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Xianhua Li, A. Ballester-Bolinches

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On Supplements of Subgroups of Finite Groups (*).

Xianhua Li - A. Ballester-Bolinches

Sunto. – *Nel presente lavoro viene introdotto e studiato il concetto di s-coppia per un sottogruppo di un gruppo finito. Esso fornisce un modo uniforme per studiare l'influenza di alcune famiglie di sottogruppi sulla struttura di un gruppo finito. Vengono dati un criterio di appartenenza per un gruppo finito ad una formazione satura e delle condizioni necessarie e sufficienti per la solubilità, la superrisolubilità e la nilpotenza di un gruppo finito.*

Summary. – *In this paper the concept of s-pair for a subgroup of a finite group is introduced and studied. It provides a uniform way to study the influence of some families of subgroups on the structure of a finite group. A criterion for a finite group to belong to a saturated formation and necessary and sufficient conditions for solubility, supersolvability and nilpotence of a finite group are given.*

1. – Introduction.

All groups considered are finite.

The relation between properties of subgroups of a group and its structure is always a question of particular interest in the theory of groups. Of the various families of subgroups that can influence on the structure of the group, those of interest to us in this paper are maximal subgroups, Sylow subgroups and projectors associated to saturated formations.

It is well-known that each maximal subgroup of a soluble group is a complement of a chief factor of G. Taking this elementary fact as starting point, Deskins [5] and Mukherjee and Bhattacharya [8] introduced the interesting concepts of normal index, completions and θ-pairs, respectively. All of them are associated with a maximal subgroup and turned out to be useful in studying the normal structure of a group (see [1] [2], [8], [10], [11] and their references). More recently, the concepts of c-normality and c-supplementation introduced in [9]

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and [3], respectively, also contribute to a better understanding of the normal structure of the groups.

In this paper, the concept of s-pair for a subgroup H of a group G is introduced and analyzed. When H is a maximal subgroup, then the θ-pairs for H in G supplementing H are exactly the s-pairs for H in G. Moreover, a subgroup is c-normal or c-supplemented if it has an s-pair of special type. Therefore s-pairs provide a uniform way to study the influence of some families of subgroups on the structure of a group.

Our main results spring from the following question: what do intrinsic properties of s-pairs for a family of subgroups of a group G imply about G?.

We investigate in the paper how some conditions imposed on s-pairs for maximal subgroups of Sylow subgroups imply that the corresponding group is solvable, supersolvable or nilpotent (Theorems 2 and 3). Note that Sylow p-subgroups for a prime p are the projectors associated with the saturated formation of all p-groups. In this direction, we obtain necessary and sufficient conditions for a group to belong to a saturated formation provided that the maximal subgroups of the associated projectors have special s-pairs (Theorem 1).

We shall adhere to [6] for notation, terminology and results.

2. – s-pairs for a subgroup.

We begin with the following definition:

DEFINITION 1. – *Let H be a subgroup of a group G. A pair (A, B) of subgroups of G is said to be an s-pair for H in G if (A, B) satisfies the following properties:*

(i) $G = HA$ *and* $B = \mathrm{Core}_G(A \cap H)$,
(ii) *if* A_1/B *is a proper subgroup of* A/B *and* $A_1/B \trianglelefteq G/B$, *then* $G \neq HA_1$.

For brevity, we shall denote $X_G = \mathrm{Core}_G(X)$ for a subgroup X of a group G. Obviously the pair (G, H_G) satisfies condition (i). Hence the set

$$\mathcal{S} = \{A \mid H_G \leq A \trianglelefteq G,\ G = HA\}$$

is non-empty. Let A be an element of $\mathcal{S}$ of minimal order. It is clear that (A, H_G) is an s-pair for H in G. Thus we have proved:

PROPOSITION 1. – *For every subgroup H of a group G, the set $s(H)$ of all s-pairs for H in G is non-empty.*

It is clear that if H is a maximal subgroup of G, then every element of $s(H)$ is a θ-pair for H in G (see [8]).

A partial order is defined in $s(H)$ by means of $(A,B) \le (C,D)$ if and only if $A \le C$. In this case $B \le D$ also. It is clear then what is meant by saying that (A,B) is a maximal s-pair for H. Maximal elements in $s(H)$ do exist. In fact, if $(A,B) \in s(H)$, there exists a maximal element $(C,D) \in s(H)$ such that $(A,B) \le (C,D)$.

The following proposition is frequently used in induction arguments. Its proof is standard.

PROPOSITION 2. – *Let H be a subgroup of G and let N be a normal subgroup of G contained in H. Let (C,D) be an s-pair for H in G such that $N \le D$. Then $(C/N, D/N)$ is an s-pair for H/N in G/N.*

We say that an s-pair (C,D) for H in G is normal if C is a normal subgroup of G.

Let $(A, H_G) \in s(H)$ be the s-pair obtained above, where A is an element of minimal order in the set of the normal supplements of H in G containing H_G. If (C,D) is an s-pair for H in G such that $(A,H_G) \le (C,D)$, then $A \le C$ and $D = H_G$. Assume that $A < C$. Then $G \ne HA$ since (C,D) is an s-pair for H in G. Consequently (A,H_G) is a maximal s-pair for H in G which is normal. In general, A/H_G is not a chief factor of G as a subgroup of order 2 of the alternating group of degree 4 shows. However A/H_G is actually a chief factor when H is a maximal subgroup of G.

Next we use s-pairs to characterize c-normality and c-supplementation.

Recall that a subgroup H of a group G is said to be c-normal (respectively, c-supplemented) in G if there exists a normal subgroup (respectively, a subgroup) K of G such that $G = HK$ and $H \cap K \le H_G$ (see [9] and [3], respectively).

PROPOSITION 3. – *Let G be a group and let H be a subgroup of G. Then:*

1. *H is c-normal in G if and only if there is a normal s-pair (A,B) for H in G such that $H \cap A = B$.*
2. *H is c-supplemented in G if and only if there is an s-pair (A,B) of H in G such that $H \cap A = B$.*

PROOF. – We only give a proof for the case 2.

Suppose that H is c-supplemented in G. Then there exists a subgroup K such that $G = HK$ and $H \cap K \le H_G$. If $C = KH_G$, then $G = HC$ and $H \cap C = H_G = B$. Suppose that $A/H_G \trianglelefteq G/H_G$ and A is a proper subgroup of C. If $G = HA$, it follows that $C = A(C \cap H) = AH_G = A$, a contradiction. Therefore we have that (C, H_G) is an s-pair for H in G.

The converse is clear.

Note that if K is normal in G, then C is normal in G. Therefore the proof for c-normality is exactly the same to the one used for c-supplementation. ∎

3. – Main results.

Our main results involve maximal subgroups of projectors associated with saturated formations. Recall that a formation is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images and such that each group G has an smallest normal subgroup N with $G/N \in \mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-residual of G and it is denoted by $G^{\mathfrak{F}}$. A formation $\mathfrak{F}$ is said to be saturated if a group $G \in \mathfrak{F}$ provided the Frattini factor group $G/\Phi(G)$ is in $\mathfrak{F}$.

If $\mathfrak{F}$ is a formation, a subgroup H of a group G is called an $\mathfrak{F}$-projector of G if HN/N is a maximal $\mathfrak{F}$-subgroup of G/N whenever N is a normal subgroup of G. It is well-known that if the formation $\mathfrak{F}$ is saturated, then every group has $\mathfrak{F}$-projectors. Moreover if N is a normal subgroup of G and P/N is an $\mathfrak{F}$-projector of G/N, then there exists an $\mathfrak{F}$-projector P_0 of G such that $P = P_0N$ (see [6, Chapter III] for details). It is also well-known that a formation $\mathfrak{F}$ is saturated if and only if it is locally defined, that is, there exists a formation function f such that $\mathfrak{F} = \mathrm{LF}(f)$. Moreover if $\mathfrak{F}$ is saturated, then $\mathfrak{F}$ is locally defined by a unique formation function F which is integrated and full; this F is called the canonical local definition of $\mathfrak{F} = \mathrm{LF}(F)$ (see [6, Chapter IV] for details).

DEFINITION 2 ([2]). – *Let $\mathfrak{F}$ be a saturated formation with canonical local definition F. Let A and B be subgroups of a group G such that $B \trianglelefteq G$ and $B \leq A$. We say that A/B is $\mathfrak{F}$-central in G if $(G/B)^{F(p)} \leq C_G(A/B)$ for each prime $p \in \pi(A/B)$, the set of primes dividing $|A/B|$.*

It is clear that if A/B is a chief factor of G, then A/B is $\mathfrak{F}$-central in G in the sense of the above definition if and only if it is $\mathfrak{F}$-central in the classical sense (see [6, IV,5.6]).

In [2], it is proved that a group G belongs to a saturated formation $\mathfrak{F}$ if and only if each maximal subgroup has a maximal θ-pair which is $\mathfrak{F}$-central in G. We prove:

THEOREM 1. – *Let $\mathfrak{F}$ be a saturated formation. A group G belongs to $\mathfrak{F}$ if and only if for each maximal subgroup H of each $\mathfrak{F}$-projector of G, there exists an s-pair (C, D) for H in G such that C/D is $\mathfrak{F}$-central in G.*

PROOF. – Assume that G is a group in $\mathfrak{F}$. Then G is the unique $\mathfrak{F}$-projector of G. Let M be a maximal subgroup of G, and let $C/\mathrm{Core}_G(M)$ be a minimal subgroup of the primitive group $G/\mathrm{Core}_G(M)$. Then $C/\mathrm{Core}_G(M)$ is an $\mathfrak{F}$-central chief factor of G supplementing M in G by [6, IV,5.7]. Consequently $(C, \mathrm{Core}_G(M))$ is a maximal s-pair for M in G such that $C/\mathrm{Core}_G(M)$ is $\mathfrak{F}$-central in G.

Suppose that for every maximal subgroup H of every $\mathfrak{F}$-projector of G there exists an s-pair (C, D) for H in G such that C/D is $\mathfrak{F}$-central in G. We prove that G

belongs to $\mathfrak{F}$ by induction on $|G|$.

Let N be a minimal normal subgroup of G. We see that G/N satisfies the hypotheses of the theorem. To see this, let T/N be an $\mathfrak{F}$-projector of G/N. Then, by [6, III, 3.7, 3.9], we can find an $\mathfrak{F}$-projector T_0 of G such that $T = T_0N$. Let A/N be a maximal subgroup of T/N. It is clear that $A = T_1N$ for some maximal subgroup T_1 of T_0 containing $N \cap T_0$. By hypothesis, there exists an s-pair (C, D) for T_1 in G such that C/D is $\mathfrak{F}$-central in G. Then $G = A(CN)$. Denote $F = \mathrm{Core}_G(A \cap CN)$. Suppose that (CN, F) is an s-pair for A in G. By Proposition 2, $(CN/N, F/N)$ is also an s-pair for A/N in G/N. Moreover $(CN/N)/(F/N)$ is $\mathfrak{F}$-central in G/N. Assume that (CN, F) is not an s-pair for A in G. Then there exists a normal subgroup S of G contained in CN such that (S, F) is an s-pair for A in G. It is clear that $\pi(S/F)$ is contained in $\pi(C/D)$. Moreover $[G^{F(p)}, S] \leq [G^{F(p)}, CN] \leq DN \leq F$ for all primes $p \in \pi(S/F)$. This means that $(S/N)/(F/N)$ is $\mathfrak{F}$-central in G/N. Consequently the s-pair $(S/N, F/N)$ of A/N in G/N has the required properties. By induction, we have that $G/N \in \mathfrak{F}$.

Consequently, every proper epimorphic image of G belongs to $\mathfrak{F}$. Assume, arguing by contradiction, that G is not in $\mathfrak{F}$, so that it has a unique minimal normal subgroup N and there exists an $\mathfrak{F}$-projector M of G such that $G = NM$. Let M_1 be a maximal subgroup of M. Then there exists an s-pair (K, L) for M_1 in G such that K/L is $\mathfrak{F}$-central in G. Since L is a normal subgroup of G contained in M and N is the unique minimal normal subgroup of G, it follows that $L = 1$ and then $1 \neq G^{F(q)} \leq C_G(K)$ for all primes q dividing $|K|$. If N were non-abelian, we would have $K \leq C_G(N) = 1$ and $M = M_1$, a contradiction. Thus N is abelian and $K \leq C_G(N) = N$. This implies that $G = MN = M_1N$, $N \cap M = N \cap M_1 = 1$ and $M = M_1$, a contradiction. Consequently $G \in \mathfrak{F}$ and the theorem is proved. ∎

It is known that a group with cyclic Sylow subgroups is metacyclic. For groups with non-cyclic Sylow subgroups, we have:

THEOREM 2. – *Let G be a group with at least a non-cyclic Sylow subgroup. Then G is solvable (respectively, supersolvable) if and only if for every maximal subgroup H of any non-cyclic Sylow subgroup of G, there exists $(A, B) \in s(H)$ such that A/B is solvable (respectively, supersolvable).*

PROOF. – Suppose that the result is false and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G. If G/N has no non-cyclic Sylow subgroups, then G/N is metacyclic and so G/N is supersolvable. Now if G/N has at least one non-cyclic Sylow subgroup, we can argue as in the above result to conclude that G/N satisfies the hypotheses to the theorem (here we apply that solvable and supersolvable groups are subgroup-closed classes). The minimal choice of G implies that G/N is solvable (respectively, supersolvable). Since both classes are saturated formations, it follows that G is a group with a unique minimal normal subgroup, N say. Moreover, N is non-Frattini. We distinguish two cases:

Solvable case. Since G/N is solvable, it follows that N is a direct product $N = N_1 \times N_2 \times N_t$, where the N_i are isomorphic non-abelian simple groups for $1 \leq i \leq t$. Suppose that $t > 1$ and let p be a prime dividing $|N|$. If G_p is a Sylow p-subgroup of G, we have that G_p is not cyclic. Let H be a maximal subgroup of G_p. We know that there exists $(C, D) \in s(H)$ such that $G = HC$ and C/D is solvable. Since N is not abelian and D is a normal p-subgroup of G, it follows that $D = 1$. Now C contains a Sylow r-subgroup of G for each prime $r \neq p$. This implies that $|N_1 : C \cap N_1|$ is a power of p. Therefore N_1 has subgroups of more than two different prime power indices because $|\pi(N_1)| \geq 3$. This contradicts the results of [7]. Consequently $t = 1$, that is, N is a non-abelian simple group. We can assume that a Sylow 2-subgroup G_2 of G is not cyclic, because otherwise the group would be 2-nilpotent. Arguing as above, for each maximal subgroup P of G_2, there exists $(C, 1) \in s(P)$ such that $G = PC$, C is solvable and $|N : C \cap N| = 2^a$. Applying [7], $N = \mathrm{PSL}(2, q)$ for a prime $q = 2^a - 1$ and $N \cap C$ is a maximal subgroup of N of index 2^a. Moreover $N \cap C$ is the normalizer in N of a Sylow q-subgroup of G (see [4]).

On the other hand, it is clear that G is isomorphic to a subgroup of $Aut(N)$. Moreover, $|Aut(N) : N| = 2$ (see [4]). Hence either $G = N$ or $G = Aut(N)$.

Assume that $G = Aut(N)$. Let Q be a Sylow q-subgroup of G. It is known that $D = N_G(Q)$ is a subgroup of order $q(q-1)$ and $G = NN_G(Q)$ (see [4]). Let D_2 be a Sylow 2-subgroup of D and let G_2 be a Sylow 2-subgroup of G containing D_2. Since D_2 is a proper subgroup of G_2, it is contained in a maximal subgroup T of G_2. By hypothesis, there exists $(A, 1) \in s(T)$ such that $G = TA$. Then $N \cap A$ is the normalizer in N of a Sylow q-subgroup of G. Without loss of generality we can assume that $N \cap A = N \cap D$. This implies that $N \cap A$ is a maximal subgroup of D contained in $N_A(Q)$. Suppose that $N \cap A = N_A(Q)$. Then, if G were equal to AN, we would have $|A| = q(q-1)$ and then $A = D$, a contradiction. Therefore A is contained in N, $A \leq D$ and so $G_2 = T$, a contradiction. Consequently, $D = N_A(Q)$ and $D = A$. But, in this case, we have that $G_2 = T$, a contradiction.

If G were equal to N, we would argue in a similar way to get the final contradiction.

Supersolvable case. We have that G is solvable by the above case. This implies that N is an abelian self-centralizing minimal normal subgroup of G and there exists a core-free maximal subgroup M of G such that $G = NM$, $N \cap M = 1$ and M is supersolvable (see [6, A, 15.2]). It is clear that N is not cyclic. Let p be the prime dividing $|N|$ and suppose that N is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of N. By hypothesis, there exists an s-pair $(C, 1)$ of P_1 such that $G = P_1C$ and C is supersolvable. Then $N \cap C$ is a non-trivial normal subgroup of G. Since N is the unique minimal normal subgroup of G, it follows that $N = N \cap C$ and $G = C$, a contradiction. Hence N is not a Sylow p-subgroup of G.

In this case, we can find a non-trivial Sylow p-subgroup M_p of M such that $P = NM_p$ is a Sylow p-subgroup of G. Let q be the largest prime dividing $|G|$ and let Q be a Sylow q-subgroup of G. If $p = q$, then N is contained in Q and so Q is normal in G because G/N is supersolvable. In particular, Q is contained in N, a contradiction. Hence $q \neq p$. We can assume that $Q \leq M$. Since M is supersolvable, it follows that $M = N_G(Q)$. Let P_3 be maximal subgroup of P containing M_p. By hypothesis, there exists an s-pair $(H, 1)$ of P_3 such that $G = P_3H$ and H is supersolvable. Then H contains s Sylow q-subgroup of G. Without loss of generality we can assume that Q is contained in H. Thus H is actually contained in M. Let $M_{p'}$ be a Hall p'-subgroup of M such that $M = M_pM_{p'}$. Then $G = P_3H = P_3M = P_3M_pM_{p'} = P_3M_{p'}$, a contradiction. ■

Nilpotent groups admit an analogous characterization.

THEOREM 3. – *Let G be a group with at least a non-cyclic Sylow subgroup. Then G is nilpotent if and only if for every maximal subgroup H of any non-cyclic Sylow subgroup of G, there exists $(A, B) \in s(H)$ such that A/B is nilpotent.*

PROOF. – Obviously every nilpotent group satisfies the required condition. Assume that G is a group with at least a non-cyclic Sylow subgroup such that for every maximal subgroup H of any non-cyclic Sylow subgroup of G, there exists $(A, B) \in s(H)$ such that A/B is nilpotent. We prove that G is nilpotent by induction on the order of G. Applying the above theorem, G is supersolvable. Suppose that r is a prime dividing the order of G and let R be a normal Sylow r-subgroup of G. Assume that R is not cyclic. Then $R/\Phi(R)$ is a non-cyclic Sylow r-subgroup of $G/\Phi(R)$ (note that $\Phi(R)$ is normal in G). If $\Phi(R) \neq 1$, then $G/\Phi(R)$ is nilpotent by induction. This implies that G is nilpotent. Therefore we may assume that $\Phi(R) = 1$. Then $R = N_1 \times N_2 \times \cdots \times N_s$, where N_i are minimal normal subgroups of G and $|N_i| = r$. Let $H_i = N_1 \times \cdots \times N_{i-1} \times N_{i+1} \times \cdots \times N_s$. It is clear that H_i is a maximal subgroup of R. By hypothesis, there exists an s-pair (A, B) of H_i such that $G = AH_i$, $B = (A \cap H_i)_G$ and A/B is nilpotent. Let $G_{r'}$ be a Hall r'-subgroup of G. Since $N_iG_{r'}$ is isomorphic to G/H_i and G/H_i is an epimorphic image of A/B, it follows that $N_iG_{r'}$ is nilpotent. This implies that $G_{r'}$ is contained in $C_G(N_i)$ for any i with $1 \leq i \leq s$. Moreover, $G_{r'}$ is nilpotent. Therefore G is nilpotent.

Consequently, we may assume that every normal Sylow subgroup of G is cyclic. Let P be the normal Sylow p-subgroup of G for the largest prime p dividing the order of G. Then P is cyclic. Let Q be a non-cyclic Sylow q-subgroup of G for some prime q. It is not difficult to prove that PG_q satisfies the hypothesis of theorem. Hence if $PG_q \neq G$, then PG_q is nilpotent. Now, by induction, G/N is nilpotent. Therefore PG_q is normal in G and so G_q is also normal in G, a contradiction.

Consequently, we have that $G = PG_q$ for a prime q such that G has a non-cyclic Sylow q-subgroup. It P is central in G, then G is nilpotent and the result is true. Assume that P is not central in G and let Q be a maximal subgroup of G_q containing the Sylow q-subgroup of $C_G(P)$. By hypothesis, there is an $s-$pair (C, D) of Q such that $G = QC$, $D = (Q \cap C)_G$ and C/D nilpotent. Then a Sylow q-subgroup C_q of C is a normal subgroup of C. On the other hand, $C_p = P$ is normal in C. Hence C is nilpotent and $G = QC_G(P)$. This implies $Q = G_q$, a contradiction. ■

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Xianhua Li: Department of Mathematics, Suzhou University,
Suzhou Jiangsu 215006, People's Republic of China
and Institut für Mathematik Potsdam Universität
E-Mail: xhli@suda.edu.cn

A. Ballester-Bolinches: Departament d'Àlgebra, Universitat de València,
c/ Dr. Moliner 50, 46100 Burjassot (València), Spain
E-Mail: Adolfo.Ballester@uv.es

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