+ for all $k > k^*$.

The symbol $\sigma(x,\xi)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is such that

$$|D_x^a D_\xi^\beta \sigma(x,\xi)| \leq C_{a,\beta} \langle\xi\rangle_\rho^{\overline{k} - \frac{\beta_1}{\rho_1} - \dots - \frac{\beta_{n-1}}{\rho_{n-1}} - \beta_n},$$

where $\overline{k} < k^*$; $\langle\xi\rangle_\rho = \sum_{i=1}^{n} (1 + |\xi_i|^{\rho_i})$ is the anisotropic norm. Let Λ be a neighborhood of the anisotropic characteristic manifold Σ, (see 1.5), and let Γ the set $\Omega \times \Lambda$, we state the following:

THEOREM 2.1. – *Assume I_{k^*} is not empty, and moreover for $(x,\xi)\in\Gamma$:*

$$i)\ \Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^*j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*} \neq 0\,,\ \xi'\neq 0\,,\ \xi_n\neq 0$$

$$ii)\ \Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^*j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*}\,\Im \sum_{\left|\frac{\beta'}{\rho'}\right|+j=k^+} a_{\beta'j}(x)\xi'^{\beta'}\xi_n^{j} \geq 0$$

(2.2) for every k_+

$$iii)\ \Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^*j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*}\,\Im \sum_{\left|\frac{a'}{\rho'}\right|=m} b_{a'}(x)\xi'^{a'} \leq 0\,,$$

$$iv)\ \Re \sum_{\left|\frac{a'}{\rho'}\right|=m} b_{a'}(x)\xi'^{a'} \neq 0\,,\ \xi'\neq 0$$

Then for all $a,\beta\in\mathbb{Z}^n_+$, for all $K\subset\subset\Omega$ we have positive constants $L_{a,\beta}$ and B such that:

$$\frac{|D_x^a D_\xi^\beta p(x,\xi)|\,\langle\xi\rangle_\rho^{\mu\sum_{i=1}^n\beta_i\frac{1}{\rho_i}-\delta\sum_{i=1}^n a_i\frac{1}{\rho_i}}}{|p(x,\xi)|} \leq L_{a,\beta}\ ,x\in K,|\xi|>B,\xi\in\mathbb{R}^n, \tag{2.3}$$

with $\mu = k^*-(m-1)$, $\delta = m-k^*$. Observe that $\delta<\mu$ since we have assumed $k^* > (m-\frac{1}{2})$

PROOF OF THEOREM 2.1. – We limit ourselves for simplicity to prove the estimate (2.3) for $|a|+|\beta| = a_1+\dots+a_n+\beta_1+\dots+\beta_n = 1$. The case $|a|+|\beta|>1$ does not involve actual complications; cf. Wakabayashi ([32], Theorem 2.6), Kajitani-Wakabayashi ([17], Theorem 1.9) for the analytic frame.

We estimate first the numerator of (2.3) and then we give some lemmas to estimate the denominator.

If $|a|=1$, we get

$$|D_{x_l}p(x,\xi)|\,\langle\xi\rangle_{\rho'}^{-\delta\frac{1}{\rho_l}} \leq L_1\Big(\big(\langle\xi'\rangle_{\rho'}^m+\langle\xi'\rangle_\rho^{\overline{m}}|\xi_n|^j\big)\langle\xi\rangle_\rho^{-\delta\frac{1}{\rho_l}}+\langle\xi\rangle_\rho^{\overline{k}-\delta\frac{1}{\rho_l}}\Big)$$

where $l=1,\dots,n-1$, $\overline{m}<m-j$ and $\overline{k}<k^*$; moreover

$$|D_{x_n}p(x,\xi)|\,\langle\xi\rangle_\rho^{-\delta} \leq L_2\Big(\big(\langle\xi'\rangle_{\rho'}^m+\langle\xi'\rangle_{\rho'}^{\overline{m}}|\xi_n|^j\big)\langle\xi\rangle_\rho^{-\delta}+\langle\xi\rangle_\rho^{\overline{k}-\delta}\Big)$$

for suitable constants $L_1,\ L_2$.

If $|\beta|=1$,

$$|D_{\xi_l}p(x,\xi)|\langle\xi\rangle_\rho^{\mu\frac{1}{\rho_l}} \leq L_3\bigg(\Big(\langle\xi'\rangle_{\rho'}^{m-\frac{1}{\rho_l}}+\langle\xi'\rangle_{\rho'}^{\overline{m}-\frac{1}{\rho_l}}|\xi_n|^j\Big)\langle\xi\rangle_\rho^{\mu\frac{1}{\rho_l}}+\langle\xi\rangle_\rho^{\overline{k}-\frac{1}{\rho_l}(1-\mu)}\bigg),$$

$$|D_{\xi_n}p(x,\xi)|\,\langle\xi\rangle_\rho^\mu \leq L_4\Big(\big(\langle\xi'\rangle_{\rho'}^{\overline{m}}|\xi_n|^{j-1}+|\xi_n|^{m-1}\big)\langle\xi\rangle_\rho^\mu+\langle\xi\rangle_\rho^{\overline{k}-1+\mu}\Big)$$

with suitable constants $L_3,\ L_4$.

Therefore, we observe that $\overline{k}-(1-\mu) \geq \overline{k}-\frac{1}{\rho_l}(1-\mu),\ l=1,...,n-1$ and $\overline{k}-(1-\mu)=\overline{k}-\delta>\overline{k}-\frac{1}{\rho_l}\delta$ since $\rho+\delta=1$. To prove (2.3), it will be then sufficient to show the boundedness in $\mathbb{R}^n$, for $|\xi|>B$, of the functions

$$Q_1(\xi) = \frac{\left(\langle\xi'\rangle_{\rho'}^{m}+\langle\xi'\rangle_{\rho'}^{\overline{m}}|\xi_n|^j\right)\langle\xi\rangle_\rho^{-\delta}}{|p(x,\xi)|},$$

$$Q_2(\xi) = \frac{\left(\langle\xi'\rangle_{\rho'}^{\overline{m}}|\xi_n|^{j-1}+|\xi_n|^{m-1}\right)\langle\xi\rangle_\rho^{\mu}}{|p(x,\xi)|},$$

$$Q_3(\xi) = \frac{\left(\langle\xi'\rangle_{\rho'}^{\overline{m}-\frac{1}{\rho_l}}|\xi_n|^j+\langle\xi'\rangle_{\rho'}^{m-\frac{1}{\rho_l}}\right)\langle\xi\rangle_\rho^{\mu\frac{1}{\rho_l}}}{|p(x,\xi)|},$$

$$Q_4(\xi) = \frac{\langle\xi\rangle_\rho^{\overline{k}-1+\mu}}{|p(x,\xi)|}.$$

First we introduce three regions:

$$\begin{aligned} R_1 &: \quad c\,\langle\xi'\rangle_{\rho'} \leq |\xi_n| \leq C\,\langle\xi'\rangle_{\rho'} \\ R_2 &: \quad |\xi_n| > C\,\langle\xi'\rangle_{\rho'} \\ R_3 &: \quad |\xi_n| < c\,\langle\xi'\rangle_{\rho'} \end{aligned}, \tag{2.4}$$

for suitable constants c, C to be determined precisely later, satisfying $0<c<<\min_{x\in K} G(x)$, $G(x)=\min\{|b_{\alpha'}(x)|\}_{\left|\frac{\alpha'}{\rho'}\right|=m}$, and $C>>\max_{x\in K} F(x)$, $F(x)=\max\{|b_{\alpha'}(x)|\}_{\left|\frac{\alpha'}{\rho'}\right|=m}$. We understand the neighborhood Λ to be the region R_1.

The following inequalities then hold:

$$\langle\xi\rangle_\rho^{-\delta} \leq \begin{cases} C^\delta\,|\xi_n|^{-\delta}, & \xi\in R_1 \quad (I) \\ |\xi_n|^{-\delta}, & \xi\in R_2 \quad (II) \\ \langle\xi'\rangle_{\rho'}^{-\delta}, & \xi\in R_3\,; \quad (III) \end{cases} \tag{2.5}$$

and,

$$\langle\xi\rangle_\rho^{\mu} \leq \begin{cases} C_1\,|\xi_n|^{\mu}, & \xi\in R_1 \\ C_2\,|\xi_n|^{\mu}, & \xi\in R_2 \\ C_3\,\langle\xi'\rangle_{\rho'}^{\mu}, & \xi\in R_3\,; \end{cases}$$

note that (II) and (III) in (2.5) hold for all $\xi\in\mathbb{R}^n$, but for our aim we may limit ourselves to consider them respectively in R_2 and in R_3. By abuse of notation, in the following we shall also denote by R_1, R_2, R_3 the sets $\Omega\times R_1, \Omega\times R_2, \Omega\times R_3$; recall that $\Gamma=\Omega\times\Lambda$.

LEMMA 2.2. – *Let* $p(x,\xi)$ *be the function* (2.1) *and assume that i), ii), iii) in* (2.2) *hold. Then there are positive constants* $K_1 < 1$, B, *such that:*

$$(2.6)\quad |p(x,\xi)| \geq K_1 \left| \Im \sum_{\left|\frac{\beta'^*}{\rho'}\right| + j^* = k^*} a_{\beta'^* j^*}(x) \xi'^{\beta'^*} \xi_n^{j^*} \right| \quad (x,\xi) \in \Gamma \cap R_1,\ |\xi| > B.$$

PROOF. – We have that

$$(2.7)\quad |p(x,\xi)|^2$$

$$= \left(\xi_n^m - \Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x)\xi'^{\alpha'} + \Re \sum_{k^* \leq \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^j + \Re\sigma(x,\xi) \right)^2$$

$$+ \left(\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right| + j^* = k^*} a_{\beta'^* j^*}(x) \xi'^{\beta'^*} \xi_n^{j^*} \right.$$

$$\left. + \Im \sum_{k^* < \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^j - \Im \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x) \xi'^{\alpha'} + \Im\sigma(x,\xi) \right)^2 .$$

By removing the terms rising from the real part of $p(x,\xi)$, we can write

$$|p(x,\xi)|^2 \geq \left(\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right| + j^* = k^*} a_{\beta'^* j^*}(x) \xi'^{\beta'^*} \xi_n^{j^*} \right)^2 + \sum_{i=1}^{4} W_i(x,\xi)$$

where

$$(2.8)\quad W_1(x,\xi) = \left(\Im \sum_{k^* < \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^j - \Im \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x) \xi'^{\alpha'} + \Im\sigma(x,\xi) \right)^2,$$

$$(2.9)\quad W_2(x,\xi) = 2\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right| + j^* = k^*} a_{\beta'^* j^*}(x) \xi'^{\beta'^*} \xi_n^{j^*} \Im \sum_{k^* < \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^j ,$$

$$(2.10)\quad W_3(x,\xi) = -2\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right| + j^* = k^*} a_{\beta'^* j^*}(x) \xi'^{\beta'^*} \xi_n^{j^*} \Im \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x) \xi'^{\alpha'} ,$$

$$(2.11)\quad W_4(x,\xi) = 2\Im\sigma(x,\xi) \Im \sum_{\left|\frac{\beta'^*}{\rho'}\right| + j^* = k^*} a_{\beta'^* j^*}(x) \xi'^{\beta'^*} \xi_n^{j^*} .$$

The function (2.8) is non-negative, (2.9) and (2.10) are also non-negative by hypotheses (ii), (iii) for all $(x,\xi)\in R_1$.

Concerning (2.11), it holds

$$\left(\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^* j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*}\right)^2 + W_4(x,\xi) \geq (1-\varepsilon)\left(\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^* j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*}\right)^2 .$$

In fact, for $|\xi|$ sufficiently large

$$\begin{aligned}\frac{|W_4(x,\xi)|}{\left(\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^* j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*}\right)^2} &\leq \text{const}\ \frac{\langle\xi'\rangle_{\rho'}^{k^*-j^*}|\xi_n|^{j^*}\langle\xi'\rangle_{\rho'}^{\overline{k}}}{\xi_n^{2j^*}\langle\xi'\rangle_{\rho'}^{2k^*-2j^*}}\\ &\leq \text{const}\ \frac{|\xi_n|^{k^*-\overline{k}}}{\xi_n^{2k^*}} < \varepsilon, \quad |\xi|>B;\end{aligned}$$

since $\overline{k}<k^*$.

Then

$$|p(x,\xi)| \geq K_1\left|\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right|+j^*=k^*} a_{\beta'^* j^*}(x)\xi'^{\beta'^*}\xi_n^{j^*}\right|, \quad (x,\xi)\in R_1,\ |\xi|>B,$$

for a suitable positive constant K_1. ■

LEMMA 2.3. – *Let $p(x,\xi)$ be the function* (2.1). *Then there are positive constants $K_2<1$, B, such that:*

$$|p(x,\xi)| \geq K_2\,|\xi_n|^m, \quad (x,\xi)\in R_2,\ |\xi|>B. \tag{2.12}$$

PROOF. – We write $|p(x,\xi)|^2$ as in (2.7); by removing the terms arising from the imaginary part of $p(x,\xi)$, we get

$$|p(x,\xi)|^2 \geq \left(\xi_n^m - \Re \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(x)\xi'^{\alpha'}\right)^2 + W_1(x,\xi)+W_2(x,\xi)+W_3(x,\xi) \tag{2.13}$$

where

$$W_1(x,\xi) = \left(\Re \sum_{k^*\leq\left|\frac{\beta'}{\rho'}\right|+j<m} a_{\beta' j}(x)\xi'^{\beta'}\xi_n^j + \Re\sigma(x,\xi)\right)^2, \tag{2.14}$$

$$(2.15)\qquad W_2 = 2\Re \sum_{k^* \le \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^{j+m} - 2\Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x) \xi'^{\alpha'} \Re \sum_{k^* \le \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^{j}$$

$$(2.16)\qquad W_3 = 2\xi_n^m \Re\sigma(x,\xi) - 2\Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x)\xi'^{\alpha'} \Re\sigma(x,\xi).$$

Observe first that for $\lambda > 0$ sufficiently small

$$\left(\xi_n^m - \Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x)\xi'^{\alpha'}\right)^2 > \lambda\, \xi_n^{2m};$$

In fact,

$$\left(\xi_n^m - \Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x)\xi'^{\alpha'}\right)^2 \ge \xi_n^{2m} - 2\xi_n^m \Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x)\xi'^{\alpha'},$$

and using (2.4) in R_2, we have

$$\frac{|\xi_n^m \Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x)\xi'^{\alpha'}|}{\xi_n^{2m}} \le \text{const}\, \frac{\max_{x\in K} F(x) \langle \xi' \rangle_{\rho'}^m}{|\xi_n^m|} \le \text{const}\, \frac{\max_{x\in K} F(x)}{C^m}$$

and

$$\xi_n^{2m} - 2\xi_n^m \Re \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x)\xi'^{\alpha'} \ge \left(1 - \text{const}\, \frac{\max_{x\in K} F(x)}{C^m}\right) \xi_n^{2m} > \lambda\, \xi_n^{2m},$$

since $C >> \max_{x\in K} F(x)$.

(2.14) is non-negative. We denote (2.15) by $\Upsilon_1(x,\xi) - \Upsilon_2(x,\xi)$ and (2.16) by $\Upsilon_3(x,\xi) - \Upsilon_4(x,\xi)$. Then

$$|p(x,\xi)|^2 \ge \lambda \xi_n^{2m} + \Upsilon_1(x,\xi) - \Upsilon_2(x,\xi) + \Upsilon_3(x,\xi) - \Upsilon_4(x,\xi).$$

Arguing on $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ in the same way as we have done in Lemma 2.2, it is possible to show that for all $\varepsilon > 0$

$$\frac{\lambda}{2}\xi_n^{2m} + \Upsilon_1(x,\xi) - \Upsilon_2(x,\xi) \ge \frac{(\lambda - \varepsilon)}{2}\xi_n^{2m}, \quad (x,\xi) \in R_2,\ |\xi| > B,$$

and

$$\frac{\lambda}{2}\xi_n^{2m} + \Upsilon_3(x,\xi) - \Upsilon_4(x,\xi) \geq \frac{(\lambda-\varepsilon)}{2}\xi_n^{2m}, \quad (x,\xi) \in R_2,\ |\xi| > B\,.$$

Thus

$$|p(x,\xi)| \geq K_2\,\xi_n^m\,, \quad (x,\xi) \in R_2,\ |\xi| > B\,.$$

■

LEMMA 2.4. – *Let* $p(x,\xi)$ *be the function* (2.1), *such that* iv) *in* (2.2) *holds. Then there are positive constants* $K_3 < 1$, B, *such that:*

$$|p(x,\xi)| \geq K_3\,\langle\xi'\rangle_{\rho'}^m\,, \quad (x,\xi) \in R_3,\ |\xi| > B. \tag{2.17}$$

PROOF. – We apply again (2.13), (2.14), (2.15), (2.16) to $|p(x,\xi)|^2$. Observe that in R_3, arguing as above, since $c << \min_{x\in K} G(x)$, we obtain for a suitable constant $\mu > 0$

$$\left(\xi_n^m - \Re \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(x)\xi'^{\alpha'}\right)^2 > \mu\,\langle\xi'\rangle_{\rho'}^{2m}\,.$$

About the terms in (2.14), (2.15) and (2.16), the remarks we have done in Lemma 2.3 hold by replacing $\lambda\,\xi_n^{2m}$ with $\mu\,\langle\xi'\rangle_{\rho'}^{2m}$. Then we have

$$|p(x,\xi)| \geq K_3\,\langle\xi'\rangle_{\rho'}^m\,, \quad (x,\xi) \in R_3,\ |\xi| > B.$$

■

REMARK 2.5. – *In the previous lemmas we have estimated the symbol function* $|p(x,\xi)|$ *in* (2.1), *separately in the three regions* (2.4). *It is possible to obtain the following global result on* $|p(x,\xi)|$: *there exists* $m' \in \mathbb{R}$, $a > 0$, $B > 0$, *such that*

$$|p(x,\xi)| \geq a\langle\xi\rangle_\rho^{m'}\ \text{, in } \Gamma \text{ for } |\xi| > B. \tag{2.18}$$

In fact, by remembering that under assumptions i), ii), iii), iv) *in theorem* 2.1 *the estimates* (2.6), (2.12), (2.17) *hold, we obtain that* $|p(x,\xi)| \geq \text{const}\,\xi_n^{k^*}$ *in* R_1 *and* R_2. *Since in these regions* $\xi_n^{k^*} = \frac{1}{2}\xi_n^{k^*} + \frac{1}{2}\xi_n^{k^*} \geq c\langle\xi'\rangle_{\rho'}^{k^*} + \frac{1}{2}\xi_n^{k^*} \sim \text{const}\,(\langle\xi'\rangle_{\rho'} + \xi_n)^{k^*} = \text{const}\,\langle\xi\rangle_\rho^{k^*}$ *so*

$$|p(x,\xi)| \geq a\,\langle\xi\rangle_\rho^{k^*}.$$

In the same way we get

$$|p(x,\xi)| \geq a\,\langle\xi\rangle_\rho^m$$

in R_3. *Because* $k^* < m$, *we have* $m' = k^*$ *in* (2.18).

We first consider $Q_1(\xi)$ separately in the regions R_1, R_2, R_3, to prove boundedness.

In R_1 by (2.5), (2.6) we get easily:

$$Q_1(\xi) \le \text{const} \frac{|\xi_n|^{\overline{m}+j-\delta} + |\xi_n|^{m-\delta}}{|\xi_n|^{k^*}} \le L \ , |\xi| > B$$

since $\delta \ge m - k^*$. We recall that $\overline{m} + j < m$.

In the region R_2 we have that $|p(x,\xi)| \ge |\xi_n|^m > |\xi_n|^{k^*}$. In R_3, by using (2.5), (2.17), we have for a constant $\varepsilon > 0$ which we may take as small as we want by fixing B sufficiently large:

$$Q_1(\xi) \le \text{const} \frac{\langle\xi'\rangle_{\rho'}^{\overline{m}+j-\delta} + \langle\xi'\rangle_{\rho'}^{m-\delta}}{\langle\xi'\rangle_{\rho'}^{m}} < \varepsilon \ , |\xi| > B$$

We have therefore proved that $Q_1(\xi)$ is bounded. Let us estimate $Q_2(\xi)$ and $Q_3(\xi)$. In the region R_2 we argue as before; in the regions R_1, R_3 we obtain respectively

$$Q_2(\xi) \le \text{const} \frac{|\xi_n|^{\overline{m}-\frac{1}{\rho_l}+j+\mu\frac{1}{\rho_l}} + |\xi_n|^{m-\frac{1}{\rho_l}+\mu\frac{1}{\rho_l}}}{|\xi_n|^{k^*}} < \varepsilon \ ,$$

in R_1 for $|\xi| > B$, since $\mu \le k^* - (m-1)$, $\overline{m} < m - j$,

$$Q_2(\xi) \le \text{const} \frac{\langle\xi'\rangle_{\rho'}^{\overline{m}-\frac{1}{\rho_l}+j+\mu\frac{1}{\rho_l}} + \langle\xi'\rangle_{\rho'}^{m-\frac{1}{\rho_l}+\mu\frac{1}{\rho_l}}}{\langle\xi'\rangle_{\rho'}^{k^*}} < \varepsilon \ ,$$

in R_3 for $|\xi| > B$.

For Q_3 we obtain that

$$Q_3(\xi) \le \text{const} \frac{|\xi_n|^{\overline{m}+j-1+\mu} + |\xi_n|^{m-1+\mu}}{|\xi_n|^{k^*}} \le L \ , |\xi| > B \tag{2.19}$$

since $\mu \le k^* - m + 1$ in R_1 and

$$Q_3(\xi) \le \text{const} \frac{\langle\xi'\rangle_{\rho'}^{\overline{m}+j-1+\mu} + \langle\xi'\rangle_{\rho'}^{m-1+\mu}}{\langle\xi'\rangle_{\rho'}^{m}} < \varepsilon \ , \tag{2.20}$$

in R_3 for $|\xi| > B$.

For Q_4, we get $\frac{|\xi_n|^{\overline{k}-(1-\mu)}}{|\xi_n|^{k^*}} < \varepsilon$ in R_1 since $\overline{k} < k^*$, $\mu < 1$. In R_3 $\frac{\langle\xi'\rangle_{\rho'}^{\overline{k}-(1-\mu)}}{\langle\xi'\rangle_{\rho'}^{k^*}} < \varepsilon$, $|\xi| > B$

Now Lemma 2.2, Lemma 2.3, Lemma 2.4 and the estimate (2.18) complete the proof. ∎

We shall use also the following variant of Theorem 2.3, where the role of I_{k^*}-terms is taken by the pseudo-differential term $\sigma(x,\xi)$. Namely, we fix now t with

$0 < t < \frac{1}{2}$ and assume that $|\Im\sigma(x,\xi)| \geq \langle\xi\rangle_\rho^{\overline{k}}$ where $m - \frac{1}{2} < \overline{k} < m - t$, considering a symbol of the form:

$$p(x,\xi) = \xi_n^m - \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(x)\xi'^{\alpha'} + \sum_{m-t\leq\left|\frac{\beta'}{\rho'}\right|+j<m} a_{\beta' j}(x)\xi'^{\beta'}\xi_n^j + \sigma(x,\xi)\,, \tag{2.21}$$

THEOREM 2.6. – *Let $p(x,\xi)$ be the function* (2.21) *such that for* $(x,\xi) \in \Gamma$

$$\begin{aligned} &i)\ |\Im\sigma(x,\xi)| \geq \langle\xi\rangle_\rho^{\overline{k}}\,,\\ &ii)\ \Im\sigma(x,\xi)\Im \sum_{\left|\frac{\beta'}{\rho'}\right|+j=k^+} a_{\beta' j}(x)\xi'^{\beta'}\xi_n^j \geq 0\,, \text{ for every } k_+ \geq m-t,\\ &iii)\ \Im\sigma(x,\xi)\Im \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(x)\xi'^{\alpha'} \leq 0\,,\\ &iv)\ \Re \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(x)\xi'^{\alpha'} \neq 0\,,\ \xi' \neq 0 \end{aligned} \tag{2.22}$$

Then for all $\alpha,\beta \in \mathbb{Z}_+^n$, for all $K \subset\subset \Omega$ we have for positive constants $L_{\alpha,\beta}$ and B that:

$$\frac{|D_x^\alpha D_\xi^\beta p(x,\xi)|\,\langle\xi\rangle_\rho^{\mu\sum_{i=1}^n \beta_i\frac{1}{\rho_i} - \delta\sum_{i=1}^n \alpha_i\frac{1}{\rho_i}}}{|p(x,\xi)|} \leq L_{\alpha,\beta}\,,\ x \in K\,, |\xi| > B,\ \xi \in \mathbb{R}^n, \tag{2.23}$$

with $\mu = \overline{k} - (m-1)$, $\delta = m - \overline{k}$. Observe that is $\delta < \mu$ since we have assumed $\overline{k} > (m - \frac{1}{2})$.

PROOF. – We have $\overline{k}$ in the role of k^* in the proof of Theorem 2.1, by observing that in R_1

$$\begin{aligned} |p(x,\xi)|^2 &\geq \left(\Im \sum_{m-t<\left|\frac{\beta'}{\rho'}\right|+j<m} a_{\beta' j}(x)\xi'^{\beta'}\xi_n^j - \Im \sum_{\left|\frac{\alpha'}{\rho'}\right|=m} b_{\alpha'}(x)\xi'^{\alpha'} + \Im\sigma(x,\xi)\right)^2\\ &\geq (\Im\sigma(x,\xi))^2 \geq \langle\xi\rangle_\rho^{2\overline{k}} \geq \xi_n^{2\overline{k}} \end{aligned} \tag{2.24}$$

since $ii), iii)$ and $i)$ hold; then arguing like in the proof of Theorem 2.1 we obtain our result. Of course the power m' in Remark 2.5 is given by $\overline{k}$:

$$|p(x,\xi)| \geq a\langle\xi\rangle_\rho^{\overline{k}}\,,\ |\xi| > B. \tag{2.25}$$

■

The Theorem 2.6 allows us to prove the C^∞-local solvability for semilinear equations. We need a definiton and some remarks:

DEFINITION 2.7. – *Let* $\rho := (\rho_1, \dots, \rho_{n-1}, 1)$, *with* $\rho_j \le 1$ *for every* $j = 1, \dots, n-1$. *We say that* $p(x, \xi) \in S^m_\rho(\Omega \times \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ *open, if* $p(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ *and for all multi-indices* $\alpha, \beta \in \mathbb{Z}^n_+$ *there exists a constant* $C_{\alpha\beta}$ *such that*

$$|D^\alpha_x D^\beta_\xi p(x, \xi)| \le C_{\alpha\beta} \langle \xi \rangle_\rho^{m - \frac{\beta_1}{\rho_1} - \dots - \frac{\beta_{n-1}}{\rho_{n-1}} - \beta_n},$$

for $(x, \xi) \in \Omega \times \mathbb{R}^n$.

As usual we can also consider the microlocal class $S^m_\rho(\Gamma)$, *where* $\Gamma \ni (x, \xi)$ *is conical with respect to the* ξ*-variables.*

The class of the pseudo-differential operators with symbol in $S^m_\rho(\Omega \times \mathbb{R}^n)$ has been studied in Hunt-Piriou [16], [15]: the basic results that hold in the standard isotropic class regarding composition, adjoint, change of variables can be adapted for the symbols in $S^m_\rho(\Omega \times \mathbb{R}^n)$, cf. also [23, Proposition 3.1]. In particular the estimates (2.3), (2.18) imply the existence of a microlocal parametrix, localized in a neighbourhood of the anisotropic characteristic manifold. Then by a construction as in [12] we get a global parametrix for P in the spaces $H^s_{\rho'}$. This fact, combined with the results in [7], [8], [9] and [10], yields the solvability part of Theorem 1.1.

3. – Local solvability in Gevrey classes.

Till now we have studied equations of the form (1.1)-(1.2) in which we have supposed that, for a certain k^*,

$$\Im \sum_{\left|\frac{\beta'^*}{\rho'}\right| + j^* = k^*} a_{\beta'^* j^*}(x) \xi'^{\beta'^*} \xi_n^{j^*} \ne 0. \tag{3.1}$$

For purposes below we rewrite the equation (1.1) in the form

$$P(x', x_n, D_{x'}, D_{x_n})u + J(u) = \mu f(x), \tag{3.2}$$

where $P(x', x_n, D_{x'}, D_{x_n})$ is as in (1.2) and

$$J(u) := F(x', x_n; \Re(\partial^{\gamma'}_{x'} \partial^j_{x_n} u), \Im(\partial^{\gamma'}_{x'} \partial^j_{x_n} u))|_{\left|\frac{\gamma'}{\rho'}\right| + j < k^*}. \tag{3.3}$$

In this section we deal with the local solvability of the equation (3.2) in the case in which the hypothesis (3.1) is not satisfied, and prove Theorem 1.3. The basic idea is to transform the operator $P(x, D)$ into another operator $\widetilde{P}(x, D)$ (see Section 3.2 below) that satisfies (3.1); by applying Theorem 2.1 to $\widetilde{P}(x, D)$ we shall find a parametrix $\widetilde{E}$ for $\widetilde{P}(x, D)$ in the Sobolev anisotropic spaces. Starting from $\widetilde{E}$ we shall finally construct a parametrix E for $P(x, D)$ in a suitable functional setting that we are now going to describe.

3.1 – *The functional setting.*

We shall use the notation $(\mathcal{F}_{x'\to\xi'}f)(x_n,\xi')$ for the partial Fourier transform of $f(x)$ with respect to x', i.e. $(\mathcal{F}_{x'\to\xi'}f)(x_n,\xi') := \int e^{-ix'\xi'}f(x)\,dx'$. In this section we limit ourselves to the analysis of the algebra property of $\mathrm{H}^{s,\psi}_{\tau,\sigma',r}$ introduced in section 1 formula (1.8), since the result that we need in this frame in order to treat the Gevrey nonlinearity (cf. Section 3.3 below) is more general than the one proved in [23].

THEOREM 3.1. – *Let the weight function $\psi(x_n,\xi')$ in (1.8) of order (r,ρ') satisfy the following condition:*

$$\psi(x_n,\xi') - \psi(x_n,\xi'-\eta') - \psi(x_n,\eta') \le -b\ \min\{\langle\xi'-\eta'\rangle_{\rho'}, \langle\eta'\rangle_{\rho'}\}^r, \tag{3.4}$$

with $b>0$ independent of $\xi',\eta'\in\mathbb{R}^{n-1}$ and $x_n\in(-\delta,\delta)$. Assume ρ_j is rational and let s_0 be an integer such that $\rho_j s_0$ is an integer for all $j=1,\dots,n-1$. Then for every $s\ge s_0$ the space $\mathrm{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1}\times(-\delta,\delta))$ is an algebra and there exists a constant $C(s)$ of the form $C(s)=\overline{C}(s)\big(\int e^{-2\tau b\langle\xi'\rangle^r_{\rho'}}\,d\xi'\big)^{\frac12}$ such that

$$\|uv\|_{\mathrm{H}^{s,\psi}_{\tau,\sigma',r}} \le C(s)\|u\|_{\mathrm{H}^{s,\psi}_{\tau,\sigma',r}}\|v\|_{\mathrm{H}^{s,\psi}_{\tau,\sigma',r}}. \tag{3.5}$$

PROOF. – We prove this theorem for $s\ge s_0$ satisfying the same assumptions as s_0; the result will extend by interpolation to all $s\ge s_0$ in view of the fact that $\mathrm{H}^{s,\psi}_{\tau,\sigma',r}$ are isometric to $H^s_{\rho'}$. Let now $k(x_n,\xi') := e^{\tau\psi(x_n,\xi')}\langle\xi'\rangle^{s-a_n}_{\rho'}$. Under the assumptions that we have made on s we know that $\|uv\|_{\mathrm{H}^{s,\psi}_{\tau,\sigma',r}}$ is equivalent to $\sum_{\sum_{j=1}^n a_j\frac{1}{\rho_j}\le s}\|e^{\tau\psi(x_n,D')}D^a_x(uv)\|_{L^2}$, $\rho:=(\rho',1)$; now, writing $\tilde u(x_n,\xi')=\mathcal{F}_{x'\to\xi'}u(x_n,\xi')$ and $\tilde v(x_n,\xi')=\mathcal{F}_{x'\to\xi'}v(x_n,\xi')$ we get easily from Leibnitz rule and the basic properties of the Fourier trasform that, for $\sum_{j=1}^n a_j\frac{1}{\rho_j}\le s$:

$$\begin{aligned}
\|e^{\tau\psi(x_n,D')}D^a_x(uv)\|^2_{L^2} &\le C_1\sum_{j+h=a_n}\Bigg\{\int\Bigg|\int_{\langle\eta'-\xi'\rangle_{\rho'}\le\langle\eta'\rangle_{\rho'}}(D^j_{x_n}\tilde u)(x_n,\xi'-\eta')\\
&\qquad\cdot(D^h_{x_n}\tilde v)(x_n,\eta')\,d\eta'\Bigg|^2 k^2(x_n,\xi')\,d\xi'\,dx_n\\
&\quad+\int\Bigg|\int_{\langle\eta'-\xi'\rangle_{\rho'}>\langle\eta'\rangle_{\rho'}}(D^j_{x_n}\tilde u)(x_n,\xi'-\eta')(D^h_{x_n}\tilde v)(x_n,\eta')\,d\eta'\Bigg|^2 k^2(x_n,\xi')\,d\xi'\,dx_n\Bigg\}\\
&= C_1(s)\sum_{j+h=a_n}\big\{I^{(a)}_1+I^{(a)}_2\big\}.
\end{aligned}$$

Let us now estimate $I^{(a)}_1$; observe that, due to the inequality $\frac{\langle\xi'\rangle_{\rho'}}{\langle\xi'-\eta'\rangle_{\rho'}\langle\eta'\rangle_{\rho'}}\le 1$ and to (3.4) we get $\frac{k(x_n,\xi')}{k(x_n,\xi'-\eta')k(x_n,\eta')} \le e^{-\tau b \min\{\langle\xi'-\eta'\rangle_{\rho'},\langle\eta'\rangle_{\rho'}\}^r}$. So we obtain:

$$I_1^{(a)} \le \Big\| \int |(D^j_{x_n}\tilde{u})(x_n,\xi'-\eta')k(x_n,\xi'-\eta')e^{-\tau b\langle\xi'-\eta'\rangle^r_{\rho'}}| \cdot |(D^h_{x_n}\tilde{v})(x_n,\eta')k(x_n,\eta')|\, d\eta' \Big\|^2_{L^2(\mathbb{R}^{n-1}\times(-\delta,\delta))};$$

by Young and Hölder estimates it follows that

$$\begin{aligned} I_1^{(a)} &\le \overline{C}(s)\|k(x_n,\xi')(D^h_{x_n}\tilde{v})(x_n,\xi')\|^2_{L^2}\ \|k(x_n,\xi')(D^j_{x_n}\tilde{u})(x_n,\xi')e^{-\tau b\langle\xi'\rangle^r_{\rho'}}\|^2_{L^1} \\ &\le \overline{C}(s)\|v\|^2_{\mathbb{H}^{s,\psi}_{\tau,\sigma',r}}\ \|k(x_n,\xi')(D^j_{x_n}\tilde{u})(x_n,\xi')\|^2_{L^2}\int e^{-2\tau b\langle\xi'\rangle^r_{\rho'}}\, d\xi' \\ &\le C_2(s)^2\|u\|^2_{\mathbb{H}^{s,\psi}_{\tau,\sigma',r}}\ \|v\|^2_{\mathbb{H}^{s,\psi}_{\tau,\sigma',r}}, \end{aligned}$$

where $C_2(s) = C_3(s)\big(\int e^{-2\tau b\langle\xi'\rangle^r_{\rho'}}\, d\xi'\big)^{\frac{1}{2}}$ and the norms are in $\mathbb{R}^{n-1}\times(-\delta,\delta)$.

$I_2^{(a)}$ can be treated in the same way. ∎

3.2 – *Analysis of the "conjugate operator"*

Let us consider now the linear operator $P(x,D)$, cf. (1.2). Since we want to analyze the local solvability at the origin, it is not restrictive to suppose that the coefficients are in the space $G_0^{\sigma^{(r)}}(\mathbb{R}^{n-1}\times(-\delta,\delta))$, with $\sigma^{(r)} = (\sigma_1^{(r)},\dots,\sigma_n^{(r)})$, $1 < \sigma_j^{(r)} < \frac{1}{\rho_j r}$ for $j = 1,\dots,n-1$, $\sigma_n^{(r)} > 1$. Since $G_0^{\sigma^{(r)}}(\mathbb{R}^{n-1}\times(-\delta,\delta)) \subset \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1}\times(-\delta,\delta))$, cf. [23, Lemma 2.1], using Theorem 3.1 we have that $P(x,D) : \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1}\times(-\delta,\delta)) \to \mathbb{H}^{s-m,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1}\times(-\delta,\delta))$. Now let us consider the following operator (that we shall call the "conjugate operator"):

$$\widetilde{P}(x,D) := e^{\tau\psi(x_n,D')}P(x,D)e^{-\tau\psi(x_n,D')}. \tag{3.6}$$

It is easy to see that $P(x,D) = e^{-\tau\psi(x_n,D')}\widetilde{P}(x,D)e^{\tau\psi(x_n,D')}$; moreover the conjugate operator acts in the following way: $\widetilde{P}(x,D) : H^s_{\rho'}(\mathbb{R}^{n-1}\times(-\delta,\delta)) \to H^{s-m}_{\rho'}(\mathbb{R}^{n-1}\times(-\delta,\delta))$.

Our purpose is now to analyze $\widetilde{P}(x,D)$, in order to write it explicitly modulo a remainder of order sufficiently small. We first introduce the following class of symbols.

DEFINITION 3.2. – *Let us fix $\rho = (\rho',1) = (\rho_1,\dots,\rho_{n-1},1)$ with $\rho_j \le 1$ for every $j = 1,\dots,n-1$. For $\mu, m \in \mathbb{R}$ the symbol space $S^{\mu,m}_\rho(\Omega\times\mathbb{R}^n)$ is defined to be the set of all the functions $p(x,\xi) \in C^\infty(\Omega\times\mathbb{R}^n)$ such that*

$$|D^\alpha_x D^\beta_\xi p(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^\mu_\rho\langle\xi'\rangle_{\rho'}^{m-\frac{\beta_1}{\rho_1}-\dots-\frac{\beta_{n-1}}{\rho_{n-1}}-\beta_n}$$

for all multi-indices $\alpha,\beta \in \mathbb{Z}^n_+$.

Let us fix now the weight function $\psi(x_n, \xi')$ of order (r, ρ') in the following way:

$$\psi(x_n, \xi') = \Big(1 + \frac{x_n}{2\delta}\Big) \langle \xi' \rangle_{\rho'}^{r}. \tag{3.7}$$

The following result can be obtained by applying the general calculus in [23, Theorem 3.1].

PROPOSITION 3.3. – *Let $P(x, D)$ and the weight function $\psi(x_n, \xi')$ be given by* (1.2) *and* (3.7) *respectively; we suppose that the coefficients of $P(x, D)$ are in the class $G_0^{\sigma^{(r)}}(\Omega)$, $1 < \sigma_j^{(r)} < \frac{1}{\rho_j r}$ for every $j = 1, \dots, n-1$, $\sigma_n^{(r)} > 1$. Then the symbol $\tilde{p}(x, \xi)$ of the conjugate operator $\widetilde{P}(x, D)$ is given by*

$$\tilde{p}(x, \xi) = p(x, \xi) + p_{m,-(1-r)}(x, \xi) + p_{m,-(1-r)-\upsilon}(x, \xi), \tag{3.8}$$

where:

- *$p(x, \xi)$ is the symbol of $P(x, D)$;*
- *$p_{m,-(1-r)}(x, \xi) \in S_\rho^{m,-(1-r)}(\Omega \times \mathbb{R}^n)$; it is given by*

$$im \frac{\tau}{2\delta} \langle \xi' \rangle_{\rho'}^{r} \xi_n^{m-1} - \tau \Big(1 + \frac{x_n}{2\delta}\Big) \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} \sum_{l=1}^{n-1} D_{x_l} b_{\alpha'}(x) \partial_{\xi_l} \big(\langle \xi' \rangle_{\rho'}^{r}\big) (\xi')^{\alpha'};$$

- *$p_{m,-(1-r)-\upsilon}(x, \xi) \in S_\rho^{m,-(1-r)-\upsilon}(\Omega \times \mathbb{R}^n)$, $\upsilon > 0$.*

REMARK 3.4. – *If in the previous proposition the weight function $\psi(x_n, \xi')$ keeps the form*

$$\psi(x_n, \xi') = \Big(1 - \frac{x_n}{2\delta}\Big) \langle \xi' \rangle_{\rho'}^{r} \tag{3.9}$$

we obtain the same result as before, where in (3.8) *we have* $p_{m,-(1-r)}(x, \xi) = -im \frac{\tau}{2\delta} \langle \xi' \rangle_{\rho'}^{r} \xi_n^{m-1} - \tau \big(1 - \frac{x_n}{2\delta}\big) \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} \sum_{l=1}^{n-1} D_{x_l} b_{\alpha'}(x) \partial_{\xi_l} \big(\langle \xi' \rangle_{\rho'}^{r}\big) (\xi')^{\alpha'}$.

3.3 – *The nonlinearity: composition in Gevrey-Sobolev spaces.*

We want to analyze the case in which the nonlinearity $F(x; y)$ in (3.3) is Gevrey in all the variables: to this aim we are going to study in this section the composition $F(x; u(x))$, where $F(x; y)$ is supposed to be Gevrey both in x and y and $u \in \mathbb{H}_{\tau,\sigma',r}^{s,\psi}$. We shall follow in our discussion the lines of Bourdaud-Reissig-Sickel [1], where this kind of composition is discussed in the frame of spaces that are similar to $\mathbb{H}_{\tau,\sigma',r}^{s,\psi}$.

First of all let us introduce the set

$$P_R := \{\xi' \in \mathbb{R}^{n-1} : |\xi_j| \le R,\ j = 1, \dots, n-1\}$$

for $R \geq 0$; moreover let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$, $\varepsilon_j \in \{0, 1\}$ for $j = 1, \ldots, n-1$, and define

$$P_R(\varepsilon) := \{\xi' \in \mathbb{R}^{n-1} : \operatorname{sign} \xi_j = (-1)^{\varepsilon_j},\ j = 1, \ldots, n-1\} \setminus P_R.$$

We shall use in the following the notation

$$\rho_{\min} := \min_{j=1,\ldots,n-1} \rho_j, \quad \rho_{\max} := \max_{j=1,\ldots,n-1} \rho_j \tag{3.10}$$

REMARK 3.5. – *Let us suppose that all the hypotheses of Theorem* 3.1 *are satisfied; let in addition* $\operatorname{supp}(\mathcal{F}_{x' \to \xi'} u)(x_n, \xi') \subset P_R(\varepsilon)$ *and* $\operatorname{supp}(\mathcal{F}_{x' \to \xi'} v)(x_n, \xi') \subset P_R(\varepsilon)$ *for a certain* $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$ *and for all* $x_n \in (-\delta, \delta)$*; then we have that* $\operatorname{supp}(\mathcal{F}_{x' \to \xi'}(uv))(x_n, \xi') \subset P_R(\varepsilon)$*; moreover by the proof of Theorem* 3.1 *the constant* $C(s)$ *in* (3.5) *can be estimated by*

$$C(s) \leq \widetilde{C}(s) \Big(\int_{R_\rho}^{+\infty} e^{-2\tau b y^r} y^{k_\sigma - 1}\, dy \Big)^{\frac{1}{2}} := C_R(s), \tag{3.11}$$

where $k_\sigma := \sum_{j=1}^{n-1} \sigma_j$, $\sigma_j = \frac{1}{\rho_j}$ *and*

$$R_\rho := \begin{cases} R^{\rho_{\min}} & R \geq 1 \\ R & 0 \leq R < 1 \end{cases}. \tag{3.12}$$

Observe that we can choose $\widetilde{C}(s)$ *sufficiently large in such a way that* $C_0(s) > 1$*; moreover* $C_R(s) \to 0$ *for* $R \to +\infty$.

In the following part of this section we shall write for simplicity of notation $\| \cdot \|_s$ in way of $\| \cdot \|_{\mathrm{H}^{s,\psi}_{\tau,\sigma',r}}$.

PROPOSITION 3.6. – *Let us suppose that* $s > s_0$, *cf. Theorem* 3.1; *we take the weight function* $\psi(x_n, \xi')$ *satisfying* (3.4). *Then there exist c and a such that*

$$\|e^{iu(x)} - 1\|_s \leq \begin{cases} c\, e^{a\|u\|_s^{r\rho_{\max}} (\log \|u\|_s)^{\rho_{\max}/\rho_{\min}}} & \text{if } \|u\|_s > 1 \\ c\, \|u\|_s & \text{if } \|u\|_s \leq 1 \end{cases} \tag{3.13}$$

for every real-valued function $u \in \mathrm{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta))$*; the constants c and a depend only on n and s.*

PROOF. – We shall prove the proposition in three steps.

First step. Let $\operatorname{supp}\big((\mathcal{F}_{x' \to \xi'} u)(x_n, \xi')\big) \subset P_R(\varepsilon)$ for all $x_n \in (-\delta, \delta)$ and some $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$. By Remark 3.5 we have:

$$\|e^{iu} - 1\|_s = \Big\| \sum_{l=1}^{\infty} \frac{(iu)^l}{l!} \Big\|_s \leq \sum_{l=1}^{\infty} \frac{C_R(s)^{l-1} \|u\|_s^l}{l!} \leq \frac{1}{C_R(s)} \big(e^{C_R(s)\|u\|_s} - 1\big). \tag{3.14}$$

Second step. Now we suppose that $\operatorname{supp}\big((\mathcal{F}_{x'\to\xi'}u)(x_n,\xi')\big)\subset P_R$ for $x_n\in(-\delta,\delta)$. Observe that for every $h\in\mathbb{N}$ we can write $e^{iu(x)}-1=\sum_{l=1}^{h}\frac{(iu(x))^l}{l!}+\sum_{l=h+1}^{\infty}\frac{(iu(x))^l}{l!}=u_1(x)+u_2(x)$; by the standard properties of the Fourier transform and of the convolution product we get $\operatorname{supp}\big((\mathcal{F}_{x'\to\xi'}u_1)(x_n,\xi')\big)\subset P_{Rh}$. So, since $\|f\|_s$ is equivalent to $\sum_{\sum_{j=1}^{n-1}a_j\frac{1}{\rho_j}+a_n\le s}\|e^{\tau\psi(x_n,D')}D^\alpha f\|_{L^2}$, cf. [23], writing $k(x_n,\xi')=e^{\tau\psi(x_n,\xi')}\langle\xi'\rangle_{\rho'}^{s-a_n}$ we obtain:

$$
\begin{aligned}
(3.15)\quad \|e^{iu}-1\|_s\le \sum_{\sum_{j=1}^{n}a_j\frac{1}{\rho_j}\le s}\Bigg\{&\Big\|k(x_n,\xi')\sum_{l=h+1}^{\infty}\frac{i^l\mathcal{F}_{x'\to\xi'}(D_{x_n}^{a_n}u^l)(x_n,\xi')}{l!}\Big\|_{L^2(P_{Rh}(\varepsilon)\times(-\delta,\delta))}\\
&+\|\mathcal{F}_{x'\to\xi'}(D_{x_n}^{a_n}(e^{iu(x)}-1))(x_n,\xi')k(x_n,\xi')\|_{L^2(P_{Rh}\times(-\delta,\delta))}\Bigg\}\\
=\sum_{\sum_{j=1}^{n}a_j\frac{1}{\rho_j}\le s}\{T_1+T_2\}.&
\end{aligned}
$$

Let us analyze separately T_1 and T_2.

Regarding T_2 we proceed in different ways depending on a_n.
Suppose first that $a_n\neq 0$: by definition of weight function, in the set P_{Rh} we have: $k(x_n,\xi')\le e^{C\langle\xi'\rangle_{\rho'}^{r}}\langle\xi'\rangle_{\rho'}^{s}\le e^{C(1+Rh)^{r\rho_{\max}}}(1+Rh)^{s\rho_{\max}}$; so using Faà di Bruno formula and the fact that u is real-valued we get:

$$
(3.16)\quad T_2\le e^{C(1+Rh)^{r\rho_{\max}}}(1+Rh)^{s\rho_{\max}}\sum_{0<k\le a_n}\;\sum_{a_n^{(1)}+\ldots+a_n^{(k)}=a_n}\|\partial_{x_n}^{a_n^{(1)}}u(x)\cdots\partial_{x_n}^{a_n^{(k)}}u(x)\|_{L^2},
$$

where the norm is in $L^2(\mathbb{R}^{n-1}\times(-\delta,\delta))$; by Young estimate it follows that:

$$
\begin{aligned}
&\|\partial_{x_n}^{a_n^{(1)}}u(x)\cdots\partial_{x_n}^{a_n^{(k)}}u(x)\|^2_{L^2(\mathbb{R}^{n-1}\times(-\delta,\delta))}\\
&\le C\big\|\mathcal{F}_{x'\to\xi'}\big(\partial_{x_n}^{a_n^{(1)}}u(x)\big)*_{\xi'}\mathcal{F}_{x'\to\xi'}\big(\partial_{x_n}^{a_n^{(2)}}u(x)\cdots\partial_{x_n}^{a_n^{(k)}}u(x)\big)\big\|^2_{L^2(\mathbb{R}^{n-1}\times(-\delta,\delta))}\\
&\le C_1\big\|\mathcal{F}_{x'\to\xi'}\big(\partial_{x_n}^{a_n^{(1)}}u(x)\big)\big\|^2_{L^1(P_R\times(-\delta,\delta))}\big\|\mathcal{F}_{x'\to\xi'}\big(\partial_{x_n}^{a_n^{(2)}}u(x)\cdots\partial_{x_n}^{a_n^{(k)}}u(x)\big)\big\|^2_{L^2},
\end{aligned}
$$

where $*_{\xi'}$ stands for the convolution in the ξ' variables; repeating this procedure k times we get:

$$
\|\partial_{x_n}^{a_n^{(1)}}u(x)\cdots\partial_{x_n}^{a_n^{(k)}}u(x)\|^2_{L^2(\mathbb{R}^{n-1}_{x'}\times(-\delta,\delta))}\le C\prod_{j=1}^{k}\big\|\mathcal{F}_{x'\to\xi'}\big(\partial_{x_n}^{a_n^{(j)}}u(x)\big)\big\|^2_{L^1},
$$

where the norm in the right-hand side is in $L^1(P_R\times(-\delta,\delta))$. Since $a_n^{(j)}\le s$ for every j we obtain that $\big\|\mathcal{F}_{x'\to\xi'}(\partial_{x_n}^{a_n^{(j)}}u(x))\big\|^2_{L^1(P_R\times(-\delta,\delta))}\le K\|u\|_s$ (as it is easy to deduce by Hölder's inequality); then we can estimate T_2 in the case $a_n \neq 0$:

$$T_2 \leq e^{C(1+Rh)^{r\rho_{\max}}}(1+Rh)^{s\rho_{\max}} \sum_{0<k\leq s} \|u\|_s^k. \tag{3.17}$$

Now let us analyze T_2 when $a_n = 0$. By (3.15) we have $T_2 \leq e^{C(1+Rh)^{r\rho_{\max}}}(1+Rh)^{s\rho_{\max}}\|\mathcal{F}_{x'\to\xi'}(e^{iu(x)}-1)\|_{L^2(\mathbb{R}^{n-1}_{\xi'}\times(-\delta,\delta))}$. Since the function $g(t) = e^{it}-1$ is Lipschitz continuous and $g(0)=0$ we obtain that $T_2 \leq e^{C(1+Rh)^{r\rho_{\max}}}(1+Rh)^{s\rho_{\max}}\|u\|_{L^2(\mathbb{R}^{n-1}\times(-\delta,\delta))} \leq e^{C(1+Rh)^{r\rho_{\max}}}(1+Rh)^{s\rho_{\max}}\|u\|_s$. So the estimate (3.17) holds for every $a_n = 0,\dots,s$.

Let us now estimate T_1, cf. (3.15): by Theorem 3.1 we get:

$$T_1 \leq \sum_{l=h+1}^{\infty} \frac{1}{l!}\|u^l(x)\|_s \leq \frac{1}{C(s)} \sum_{l=h+1}^{\infty} \frac{(C(s)\|u\|_s)^l}{l!}.$$

Consider first $C(s)\|u\|_s > 1$, and choose h in (3.15) in such a way that

$$3C(s)\|u\|_s \leq h \leq 3C(s)\|u\|_s + 1. \tag{3.18}$$

By Stirling's formula we have that $\sum\limits_{l=h+1}^{\infty} \frac{(C(s)\|u\|_s)^l}{l!} \leq \frac{3}{3-e}$ and so we have

$$T_1 \leq C. \tag{3.19}$$

Assume now $C(s)\|u\|_s \leq 1$, and choose $h=0$ (that means $T_2 = 0$); we obtain:

$$T_1 \leq \frac{1}{C(s)} \sum_{l=1}^{\infty} \frac{(C(s)\|u\|_s)^l}{l!} = \frac{1}{C(s)}\left(e^{C(s)\|u\|_s} - 1\right) \leq M\|u\|_s. \tag{3.20}$$

So we have that, in the case $\operatorname{supp}((\mathcal{F}_{x'\to\xi'}u)(x_n,\xi')) \subset P_R$, (3.15), (3.17), (3.19) and (3.20), together with (3.18), imply:

$$\|e^{iu(x)} - 1\|_s \leq c\|u\|_s \sum_{k=0}^{s-1} \|u\|_s^k \left(1 + e^{bR^{r\rho_{\max}}\|u\|_s^{r\rho_{\max}}}\right). \tag{3.21}$$

Third step. Now we want to estimate $\|e^{iu(x)} - 1\|_s$ for a generic function $u \in \mathrm{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1}\times(-\delta,\delta))$. Let $\chi_\varepsilon(\xi')$ and $\chi_R(\xi')$ be the characteristic functions of the sets $P_R(\varepsilon)$ and P_R respectively. Setting $u_\varepsilon(x) := \mathcal{F}^{-1}_{\xi'\to x'}\left[\chi_\varepsilon(\xi')(\mathcal{F}_{x'\to\xi'}u)(x_n,\xi')\right]$ and $u_0(x) := \mathcal{F}^{-1}_{\xi'\to x'}\left[\chi_R(\xi')(\mathcal{F}_{x'\to\xi'}u)(x_n,\xi')\right]$ we have:

$$u(x) = u_0(x) + \sum_{\varepsilon} u_\varepsilon(x); \tag{3.22}$$

observe moreover that for every ε we obtain:

$$\|u_0\|_s \leq \|u\|_s, \quad \|u_\varepsilon\|_s \leq \|u\|_s. \tag{3.23}$$

It's not difficult to prove, by induction on N, that for every tuple of complex numbers $(a_1, \dots, a_N)$ the following identity holds:

$$a_1 \cdots a_N - 1 = \sum_{l=1}^{N} \sum_{0<j_1<\dots<j_l\le N} (a_{j_1} - 1)\cdots(a_{j_l} - 1). \tag{3.24}$$

Since the set $\{\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1}),\ \varepsilon_j \in \{0,1\}$ for $j = 1, \dots, n-1\}$ contains 2^{n-1} elements, using (3.22), (3.24) and Theorem 3.1 we get:

$$\|e^{iu(x)} - 1\|_s \le \sum_{l=1}^{2^{n-1}+1} \sum_{0\le j_1<\dots<j_l\le 2^{n-1}+1} C(s)^{l-1} \|e^{iu_{j_1}(x)} - 1\|_s \cdots \|e^{iu_{j_l}(x)} - 1\|_s. \tag{3.25}$$

We can estimate $\|e^{iu_{j_t}(x)} - 1\|_s$ in (3.25) using (3.14) in the case $j_t \ne 0$, and applying (3.21) if $j_t = 0$; we may now choose R in a convenient way.

Let us consider first u such that $\|u\|_s > 1$, and fix consequently R by imposing

$$C_R(s) = \|u\|_s^{r\rho_{\max}-1} : \tag{3.26}$$

observe that R is well defined by (3.26) since $\|u\|_s^{r\rho_{\max}-1} < 1$ and we have choosen $C_0(s) > 1$, cf. Remark 3.5.

Let us fix now $a > 0$, $t \ge 0$ and define $f(t) := \int_t^\infty e^{-y} y^{a-1}\, dy$; then putting $g(w) := f^{-1}(w)$ we obtain by de l'Hospital's rule:

$$\lim_{w\to 0} \frac{g(w)}{\log(1/w)} = 1. \tag{3.27}$$

The constant $C_R(s)$, cf. (3.11), can be written with a change of variables as $C_R(s) = \widetilde{C}(s)\big(\frac{1}{r}\big(\frac{1}{2\tau b}\big)^{2/r} f(2\tau b R_\rho^r)\big)^{1/2}$, where R_ρ is given by (3.12). Using (3.26) we can write, with the previous notation, R_ρ in the following form: $R_\rho = (2\tau b)^{-\frac{1}{r}}\big(g(r(2\tau b)^{2/r}\widetilde{C}(s)^{-2}\|u\|_s^{2(r\rho_{\max}-1)})\big)^{1/r}$; then (3.27) gives us:

$$R \le C\big(2(1-r)\log\|u\|_s + \log d\big)^{\frac{1}{r\rho_{\min}}}, \tag{3.28}$$

for suitable constants C and d with d sufficiently large. Now (3.13), in the case $\|u\|_s > 1$, follows from (3.25), (3.26), (3.28) and (3.23).

Let us suppose now that $\|u\|_s \le 1$. As for the case $\|u\|_s > 1$ we obtain (3.25); we choose in this case $R = 1$. By (3.14), (3.21) and (3.23) we easily obtain that $\|e^{iu(x)} - 1\|_s \le c\,\|u\|_s$, and so the proof is complete.

Now we analyze the behaviour of the composition $F(x; u(x))$ between a Gevrey function F and elements in the space $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta))$.

THEOREM 3.7. – *Let us consider a function $F(x; y)$ satisfying:*

– $F(x; y_0) \in G^{\sigma^{(r)}}(\mathbb{R}^{n-1} \times (-\delta, \delta))$, $1 < \sigma_j^{(r)} < \frac{\sigma_j}{r}$, $j = 1, \dots, n-1$, $\sigma_n^{(r)} > 1$, *for every* $y_0 \in \mathbb{R}^N$.

– $F(x_0; y) \in \bigcup_{r' < \frac{1}{r\rho_{\max}}} G^{r'}(\mathbb{R}^N)$, *for every* $x_0 \in \mathbb{R}^{n-1} \times (-\delta, \delta)$, *where* $\rho_{\max}$ *is* *given by* (3.10);

– $F(x; 0) = 0$.

Moreover we consider $\mathbb{B} \subset \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta))$, *bounded with respect to* $\|\cdot\|_s$, *and we take* $u_1(x), \dots, u_N(x) \in \mathbb{B}$, *where we suppose that the weight function* $\psi(x_n, \xi')$ *satisfies* (3.4) *and* $s \geq s_0$, *cf. Theorem* 3.1; $u_j(x)$ *are supposed to be real-valued for all* $j = 1, \dots, N$. *Then, writing* $u(x) := (u_1(x), \dots, u_N(x))$, *we have that* $F(x; u(x)) \in \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta))$, *and the following estimate holds:*

$$\|F(x; u(x))\|_s \leq \Psi(\|u_1\|_s, \dots, \|u_N\|_s), \tag{3.29}$$

where Ψ is a continuous function satisfying $\Psi(y) \to 0$ for $y \to 0$. Taking $v_1(x), \dots, v_N(x) \in \mathbb{B}$ we also have:

$$\|F(x; u(x)) - F(x; v(x))\|_s \leq C_{\mathbb{B}} \sum_{j=1}^{N} \|u_j - v_j\|_s, \tag{3.30}$$

where the constant $C_{\mathbb{B}}$ depends on the bounded set $\mathbb{B}$.

PROOF. – Observe first that $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta)) \hookrightarrow L^\infty(\mathbb{R}^{n-1} \times (-\delta, \delta))$:

$$\|u\|_{L^\infty} \leq \sup_{x_n} \|\mathcal{F}_{x' \to \xi'} u(x_n, \xi')\|_{L^1(\mathbb{R}^{n-1}_{\xi'})} \leq C \sup_{x_n} \|e^{\tau\psi(x_n, D')} u\|_{L^2(\mathbb{R}^{n-1}_{x'})} \leq C_1 \|u\|_s.$$

Then $\mathbb{B}$ is bounded in $L^\infty(\mathbb{R}^{n-1} \times (-\delta, \delta))$. So we can fix a function $\varphi(y) \in \bigcup_{r' < \frac{1}{r\rho_{\max}}} G_0^{r'}(\mathbb{R}^N)$ such that $F(x; u(x)) = (\varphi F)(x; u(x))$. This means that we can suppose without loss of generality that, for every $x_0 \in (-\delta, \delta)$, $F(x_0; y) \in \bigcup_{r' < \frac{1}{r\rho_{\max}}} G_0^{r'}(\mathbb{R}^N)$.

Since $F(x; 0) = 0$ we have $F(x; y) = \int e^{ix'\xi'} (e^{iy\eta} - 1) \big(\mathcal{F}_{\substack{x' \to \xi' \\ y \to \eta}} F\big)(x_n, \xi'; \eta)\, \bar{d}\xi'\, \bar{d}\eta$, and so

$$\|F(x; u(x))\|_s \leq C \sum_{j=0}^{s} \int \|e^{ix'\xi'} (e^{i\eta u(x)} - 1)\|_s \sup_{x_n} \big|D^j_{x_n} \big(\mathcal{F}_{\substack{x' \to \xi' \\ y \to \eta}} F\big)(x_n, \xi'; \eta)\big| \bar{d}\xi' \bar{d}\eta.$$

As in the standard isotropic case we can deduce for every $x_n \in (-\delta, \delta)$ and $j = 1, \dots, s$ that:

$$\big|D^j_{x_n} \big(\mathcal{F}_{\substack{x' \to \xi' \\ y \to \eta}} F\big)(x_n, \xi'; \eta)\big| \leq C e^{-\varepsilon \sum_{j=1}^{n-1} (1 + |\xi_j|)^{\rho_j^{(r)}}} e^{-\varepsilon |\eta|^{\frac{1}{r'}}}. \tag{3.31}$$

Moreover it follows by the definition of $\|\cdot\|_s$ that $\|e^{ix'\xi'} f(x)\|_s \leq C e^{\langle \xi' \rangle^r_{\rho'}} \|f\|_s$; now by (3.31) and taking into account the condition $\rho_j^{(r)} > r\rho_j$ for $j = 1, \dots, n-1$, we have:

$$\|F(x;u(x))\|_s \leq C\int e^{\langle\xi'\rangle^r_{\rho'}} e^{-\varepsilon\sum_{j=1}^{n-1}(1+|\xi_j|)^{\rho_j^{(r)}}} \bar{d}\xi' \cdot \int \|e^{i\eta u(x)}-1\|_s e^{-\varepsilon|\eta|^{\frac{1}{r'}}}\bar{d}\eta \tag{3.32}$$

$$= C'\int \|e^{i\eta u(x)}-1\|_s e^{-\varepsilon|\eta|^{\frac{1}{r'}}}\,\bar{d}\eta.$$

Now (3.24) and Theorem 3.1 imply:

$$\|e^{i\eta u(x)}-1\|_s \leq C\sum_{l=1}^{N}\sum_{0<j_1<\ldots<j_l\leq N}\|e^{i\eta_{j_1}u_{j_1}(x)}-1\|_s\cdots\|e^{i\eta_{j_l}u_{j_l}(x)}-1\|_s; \tag{3.33}$$

recall that $\frac{1}{r'} > r\rho_{\max}$: then, using Proposition 3.6, we can deduce by (3.32) and (3.33) that $\|F(x;u(x))\|_s \leq \Psi(\|u_1\|_s,\ldots,\|u_N\|_s)$. Observe that in the case $\|u_j\|_s \leq 1$ for every $j=1,\ldots,N$, (3.33), (3.13) and (3.32) imply:

$$\|F(x;u(x))\|_s \leq \widetilde{C}\sum_{l=1}^{N}\sum_{0<j_1<\ldots<j_l\leq N}\|u_{j_1}\|_s\cdots\|u_{j_l}\|_s;$$

so if $\|u_j\|_s \to 0$ for every $j=1,\ldots,N$, $\Psi(\|u_1\|_s,\ldots,\|u_N\|_s)\to 0$. This proves (3.29).

In order to verify (3.30) we write:

$$F(x;y)-F(x;z) = \sum_{j=1}^{N}(y_j-z_j)\int_0^1\big((\partial_{y_j}F)(x;y+tz)-(\partial_{y_j}F)(x;0)\big)\,dt$$
$$+\sum_{j=1}^{N}(y_j-z_j)(\partial_{y_j}F)(x;0).$$

So we have that

$$\|F(x;u(x))-F(x;v(x))\|_s \leq C\sum_{j=1}^{N}\|u_j-v_j\|_s$$
$$\cdot\left[\int_0^1\|(\partial_{y_j}F)(x;u(x)+tv(x))-(\partial_{y_j}F)(x;0)\|_s\,dt+\|(\partial_{y_j}F)(x;0)\|_s\right];$$

since $G(x;y) := (\partial_{y_j}F)(x;y)-(\partial_{y_j}F)(x;0)$ satisfies all the hypotheses of the theorem, we can apply (3.29) to $G(x;u+tv)$ and we get

$$\|F(x;u(x))-F(x;v(x))\|_s$$
$$\leq C\sum_{j=1}^{N}\|u_j-v_j\|_s\left[\int_0^1\Phi(\|u_j+tv_j\|_s)\,dt+\|(\partial_{y_j}F)(x;0)\|_s\right];$$

now (3.30) follows fron the fact that $u_j(x)$ and $v_j(x)$ are in $\mathbb{B}$ and the function Ψ is continuous. ■

COROLLARY 3.8. – *Let us consider now a function* $u(x) \in \mathbb{B} \subset \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta))$, $\mathbb{B}$ *being a bounded set as before, and define* $J(u)$ *as in* (3.3). *Suppose that* $F(x; y)$ *satisfies the hypotheses of the previous theorem; then there exists a continuous non-decreasing function* Φ *such that*

$$\|J(u)\|_s \le \Phi(\|u\|_{s+k^*-\varepsilon_{k^*}}),$$

$\Phi : [0, +\infty) \to [0, +\infty)$, $\Phi(0) = 0$; *moreover for* $u, v, \in \mathbb{B}$ *by* (3.30) *we get:*

$$\|J(u) - J(v)\|_s \le C_{\mathbb{B}} \|u - v\|_{s+k^*-\varepsilon_{k^*}}.$$

3.4 – *Gevrey-local solvability for semilinear equations.*

In this section we deal with the local solvability of the semilinear equation (3.2) in the frame of the spaces $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta, \delta))$; in particular we want to prove Theorem 1.3. Regarding the linear part we shall apply the results of Section 2 in order to find a (micro)-parametrix for the conjugate operator $\widetilde{P}(x, D)$, cf. (3.6); regarding the nonlinear term we shall use the arguments of Section 3.3.

We begin by proving the existence of a parametrix for the linear operator $P(x, D)$, cf. (1.2), in the frame of the Gevrey-Sobolev spaces $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}$. First of all, for $t > 0$ we define ε_t as follows:

$$\varepsilon_t \text{ is the largest real number such that}$$
$$\{(\gamma', j) \in \mathbb{Z}^n_+ : \left|\tfrac{\gamma'}{\rho'}\right| + j < t\} = \{(\gamma', j) \in \mathbb{Z}^n_+ : \left|\tfrac{\gamma'}{\rho'}\right| + j \le t - \varepsilon_t\}.$$

THEOREM 3.9. – *Let us consider the operator* $P(x, D)$, *given by* (1.2); *we suppose that* $k^* > m - \frac{1}{2}$ *and we fix* $r \in (0, 1)$ *such that* $r > \max\{\frac{1}{2}, 1 + k^* - m - \varepsilon_{k^*}\}$. *Moreover we require that for every point* $(x_0, \xi_0) \in \Sigma$, *cf.* (1.5), *there exists a neighbourhood* Γ *of* (x_0, ξ_0), *quasi-conical with respect to the* ξ*-variables, in which* (1.3) *and one of the following conditions holds (the same condition in a neighbourhood of every point* $(x_0, \xi_0) \in \Sigma$*):*

(a) $\Im \sum_{k^* \le \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^{j+m-1} \ge 0$ *and* $\Im \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x) \xi'^{\alpha'} \xi_n^{m-1} \le 0$;

(b) $\Im \sum_{k^* \le \left|\frac{\beta'}{\rho'}\right| + j < m} a_{\beta' j}(x) \xi'^{\beta'} \xi_n^{j+m-1} \le 0$ *and* $\Im \sum_{\left|\frac{\alpha'}{\rho'}\right| = m} b_{\alpha'}(x) \xi'^{\alpha'} \xi_n^{m-1} \ge 0$.

Then there exists a linear map $E : \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta,\delta)) \to \mathbb{H}^{s+m-(1-r),\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta,\delta))$ *such that* $P(x,D)Eu = \chi(x)u + Ru$, $\chi(x) \in G_0^{\sigma^{(r)}}(\Omega)$, *where* $1 < \sigma_j^{(r)} < \frac{\sigma_j}{r}$ *for* $j = 1,\dots,n-1$, $\sigma_n^{(r)} > 1$; *moreover R is regularizing, in the sense that* $R : \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta,\delta)) \to \mathbb{H}^{t,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta,\delta))$ *for every* $t \geq 0$.

PROOF. – Let us consider the conjugate operator $\widetilde{P}(x,D)$, defined by (3.6), where the weight function $\psi(x_n,\xi')$ is fixed as in (3.7) or (3.9) depending on the conditions (a) or (b). By Proposition 3.3 and Remark 3.4 we have that the symbol $\tilde{p}(x,\xi)$ of the conjugate operator $\widetilde{P}(x,D)$ is given by

$$\tilde{p}(x,\xi) = p(x,\xi) + p_{m,-(1-r)}(x,\xi) + p_{m,-(1-r)-\upsilon}(x,\xi).$$

Observe that taking δ sufficiently small we have that, at least microlocally, $|\Im\, p_{m,-(1-r)}(x,\xi)| \geq \langle\xi\rangle_\rho^{m-(1-r)}$, and so $|\Im\left(p_{m,-(1-r)}(x,\xi) + p_{m,-(1-r)-\upsilon}(x,\xi)\right)| \geq \langle\xi\rangle_\rho^{m-(1-r)}$, for $\langle\xi\rangle_\rho \gg 0$. Applying Theorem 2.6 with $\overline{k} = m-(1-r)$ and $\sigma(x,\xi) = p_{m,-(1-r)}(x,\xi) + p_{m,-(1-r)-\upsilon}(x,\xi)$, by the same technique used in [12] we can find an operator $\widetilde{E}$ such that $\widetilde{P}\widetilde{E} = \chi(x) + \widetilde{R}$, $\widetilde{R} : H^s_{\rho'}(\mathbb{R}^{n-1}\times(-\delta,\delta)) \to H^t_{\rho'}(\mathbb{R}^{n-1} \times (-\delta,\delta))$ for every $t \geq 0$. We then obtain $PE = e^{-\tau\psi(x_n,D')}\chi(x)e^{\tau\psi(x_n,D')} + R$. Taking a function $\chi_0(x) \in G_0^{\sigma^{(r)}}(\Omega)$ such that $\chi_0(x) = 1$ for $x \in \operatorname{supp}\chi$ and replacing E with E_χ, where $E_\chi u := E(\chi u)$ we have:

$$\begin{aligned} P(x,D)E_\chi u &= e^{-\tau\psi(x_n,D')}\chi_0(x)e^{\tau\psi(x_n,D')}\chi(x)u + \overline{R}u \\ &= \chi(x)u - e^{-\tau\psi(x_n,D')}\widetilde{R}_3 e^{\tau\psi(x_n,D')} + \overline{R}u. \end{aligned}$$

Since $\widetilde{R}_3 = (1-\chi_0(x))e^{\tau\psi(x_n,D')}\chi(x)e^{-\tau\psi(x_n,D')}$ is regularizing on $H^s_{\rho'}(\mathbb{R}^{n-1}\times(-\delta,\delta))$ we have that the operator E_χ is a parametrix for $P(x,D)$. ∎

PROOF OF THEOREM 1.3. – The Gevrey local solvability of the semilinear equation (3.2), cf. Theorem 1.3, can be proved in a standard way using Theorem 3.9 and Corollary 3.8, see the proof of Theorem 1.1. ∎

REMARK 3.10. – *Since* $G_0^{\sigma^{(r)}} \subset \mathbb{H}^{s,\psi}_{\tau,\sigma',r}$, $1 < \sigma_j^{(r)} < \frac{\sigma_j}{r}$ *for* $j = 1,\dots,n-1$, $\sigma_n^{(r)} > 1$, *cf.* [23], *and taking into account the obvious inclusions of the anisotropic Gevrey spaces into the isotropic ones, under the hypotheses of the Theorem* 1.3 *we obtain the* G^s *local solvability of the equation (*3.2) *for* $s < \frac{\min_{j=1,\dots,n-1}\sigma_j}{\max\{\frac{1}{2}, 1+k^*-m-\varepsilon_{k^*}\}}$.

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REFERENCES

[1] G. BOURDAUD - M. REISSIG - W. SICKEL, *Hyperbolic equations, function spaces with exponential weights and Nemytskij operators*, Ann. Mat. Pura Appl., 4 182 (2003), no. 4, 409-455.

[2] L. CADEDDU - T. GRAMCHEV, *Nonlinear estimates in anisotropic Gevrey spaces*, Pliska Stud. Math. Bulgar. **15** (2003), 149-160.

[3] A. CORLI, *On local solvability in Gevrey classes of linear partial differential operators with multiple characteristics*, Comm. Partial Differential Equations **14** (1989), 1-25.

[4] A. CORLI, *On local solvability of linear partial differential operators with multiple characteristics*, J. Differential Equations, **81** (1989), 275-293.

[5] G. DE DONNO - A. OLIARO, *Local solvability and hypoellipticity in Gevrey classes for semilinear anisotropic partial differential equations*, Trans. Amer. Math. Soc., **355** (2003), no. 8, 3405-3432.

[6] G. DE DONNO - L. RODINO, *Gevrey hypoellipticity for partial differential equations with characteristics of higher multiplicity*, Rend. Sem. Mat. Univ. Politec. Torino, **58** (2000), no. 4, 435-448 (2003).

[7] G. GARELLO, *Inhomogeneuos paramultiplication and microlocal singularities for semilinear equations*, Boll. Un. Mat. Ital. B. (7), **10** (1996), 885-902.

[8] G. GARELLO, *Local solvability for semilinear equations with multiple characteristics*, Ann. Univ. Ferrara Sez. VII, (N.S.) **41**, (1996), 199-209, suppl.

[9] T. GRAMCHEV, *On the critical index of Gevrey solvability for some linear partial differential equations*, Workshop on Partial Differential Equations (Ferrara 1999), Ann. Univ. Ferrara Sez. VII (N.S.), suppl., **45** (2000), 139-153.

[10] T. GRAMCHEV - P. POPIVANOV, *Local Solvability of Semilinear Partial Differential Equations*, Ann. Univ. Ferrara Sez. VII - Sc. Mat., **35** (1989), 147-154.

[11] T. GRAMCHEV, P. POPIVANOV - M. YOSHINO, *Critical Gevrey Index for Hypoellipticity of Parabolic Equations and Newton Polygones*, Ann. Mat. Pura Appl., **170** (1996), 103-131.

[12] T. GRAMCHEV L. RODINO, *Gevrey solvability for semilinear partial differential equations with multiple characterisitics*, Boll. Un. Mat. Ital., B (8) **2** (1999), 65-120.

[13] L. HÖRMANDER, *The analysis of linear partial differential operators*, vol. I, II, III, IV, Springer-Verlag, Berlin, 1983-85.

[14] J. HOUNIE - P. SANTIAGO, *On the local solvability of semilinear equations*, Comm. in Partial Differential Equations, **20** (1995), 1777-1789.

[15] C. HUNT - A. PIRIOU, *Majorations L^2 et inégalité sous-elliptique pour les opérateurs pseudo-différentiels anisotropes d'ordre variable*, C. R. Acad. Sci. Paris, **268** (1969), 214-217.

[16] C. HUNT - A. PIRIOU, *Opérateurs pseudo-différentiels anisotropes d'ordre variable*, C. R. Acad. Sci. Paris, **268** (1969), 28-31.

[17] K. KAJITANI - S. WAKABAYASHI, *Hypoelliptic operators in Gevrey classes*, in "Recent developments in hyperbolic equations" L.Cattabriga, F.Colombini, M.K.V. Murthy (London) (S. Spagnolo, ed.), Longman, 1988, 115-134.

[18] H. KOMATSU, *Ultradistributions, I: Structure theorems and a characterisation; II: The kernel theorem and ultradistributions with support in a submanifold; III: Vector valued ultradistributions and the theory of kernels*, J. Fac. Sci. Univ. Tokyo, Sect. IA **20** (1973), 25-105, **24** (1977), 607-628, **29** (1982), 653-717.

[19] O. LIESS - L. RODINO, *Inhomogeneous Gevrey classes and related pseudo-differential operators*, Boll. Un. Mat. Ital., Sez. IV, 3-**C** (1984), 133-223.

[20] O. Liess - L. Rodino, *Linear partial differential equations with multiple involutive characteristics*, in "Microlocal analysis and spectral theory" (Dordrecht) (L.Rodino, ed.), Kluwer, 1997, 1-38.
[21] M. Lorenz, *Anisotropic operators with characteristics of constant multiplicity*, Math. Nachr., **124** (1985), 199-216.
[22] J.-P. Marco - D. Sauzin, *Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems*, Publ. Math. Inst. Hautes Études Sci. **96** (2002), 199-275.
[23] P. Marcolongo, A. Oliaro, *Local Solvability for Semilinear Anisotropic Partial Differential Equations*, Annali Mat. Pura Appl. (4) **179** (2001), 229-262.
[24] M. Mascarello - L. Rodino, *Partial differential equations with multiple characteristics*, Wiley-VCH, Berlin, 1997.
[25] P. R. Popivanov, *Microlocal properties of a class of pseudodifferential operators with double involutive characteristics*, Partial differential equations (Warsaw, 1984), Banach Center Publ., PWN, Warsaw, **19** (1987), 213-224.
[26] P. R. Popivanov, *Local solvability of some classes of linear differential operators with multiple characteristics*, Ann. Univ. Ferrara, VII, Sc. Mat., **45** (1999), 263-274.
[27] P. R. Popivanov - G. S. Popov, *Microlocal properties of a class of pseudo-differential operators with multiple characteristics*, Serdica, **6** (1980), 169-183.
[28] L. Rodino, *Linear partial differential operators in Gevrey spaces*, World Scientific, Singapore, 1993.
[29] C. Roumieu, *Ultra-distributions définies sur $\mathbb{R}^n$ et sur certaines classes de variétés différentiables*, J. Analyse Math., **10** 1962/1963, 153-192.
[30] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.
[31] N. A. Šananin, *The local solvability of equations of quasi-principal type*, Mat. Sb. (N.S.) **97 (139)**, no 4 (8) (1975), 503-516.
[32] S. Wakabayashi, *Singularities of solution of the cauchy problem for hyperbolic system in gevrey classes*, Japan J. Math., **11** (1985), 157-201.

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