# Bollettino
# Unione Matematica Italiana



P. Buttà, S. Caprino, G. Cavallaro, C. Marchioro


## On the dynamics of infinitely many charged particles with magnetic confinement


*Bollettino dell'Unione Matematica Italiana, Serie 8*, Vol. **9-B** (2006), n.2, p. 371–395.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2006_8_9B_2_371_0>



L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

*Articolo digitalizzato nel quadro del programma*
*bdim (Biblioteca Digitale Italiana di Matematica)*
*SIMAI & UMI*
http://www.bdim.eu/

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2006.

Bollettino U. M. I.
(8) 9-B (2006), 371-395

# On the Dynamics of Infinitely Many Charged Particles with Magnetic Confinement

P. BUTTÀ - S. CAPRINO - G. CAVALLARO - C. MARCHIORO

**Sunto.** – *Studiamo l'evoluzione temporale di un sistema di infinite particelle cariche, confinate a muoversi in un conduttore cilindrico illimitato attraverso un campo magnetico esterno e tra loro interagenti mediante un potenziale di tipo Coulomb. Dimostriamo l'esistenza, l'unicità e la quasi-località del moto. Forniamo inoltre alcune stime non banali sul comportamento del sistema per tempi lunghi.*

**Summary.** – *We study the time evolution of a system of infinitely many charged particles confined by an external magnetic field in an unbounded cylindrical conductor and mutually interacting via the Coulomb force. We prove the existence, uniqueness and quasi-locality of the motion. Moreover, we give some nontrivial bounds on its long time behavior.*

## 1. – Introduction.

The description of Nonequilibrium Statistical Mechanics by an Hamiltonian model is an appealing but very difficult task. A natural model arises by considering infinitely many particles evolving via the Newton law:

$$\ddot{\boldsymbol{x}}_i(t) = \sum_{\substack{j\in\mathbb{N}\\ j\neq i}} \boldsymbol{F}\big(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\big), \qquad i \in \mathbb{N}. \tag{1}$$

A first step for a rigorous investigation of this model is to prove the existence of the time evolution itself. This means essentially to prove that a quasi-local observable evolves remaining quasi-local. This is not trivial because we can exhibit situations with an initial bounded density that after a finite time produce infinitely many particles in a bounded region [11]. So we need a careful choice of the initial conditions in order to exclude these bad data, but at the same time to take into account all the relevant states in Nonequilibrium Statistical Mechanics. The results depend in a very sensitive way on the dimension of the space in which the particles move and on the nature of the mutual interaction. The first pioneering results have been obtained by

Lanford [11, 12] many years ago, in one dimension for bounded and finite range interactions. Then the cases in one dimension with singular interactions [8, 15] and Coulomb force [16] and the case of positive or very singular interactions in two dimensions [10] have been studied. A long period of silence on this problem follows these results and only three years ago the three dimensional case has been solved for positive, bounded interactions [5]. All these results assume a finite range interaction, while in many physical systems this assumption appears too drastic. We think of charged particles mutually interacting via the Coulomb potential, which exhibits a long range behavior. Actually the long range character of the Coulomb interaction in the whole space does not allow a well defined Statistical Mechanics. Nevertheless there are physical situations, for instance electrons moving in a neutral background, in which a screening effect is present (the «Debye screening»), weakening the very large distance interactions. Hence, it is meaningful to consider long range, fastly decaying potentials, as it is done in [2], where the authors generalize the results in [10], assuming a singular, superstable, subexponentially decreasing at infinity potential.

The above quoted papers exhibit explicit sets of initial conditions. Indeed there are other works regarding the Equilibrium (or Stationary) Dynamics, in which the initial data are full-measure with respect to the Gibbs (or Stationary Non Equilibrium) measure, but they are not costructively specified [1, 13, 14, 17, 20, 21, 22]. Let us also recall some papers dealing with the time evolution of special states [7, 18]. Up to now, there are no results for three dimensional systems, in case of a singular, long range inter-particle force.

A natural step forward would be to investigate in more detail the long time behavior of the dynamics. Unfortunately, in this kind of approaches the bounds one gets on the local density and energy are generally bad-behaving in time, so that it is difficult to say anything about the time asymptotics of the system. On the other side it is in this regime that many physical laws can be reproduced. Recently some results in this direction have been obtained in [3], [4] and [6], for some particular one-dimensional systems.

In the present paper we discuss a physical system consisting of infinitely many charged particles with the same mass ($m = 1$) and the same electric charge ($q = 1$), confined by an external magnetic field in an unbounded cylindrical conductor in $\mathbb{R}^3$, with radius $L^*$, infinitely extended in the direction of its symmetry axis, which is made of a thin shell of conducting material empty inside. We assume the conductor is kept at fixed potential (say $V = 0$). This means that the charged particles mutually interact via the Coulomb force

$$\boldsymbol{F} = -\nabla G \tag{2}$$

where $G$ is the Green function of the Laplace operator with Dirichlet boundary conditions. As it is well known (see Section 4 below) this force is singular at small distances and, due to the presence of the conductor, it decreases exponentially at large distances (see Definition (3) below).

We prove existence, uniqueness and quasi-locality of the motion and give a non-trivial bound on the growth of the velocity of each particle.

The new aspects presented by this problem are the following ones: particles move in a three dimensional domain; moreover, the magnetic field confining the particles in the cylinder is singular and so it could in principle affect very much their motion. Finally, we refine the usual time estimates on the velocities, obtaining a linear time behavior.

The plan of the paper is the following. In Section 2 we present our results on the dynamics. The proofs are given in Section 3. Some comments on the plausibility of our assumptions on the interparticle force are in Section 4, while some technical results are in Appendices A and B.

## 2. – Results.

Let us introduce notations and definitions we will deal with from now on. Let $\boldsymbol{v}$ be a fixed unit vector in $\mathbb{R}^3$. For any $\boldsymbol{\xi} \in \mathbb{R}^3$, we denote by $\boldsymbol{\xi}^{\perp} = \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{v}$ its orthogonal projection. Given $L > 0$, $L < L^*$, let $\Omega = \{\xi \in \mathbb{R}^3 : |\xi^{\perp}| < L\}$ be the infinite cylinder of radius $L$ and symmetry axis $\boldsymbol{v}$. This «mathematical» cylinder has the same symmetry axis as the cylindrical conductor, but $L < L^*$.

The charged particles interact among themselves by means of a two-body potential of the form

$$\phi(\xi) = \begin{cases} \dfrac{a}{|\xi|} + \phi_1(|\xi|) & \text{if } |\xi| \le 1 \\ \phi_2(|\xi|)e^{-\gamma|\xi|} & \text{if } |\xi| > 1 \end{cases} \tag{3}$$

where $a > 0$, $\gamma > 0$. The functions $\phi_1$,$\phi_2$ are bounded functions, together with their first and second derivatives, such that $\phi$ is non-negative and twice differentiable for all values of $|\xi| > 0$.

Moreover we assume that a magnetic field acts on the particles, in order to keep them inside the cylinder. Assuming the symmetry axis of the cylinder to be coincident with the $z$-axis of a Cartesian coordinate system in $\mathbb{R}^3$, we define this field by

$$\boldsymbol{H}(x, y, z) = (0, 0, H(x^2 + y^2)) \tag{4}$$

and

$$H(x^2 + y^2) = \frac{h(x^2 + y^2)}{(L^2 - x^2 - y^2)^{\alpha}}, \qquad \alpha > 1. \tag{5}$$

The function $h$ is a positive, differentiable function, such that $h = 1$ if $L^2 - \delta^2 \leq x^2 + y^2 < L^2$, $h = 0$ if $x^2 + y^2 \leq L^2 - 2\delta^2$ and it satisfies $0 \leq h'(r) \leq 2/\delta^2$ ; thus $\delta$ is a (small) number such that the magnetic field is different from zero only near the border of the cylinder.

The state of the system is determined by the positions and velocities $\boldsymbol{x}_i = (\boldsymbol{r}_i, \boldsymbol{v}_i)$, $i \in \mathbb{N}$, of the charged particles. Let us denote by $\boldsymbol{X} = \{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$ the state of the infinite system, which is assumed to have a locally finite density and energy. It is thus well defined, for any $\mu \in \mathbb{R}$ and $R > 0$, the function

$$Q(\boldsymbol{X};\mu,R) = \sum_i \chi_i(\mu,R)\left\{\frac{\boldsymbol{v}_i^2}{2} + \frac{1}{2}\sum_{j:j\neq i}\phi(\boldsymbol{r}_i - \boldsymbol{r}_j) + 1\right\}, \tag{6}$$

where $\chi_i(\mu,R) = \chi(|\boldsymbol{r}_i \cdot \boldsymbol{v} - \mu| \leq R)$ and $\chi(A)$ denotes the characteristic function of the set $A$.

In order to consider configurations which are typical for thermodynamical states, we allow initial data with logarithmic divergences in the velocities and local densities. More precisely, by defining

$$Q(\boldsymbol{X}) = \sup_{\mu} \sup_{R:R>\log(e+|\mu|)} \frac{Q(\boldsymbol{X};\mu,R)}{2R}, \tag{7}$$

the set

$$\Xi^{(1)} = \{\boldsymbol{X} : Q(\boldsymbol{X}) < +\infty\} \tag{8}$$

has a full measure w.r.t. any Gibbs state [8].

The time evolution $t \to \boldsymbol{X}(t)$ is defined by the solutions of the Newton equations:

$$\begin{cases} \ddot{\boldsymbol{r}}_i(t) = \boldsymbol{F}_i(\boldsymbol{X}(t)), & i \in \mathbb{N}, \\ \boldsymbol{X}(0) = \boldsymbol{X}, \end{cases} \tag{9}$$

where

$$\boldsymbol{F}_i(\boldsymbol{X}) = -\sum_{j:j\neq i} \nabla\phi(\boldsymbol{r}_i - \boldsymbol{r}_j) + \dot{\boldsymbol{r}}_i \wedge \boldsymbol{H}(\boldsymbol{r}_i) \qquad i \in \mathbb{N}. \tag{10}$$

and

$$\boldsymbol{X} \in \Xi^{(1)} \cap \Xi^{(2)} \tag{11}$$

where

$$\Xi^{(2)} = \left\{\boldsymbol{X} : \exists\, A > 0 : \left(L^2 - r_i^2\right) > \frac{1}{[A\log^5(e + |\boldsymbol{r}_i \cdot \boldsymbol{v}|)]^{\left(\frac{1}{a-1}\right)}}\right\}. \tag{12}$$

Here $r_i = |\boldsymbol{r}_i - (\boldsymbol{r}_i \cdot \boldsymbol{v})\boldsymbol{v}|$ and $a$ is the one in (5). Condition (12) means that the particles cannot be distributed arbitrarily near the border at the initial time; on the contrary they can start as closer to the border as farther they are from the origin, in accordance with (12). This condition intuitively is due to the fact that the magnetic field becomes infinite on the border, so that taking particles which are initially too close to it could make the force in (9) too large.

The solution of (9) is constructed by means of the following limiting procedure.

Given $\boldsymbol{X} \in \Xi^{(1)} \cap \Xi^{(2)}$ and $n \in \mathbb{N}$, let

$$I_n = \{i \in \mathbb{N} : \boldsymbol{r}_i \in \Omega(0, n)\}, \tag{13}$$

where $\Omega(\mu, R) = \{\boldsymbol{\xi} \in \Omega : |\boldsymbol{\xi} \cdot \boldsymbol{v} - \mu| \le R\}$.

We define the $n$-partial dynamics $t \to \boldsymbol{X}^n(t)$, $\boldsymbol{X}^n(t) = \{\boldsymbol{r}_i^n(t), \boldsymbol{v}_i^n(t)\}_{i \in I_n}$, as the solution of the differential system:

$$\begin{cases} \ddot{\boldsymbol{r}}_i^n(t) = \boldsymbol{F}_i(\boldsymbol{X}^n(t)), \qquad i \in I_n, \\ \boldsymbol{X}^n(0) = \{\boldsymbol{r}_i, \boldsymbol{v}_i\}_{i \in I_n}. \end{cases} \tag{14}$$

Our main result is the following:

THEOREM 2.1. – *If* $\boldsymbol{X} \in \Xi^{(1)} \cap \Xi^{(2)}$ *there exists a unique flow* $t \to \boldsymbol{X}(t) = \{\boldsymbol{r}_i(t), \boldsymbol{v}_i(t)\}_{i \in \mathbb{N}} \in \Xi^{(1)} \cap \Xi^{(2)}$ *satisfying:*

$$\begin{cases} \ddot{\boldsymbol{r}}_i(t) = \boldsymbol{F}_i(\boldsymbol{X}(t)), \qquad i \in \mathbb{N} \\ \boldsymbol{X}(0) = \boldsymbol{X}. \end{cases} \tag{15}$$

*Moreover, for any* $t \ge 0$ *and* $i \in \mathbb{N}$,

$$\lim_{n \to +\infty} \boldsymbol{r}_i^n(t) = \boldsymbol{r}_i(t), \qquad \lim_{n \to +\infty} \boldsymbol{v}_i^n(t) = \boldsymbol{v}_i(t), \tag{16}$$

*the convergence being uniform on compact sets.*

The growth of the velocity of a particle can be bounded as established by the following

THEOREM 2.2. – *For any* $\boldsymbol{X} \in \Xi^{(1)} \cap \Xi^{(2)}$ *there exist two positive constants* $C_1$, $C_2$, *such that*

$$|\boldsymbol{v}_i(t)| \le C_1 \sqrt{\log(e + |\boldsymbol{r}_i \cdot \boldsymbol{v}|)} + C_2 t, \qquad t \ge 0. \tag{17}$$

From now on $C_i$, $i = 3, 4, \ldots$ indicate positive constants whose numerical value may possibly depend on the interaction $\phi$, on the magnetic field $H$ and on the initial state $\boldsymbol{X}$ of the system.

We remark that the previous bound implies that the effective force acting (in average) on a very fast particle is bounded by a constant, in spite of the fact that the infinite size of the system can produce, a priori, large concentrations and then large forces.

We point out that the global solution to the finite system (15) does exist, as it can be easily seen by the conservation of the energy and estimates analogous to those of Appendix A.

## 3. – Proof of Theorem 2.1.

To prove a theorem of existence and uniqueness for the infinite system (15) we should be able to control the dynamics of system (14) uniformly in the number of particles. In other words we should be able to prove that the motion of a particle is not much influenced by far away particles. This is what we expect also from a physical point of view. Thus our proof is constructive. We prove some bounds on the partial dynamics (14), which are «weakly» depending on $n$. This allows us to construct an iterative scheme, analogous to the one used in [5], and to perform the limit as $n \to \infty$.

Before going into the proof of the theorem, we state some preliminary results. A basic tool is an estimate on the growth in time of the local energy, which is the content of the following Proposition, whose proof is given in Appendix B:

PROPOSITION 3.1. – *For each* $\boldsymbol{X} \in \Xi^{(1)} \cap \Xi^{(2)}$ *there exists a constant* $C_3 > 0$ *such that,* $\forall\, n \in \mathbb{N}$,

$$\sup_{\mu} Q(\boldsymbol{X}^n(t); \mu, R_n(t)) \leq C_3 R_n(t) \tag{18}$$

*where*

$$R_n(t) = \log(e+n) + \int_0^t ds\; V_n(s) \tag{19}$$

*and*

$$V_n(t) = \max_{i \in I_n} \sup_{s \in [0,t]} |\boldsymbol{v}_i^n(s)|. \tag{20}$$

REMARK. – Let us note that in Ref. [4] the function $R_n(t)$ was erroneously defined with $V_n(t) = \max_{i \in I_n} \sup_{s\in[0,t]} |\boldsymbol{v}_i^n(s) \cdot \boldsymbol{v}|$.

We also state the following Proposition (proved in Appendix B):

PROPOSITION 3.2. – *For each* $\boldsymbol{X} \in \Xi^{(1)} \cap \Xi^{(2)}$ *there exist* $C_4, C_5 > 0$ *such that, for any* $n \in \mathbb{N}$ *and* $i \in I_n$,

$$|\boldsymbol{v}_i^n(t)| \leq C_4 \sqrt{\log(e+n)} + C_5 t \qquad \forall t \geq 0. \tag{21}$$

After these preliminary results, we start with the proof of Theorem 2.1. In order to compare different partial dynamics we define the quantity

$$\delta_i(n,t) = |\boldsymbol{r}_i^n(t) - \boldsymbol{r}_i^{n-1}(t)| + |\boldsymbol{v}_i^n(t) - \boldsymbol{v}_i^{n-1}(t)|. \tag{22}$$

From the equations of motion in integral form we have:

$$\begin{cases} \boldsymbol{v}_i^n(t) = \boldsymbol{v}_i + \int_0^t ds\, \boldsymbol{F}_i(\boldsymbol{X}^n(s)) \\ \boldsymbol{r}_i^n(t) = \boldsymbol{r}_i + \boldsymbol{v}_i t + \int_0^t ds\, (t-s)\, \boldsymbol{F}_i(\boldsymbol{X}^n(s)). \end{cases} \tag{23}$$

From (22) and (23) it follows that, for any $i \in I_{n-1}$,

$$\begin{aligned} \delta_i(n,t) &\le (1+t)\int_0^t ds |\boldsymbol{F}_i(\boldsymbol{X}^n(s)) - \boldsymbol{F}_i(\boldsymbol{X}^{n-1}(s))| \\ &\le (1+t)\int_0^t ds \Big| \sum_{j:j\neq i} [\nabla\phi(\boldsymbol{r}_i^n(s) - \boldsymbol{r}_j^n(s)) - \nabla\phi(\boldsymbol{r}_i^{n-1}(s) - \boldsymbol{r}_j^{n-1}(s))]\Big| \\ &+ (1+t)\int_0^t ds\, |\boldsymbol{v}_i^n(s) \wedge \boldsymbol{H}(\boldsymbol{r}_i^n(s)) - \boldsymbol{v}_i^{n-1}(s) \wedge \boldsymbol{H}(\boldsymbol{r}_i^{n-1}(s))|. \end{aligned} \tag{24}$$

Let us estimate now the two terms which appear in (24). Let us begin with the second one. Omitting the dependence on $s$ and simply putting $\boldsymbol{H}^n = \boldsymbol{H}(\boldsymbol{r}_i^n)$ we have:

$$\begin{aligned} |\boldsymbol{v}_i^n \wedge \boldsymbol{H}^n - \boldsymbol{v}_i^{n-1} \wedge \boldsymbol{H}^{n-1}| &\le |\boldsymbol{v}_i^n - \boldsymbol{v}_i^{n-1}|\, |\boldsymbol{H}^n| + |\boldsymbol{v}_i^{n-1}|\, |\boldsymbol{H}^n - \boldsymbol{H}^{n-1}| \\ &\le |\boldsymbol{v}_i^n - \boldsymbol{v}_i^{n-1}|\, |\boldsymbol{H}^n| + |\boldsymbol{v}_i^{n-1}| \mathcal{L}\big(\boldsymbol{r}_i^n, \boldsymbol{r}_i^{n-1}\big) |\boldsymbol{r}_i^n - \boldsymbol{r}_i^{n-1}| \end{aligned} \tag{25}$$

where the «Lipschitz constant» $\mathcal{L}$ of the magnetic field depends on $\boldsymbol{r}_i^n$ and $\boldsymbol{r}_i^{n-1}$ and it diverges when $\boldsymbol{r}_i^n$ or $\boldsymbol{r}_i^{n-1}$ goes to the border of the cylinder, as the magnetic field $\boldsymbol{H}^n$ itself. We show, however, that along the solutions of the equations of motion $\boldsymbol{H}(\boldsymbol{r}_i^n(s))$ and $\mathcal{L}\big(\boldsymbol{r}_i^n(s), \boldsymbol{r}_i^{n-1}(s)\big)$ are controlled by the energy of a region containing the particle $i$. More precisely, we have the following estimate, whose proof is given in Appendix A:

$$|\boldsymbol{H}(\boldsymbol{r}_i^n(s))| + \mathcal{L}\big(\boldsymbol{r}_i^n(s), \boldsymbol{r}_i^{n-1}(s)\big) \le C_6[\log^5(e+n) + s^5]^{\frac{a+1}{a-1}}. \tag{26}$$

Recalling Definition (22) for $\delta_i(n,t)$, the quantity appearing in (25) can be bounded as follows (see Appendix A):

$$|\boldsymbol{v}_i^n - \boldsymbol{v}_i^{n-1}||\boldsymbol{H}^n| + |\boldsymbol{v}_i^{n-1}|\mathcal{L}\big(\boldsymbol{r}_i^n, \boldsymbol{r}_i^{n-1}\big)|\boldsymbol{r}_i^n - \boldsymbol{r}_i^{n-1}| \leq C_7[\log(e+n) + s^2]^{\frac{6a+4}{a-1}}\delta_i(n,s). \tag{27}$$

Let us now consider the first term appearing in (24). Due to the long range of the interaction, the sum involves infinite terms as $n \to \infty$. In order to handle this sum we decompose it in the following way. Put

$$\min\{|\boldsymbol{r}_i^{n-1}(s) - \boldsymbol{r}_j^{n-1}(s)|,\ |\boldsymbol{r}_i^n(s) - \boldsymbol{r}_j^n(s)|\} = m_{ij}^n(s)$$

and, fixing a particle $i$, consider the following sets of indices:

$$A_i^n(s) = \{j \neq i : m_{ij}^n(s) \leq \log(e+n)\} \tag{28}$$

$$B_i^n(s) = \{j \neq i : \log(e+n) \leq m_{ij}^n(s) \leq \sqrt{n}\} \tag{29}$$

$$C_i^n(s) = \{j \neq i : m_{ij}^n(s) \geq \sqrt{n}\}. \tag{30}$$

Before going into the computation of the sum appearing in (24) by means of the above sets, we observe that, by (3), the Lipschitz constant of the force is not finite in the set $A_i^n(s)$. Nevertheless, we have for some constant $C_8$:

$$|\nabla\phi(\boldsymbol{x}) - \nabla\phi(\boldsymbol{y})| \leq C_8\{\phi(\boldsymbol{x})^3 + \phi(\boldsymbol{y})^3\}|\boldsymbol{x} - \boldsymbol{y}|. \tag{31}$$

Then, we can write:

$$\begin{aligned} &\Big|\sum_{j:j\neq i}[\nabla\phi(\boldsymbol{r}_i^n(s) - \boldsymbol{r}_j^n(s)) - \nabla\phi(\boldsymbol{r}_i^{n-1}(s) - \boldsymbol{r}_j^{n-1}(s))]\Big| \\ &\qquad \leq \sum_{j\in A_i^n(s)} C_8[\phi^3(\boldsymbol{r}_i^n(s) - \boldsymbol{r}_j^n(s)) + \phi^3(\boldsymbol{r}_i^{n-1}(s) - \boldsymbol{r}_j^{n-1}(s))] \\ &\qquad\qquad \times (\delta_i(n,s) + \delta_j(n,s)) \\ &\qquad + C_9 e^{-\log(e+n)} \sum_{j\in B_i^n(s)} [\delta_i(n,s) + \delta_j(n,s)] \\ &\qquad + C_{10} e^{-\sqrt{n}} \sum_{j\in C_i^n(s)} (|\boldsymbol{r}_i^n(s) - \boldsymbol{r}_j^n(s)| + |\boldsymbol{r}_i^{n-1}(s) - \boldsymbol{r}_j^{n-1}(s)|) \end{aligned} \tag{32}$$

where $C_9 e^{-\log(e+n)}$ and $C_{10}e^{-\sqrt{n}}$ are bounds for the Lipschitz constants of the potential in the sets $B_i^n(s)$ and $C_i^n(s)$ (without loss of generality we take in Eq. (3) $\gamma = 1$).

Working out the r.h.s. of (32), this is bounded by

$$C_{11}\left[\sum_{\ell=n-1}^{n} \sup_{\mu} Q\big(\boldsymbol{X}^{\ell}(t);\mu,1\big)\right]^3 \sum_{j\in A_i^n(s)} \big(\delta_i(n,s)+\delta_j(n,s)\big) \tag{33}$$

$$+ C_{12}\, e^{-\log(e+n)} \sum_{j\in B_i^n(s)} [\delta_i(n,s)+\delta_j(n,s)]$$

$$+ C_{13}\, e^{-\sqrt{n}} Q(\boldsymbol{X}) n^2$$

where in the last term in (33), $nQ(\boldsymbol{X})$ is a bound for the total number of particles in $\Omega(0,n)$, as it follows from Definitions (6) and (7), while the difference $|\boldsymbol{r}_i^{\ell}(s) - \boldsymbol{r}_j^{\ell}(s)|$ has been roughly bounded by $n$, for $\ell = n-1, n$.

By (21) the quantity

$$D(n,s) = C_4 s\sqrt{\log(e+n)} + C_5 s^2 \tag{34}$$

represents a bound for the maximum displacement that a particle can undergo during time $[0,s]$. Hence the number of particles belonging to a set $\Omega(\mu,R)$ at time $s$ in the $n$-th dynamics is at most the number of particles belonging to the set $\Omega(\mu, R + D(n,s))$ at time 0. Moreover, this number can be bounded by the quantity $Q(\boldsymbol{X})[\log(e+n) + R + D(n,s)]$. This observation enables us to go on in the proof. Let us set

$$p^{(1)}(n,t) = 2\log(e+n) + 4D(n,t) \tag{35}$$

$$p^{(2)}(n,t) = 2\sqrt{n} + 4D(n,t) \tag{36}$$

and

$$g^{(h)}(n,t) = Q(\boldsymbol{X})(\log(e+n) + p^{(h)}(n,t)), \qquad h = 1,2, \tag{37}$$

and define

$$u_k(n,t) = \sup_{i\in I_k} \delta_i(n,t). \tag{38}$$

Now, fix an integer $k_0$ and for $n \gg k_0$ let

$$k_1 = Int\,[k_0 + p^{(2)}(n,t)],$$

where $Int\,[\zeta]$ denotes the integer part of $\zeta$. From (24), (33), (35), (36), and (27) we get:

$$
\begin{aligned}
(39)\qquad u_{k_0}(n,t) \le (1+t)\Bigg\{ & C_{11}\Bigg[\sum_{\ell=n-1}^{n} \sup_{\mu} Q\big(\boldsymbol{X}^{\ell}(t);\mu,1\big)\Bigg]^3 g^{(1)}(n,t) \\
& + C_{12}\, e^{-\log(e+n)} g^{(2)}(n,t) \\
& + C_7\,[\log(e+n)+t^2]^{\frac{6a+4}{a-1}}\Bigg\}\int_0^t ds\; u_{k_1}(n,s) \\
& + (1+t)\, tC_{13} e^{-\sqrt{n}} Q(\boldsymbol{X}) n^2.
\end{aligned}
$$

Now, by (18) and (21), we have

$$
(40)\qquad \sup_{\mu} Q(\boldsymbol{X}^n(s);\mu,1) \le C_3[\log(e+n) + C_4 s\sqrt{\log(e+n)} + C_5 s^2]
$$

so that from (39) and the definition of $g^{(2)}(n,t)$ we get

$$
\begin{aligned}
(41)\qquad u_{k_0}(n,t) \le\, & C_{14}\{t^9 + (1+t)[\log(e+n)+t^2]^{\frac{6a+4}{a-1}}\}\int_0^t ds\; u_{k_1}(n,s) \\
& + C_{15}(1+t)tQ(\boldsymbol{X})\, e^{-\sqrt{n}/2}.
\end{aligned}
$$

Let us set

$$
(42)\qquad q(n,t) = C_{14}\Big\{t^9 + (1+t)[\log(e+n)+t^2]^{\frac{6a+4}{a-1}}\Big\}.
$$

We now iterate (41) $m$ times, where $m$ is

$$
(43)\qquad m = Int\left[\frac{n-k_0}{p^{(2)}(n,t)}\right].
$$

In this way we obtain

$$
\begin{aligned}
(44)\qquad u_{k_0}(n,t) \le\, & C_{16}\, a(n,t)(q(n,t))^m \frac{t^m}{m!} \\
& + \frac{C_{17}(1+t)Q(\boldsymbol{X})}{e^{\sqrt{n}/2}} \sum_{h=1}^{m} (q(n,t))^{h-1}\frac{t^h}{h!}
\end{aligned}
$$

where

$$
(45)\qquad a(n,t) = \Big[C_4\sqrt{\log(e+n)} + C_5 t\Big](t+1)
$$

is a bound (by (21)) for

$$
u_{k_m}(n,s) \le \sup_{i\in I_n} \delta_i(n,s).
$$

We choose $n^*$ sufficiently large such that, for a fixed value of $t \geq 0$,

$$Int\left[\frac{n^* - k_0}{p^{(2)}(n^*, t)}\right] \geq 1$$

in accordance with Definition (43). Therefore (42), (43), and (36) prove that the series

$$\sum_{n=n^*}^{\infty} u_{k_0}(n, t) \tag{46}$$

is convergent, i.e. for any $i \in \mathbb{N}$ and $t \geq 0$ $\{\boldsymbol{r}_i^n(t), \boldsymbol{v}_i^n(t)\}$ are Cauchy sequences, hence they converge, as $n \to \infty$, to a limit $\{\boldsymbol{r}_i(t), \boldsymbol{v}_i(t)\}$ which turns out to solve Equations (15). Uniqueness can be proved by exactly the same techniques. The remaining part of the proof, i.e. the fact that $\boldsymbol{X}(t) \in \Xi^{(1)} \cap \Xi^{(2)}$, is shown in Appendix A ($\boldsymbol{X}(t) \in \Xi^{(2)}$ ) and B ($\boldsymbol{X}(t) \in \Xi^{(1)}$). □

PROOF OF THEOREM 2.2. – Fix $i \in \mathbb{N}$ and choose $k_0$ such that $k_0 - 1 \leq |\boldsymbol{r}_i \cdot \boldsymbol{v}| \leq k_0$. We observe that the behavior of the system for long times can be controlled by a different choice of $n^*$, i.e. it is sufficient to admit a dependence of $n^*$ from $t$ of the form

$$n^* = Int\,[e^t(k_0^2 + C_{18})] \tag{47}$$

in such a way that, for $n \geq n^*$

$$t < \log(e + n). \tag{48}$$

The dependence from $k_0$ in the r.h.s. of (47) is chosen in order to have a uniform convergence of (46) for $k_0 \in \mathbb{R}^+$ (as it appears evident from (43)).

Now we have, by (38),

$$|\boldsymbol{v}_i(t) - \boldsymbol{v}_i^{n^*}(t)| \leq \sum_{n \geq n^*} u_{k_0}(n, t) \tag{49}$$

hence,

$$|\boldsymbol{v}_i(t)| \leq |\boldsymbol{v}_i^{n^*}(t)| + C_{19} \tag{50}$$

where by (47) the constant $C_{19}$ is independent from $t$ and $k_0$. Then from (21)

$$\begin{aligned} |\boldsymbol{v}_i^{n^*}(t)| &\leq C_4\sqrt{\log(e + n^*)} + C_5 t \\ &\leq C_4\sqrt{\log\big(e + e^t\big(k_0^2 + C_{17}\big)\big)} + C_5 t \\ &\leq C_{20}\sqrt{\log(e + |\boldsymbol{r}_i \cdot \boldsymbol{v}|)} + C_{21} t, \end{aligned} \tag{51}$$

so that from (50) and (51) Eq. (17) follows. □

## 4. – Some final comments.

We discuss in some detail the nature of the interaction. The assumptions we do, in particular its exponential decay, are valid in some physical systems. We have already mentioned in the Introduction the Debye screening effect, which gives rise to this behavior at large distance. Moreover in our model, due to the fact that the particles are confined in a cylinder, the inter-particle force is of the type we have considered, as we will see at a heuristic level, without going into rigorous proofs.

We recall that the domain $D$ in which the charged particles move has the form:

$$D = \{\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < L^2\}. \tag{52}$$

Let us consider the Green function $G(\boldsymbol{x} - \boldsymbol{x}')$ of the Laplace equation with vanishing boundary conditions on $\partial D$ and at infinity. We study $G(\boldsymbol{x} - \boldsymbol{x}')$ where $\boldsymbol{x}'$ is the position of a fixed charged particle and $\boldsymbol{x}$ is a generic point belonging to $D$. For $\boldsymbol{x}$ close to $\boldsymbol{x}'$ it is well known that $G$ behaves as in the Coulomb case, i.e. it exhibits a singularity like $|\boldsymbol{x} - \boldsymbol{x}'|^{-1}$. The positivity of $G$ follows from the maximum principle. It remains to prove that it decreases exponentially as $|\boldsymbol{x}|$ goes to infinity. A way to see this fact is to use the well known probabilistic interpretation of the solution of the Laplace equation (see for instance [9]).

Let $u(\boldsymbol{x})$ be the solution of the Laplace equation $\triangle u = 0$ with the boundary conditions $u(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\boldsymbol{x} \in \partial D$. Then:

$$u(\boldsymbol{x}) = \int_{\partial D} P(\boldsymbol{z}, \boldsymbol{x}) f(\boldsymbol{z}) d\boldsymbol{z}$$

where $P(\boldsymbol{z}, \boldsymbol{x})$ is the probability that a Brownian motion starting from $\boldsymbol{x}$ reaches $\boldsymbol{z} \in \partial D$ the first time, while the integral represents the average on all possible exit points $\boldsymbol{z} \in \partial D$. Having this in mind, the exponential decreasing of $G$ becomes evident. Indeed, let us consider a small sphere $S$ around $\boldsymbol{x}'$: then, the value of $G$ on $\partial S$ is large but bounded. We cover $D$ by the union of $D_i$, $i = 0, 1, 2, \dots$, being $D_i = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < L^2,\ i \le z \le i + 1\}$. Without loss of generality we take $S \subset D_0$. Now, choose $\boldsymbol{x}$ far from the origin, i.e. belonging to some $D_i$ with a large $i$ and consider the Brownian motion starting from $\boldsymbol{x}$. Notice that when it hits the boundary, by the boundary conditions it does not contribute to $G$, so that the only possibility to have a nonvanishing contribution is to arrive to $\partial S$. However, in this case it must pass through $D_{i-1}$ without reaching the border and the probability to do this is less than 1. Then it must pass through $D_{i-2}$ without reaching the border and so on up to $D_0$. The probability for this to occur is the product of $i - 1$ numbers strictly less than 1, that is exponentially decaying. The study of the behavior at large distance of the derivatives of $G$ can be done in a similar way.

Another physical situation in which the choice (3) for the potential is the right one is the following one. Consider a system of charged particles, confined by an external magnetic field to stay at any time within a layer, that is a domain $D$ of the form:

$$D = \{\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3 : 0 \le z \le L\}. \tag{53}$$

For analogous reasons to the ones described before, the Green function of the Laplace equation with vanishing boundary conditions on the plains $z = 0$, $z = L$ and at infinity, has the same behavior as in the cylindrical case. Moreover, by a straightforward generalization of our techniques, we can obtain existence and uniqueness of the time evolution, relative to a set of initial data belonging to $\Xi^{(1)} \cap \hat{\Xi}^{(2)}$, where $\hat{\Xi}^{(2)}$ is a set slightly different from $\Xi^{(2)}$, suitably defined according to the different geometry of the problem.

To conclude, we remark that our approach also works with any kind of domain having smooth boundaries and possibly unbounded only in one or two dimensions. The magnetic field component parallel to the boundary has to diverge at the boundary itself in order to confine the charged particles inside the domain.

## Appendix A.

In this appendix we prove **a)** the estimates (26) and (27), and **b)** that $\boldsymbol{X}(t) \in \boldsymbol{\Xi}^{(2)}$.

**a)** Firstly, let us start with a rough proof of the fact that the magnetic field confines the system. Consider one particle alone subject to an external electric field $\boldsymbol{E}(t)$ (which simulates the effect of the other particles) and a magnetic field like (5). If the electric field pushes the particle close to the border, the magnetic field obliges it to turn around fastly and, in average, the motion becomes parallel to the border. Mathematically this effect appears in the existence of the function $P$ (see below) that plays the role of a Liapunov function, slowly varying during the motion and exploding on the border. As a consequence the boundary cannot be reached at any finite time.

Now we give the rigorous proof. For the sake of conciseness, we skip the dependence on $n$. For a generic particle with position $\boldsymbol{r}_i(t) = (x_i(t), y_i(t), z_i(t))$, the equations of motion are:

$$\ddot{\boldsymbol{r}}_i(t) = \boldsymbol{E}_i(t) + \dot{\boldsymbol{r}}_i(t) \wedge \boldsymbol{H}(\boldsymbol{r}_i(t)), \tag{A1}$$

denoting by

$$\boldsymbol{E}_i(t) = (E_{i1}(t), E_{i2}(t), E_{i3}(t)) = -\sum_{j:j\neq i} \nabla\phi\big(\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)\big). \tag{A2}$$

The scalar form of (A1) reads as

$$\text{(A3)}\qquad \begin{cases} \ddot{x}_i(t) = \dot{y}_i(t) H(x_i^2(t) + y_i^2(t)) + E_{i1}(t) \\ \ddot{y}_i(t) = E_{i2}(t) - \dot{x}_i(t) H(x_i^2(t) + y_i^2(t)) \\ \ddot{z}_i(t) = E_{i3}(t). \end{cases}$$

From (A3) we get, for a generic $T > 0$,

$$\text{(A4)}\quad \int_0^T [\dot{x}_i(t)x_i(t) + \dot{y}_i(t)y_i(t)] H(x_i^2(t) + y_i^2(t))\,dt$$
$$= \int_0^T [\ddot{x}_i(t)y_i(t) - x_i(t)\ddot{y}_i(t) + x_i(t)E_{i2}(t) - y_i(t)E_{i1}(t)]\,dt,$$

and denoting by $P(\,\cdot\,)$ a primitive of $H(\,\cdot\,)$ the l.h.s. of (A4) can be rewritten as

$$\text{(A5)}\qquad \int_0^T \frac{d}{dt}\left[\frac{1}{2} P\big(x_i^2(t) + y_i^2(t)\big)\right] dt = \left[\frac{1}{2} P\big(x_i^2 + y_i^2\big)\right]_{x_i^2+y_i^2=r_i^2(0)}^{x_i^2+y_i^2=r_i^2(T)}$$

where $r_i(T) = \sqrt{x_i^2(T) + y_i^2(T)}$ is the radial position of the particle. Notice that (A5) diverges to $+\infty$ for $r_i(T) \to L$ (as it comes from (5)) while the r.h.s. of (A4) cannot diverge for finite times (as we are going to see), and this is the reason why the particles remain inside the cylinder of radius $L$ for any positive time.

By (5) and (A5), (A4) can be written as

$$\text{(A6)}\quad \frac{1}{2} P\big(r_i^2(T)\big) - \frac{1}{2} P\big(r_i^2(0)\big)$$
$$= \int_0^T [\ddot{x}_i(t)\, y_i(t) - x_i(t)\, \ddot{y}_i(t) + x_i(t)\, E_{i2}(t) - y_i(t)\, E_{i1}(t)]\,dt.$$

After integration by parts of the r.h.s. of (A6) we can write

$$\text{(A7)}\quad \frac{1}{2} P\big(r_i^2(T)\big) - \frac{1}{2} P\big(r_i^2(0)\big)$$
$$\le L(|\dot{x}_i(T)| + |\dot{x}_i(0)| + |\dot{y}_i(T)| + |\dot{y}_i(0)|) + L \int_0^T [|E_{i2}(t)| + |E_{i1}(t)|] dt.$$

Let us now give suitable bounds to each term in the r.h.s. of (A7): observe that, by (6),

$$\text{(A8)}\qquad |\dot{\boldsymbol{r}}_i^n(s)| \le 2 \sup_\mu Q(\boldsymbol{X}^n(s); \mu, 1)$$

which implies (remember the omitted dependence on $n$)

$$|\dot{x}_i(T)| \leq 2 \sup_\mu Q(\boldsymbol{X}^n(T); \mu, 1), \tag{A9}$$

$$|\dot{x}_i(0)| \leq 2 \sup_\mu Q(\boldsymbol{X}^n(0); \mu, 1), \tag{A10}$$

and analogously for $|\dot{y}_i(T)|$ and $|\dot{y}_i(0)|$. Moreover:

$$\int_0^T |\boldsymbol{E}_i(t)| dt \leq \int_0^T \Big| - \sum_{j:j\neq i} \nabla\phi(\boldsymbol{r}_i^n(t) - \boldsymbol{r}_j^n(t)) \Big| dt \tag{A11}$$

and, by the form of the potential (3), for a positive constant $C_{22}$,

$$|\nabla\phi(\xi)| \leq C_{22}(\phi(\xi) + \phi^2(\xi)). \tag{A12}$$

Then the r.h.s. of (A11) can be bounded by:

$$\begin{aligned} &\int_0^T \Big| \sum_{j:j\neq i} C_{22}\phi(\boldsymbol{r}_i^n(t) - \boldsymbol{r}_j^n(t)) \Big| dt + \int_0^T \Big| \sum_{j:j\neq i} C_{22}\phi^2(\boldsymbol{r}_i^n(t) - \boldsymbol{r}_j^n(t)) \Big| dt \\ &\leq 2C_{22} \int_0^T \sup_\mu Q(\boldsymbol{X}^n(t); \mu, 1)\, dt + 4C_{22} \int_0^T \left( \sup_\mu Q(\boldsymbol{X}^n(t); \mu, 1) \right)^2 dt \\ &\leq 6C_{22} \int_0^T \left( \sup_\mu Q(\boldsymbol{X}^n(t); \mu, 1) \right)^2 dt. \end{aligned} \tag{A13}$$

Putting (A13), (A10), and (A9) in (A7) we obtain:

$$\begin{aligned} \frac{1}{2}P(r_i^2(T)) - \frac{1}{2}P(r_i^2(0)) &\leq 4L \sup_\mu Q(\boldsymbol{X}^n(T); \mu, 1) + 4L \sup_\mu Q(\boldsymbol{X}^n(0); \mu, 1) \\ &\quad + 12LC_{22} \int_0^T \left( \sup_\mu Q(\boldsymbol{X}^n(t); \mu, 1) \right)^2 dt. \end{aligned} \tag{A14}$$

In order to get an analogous bound for the magnetic field $H$, let us make the following considerations. If both $r_i(T)$ and $r_i(0)$ belong to the region in which $L^2 - \delta^2 \leq x^2 + y^2 < L^2$ (let us call it region I), by the explicit form (5) of the function $H$, we get immediatly

$$\frac{1}{2}P(r_i^2(T)) - \frac{1}{2}P\big(r_i^2(0)\big) = \frac{1}{2}\frac{1}{a-1}H\big(r_i^2(T)\big)^{\frac{a-1}{a}} - \frac{1}{2}\frac{1}{a-1}H(r_i^2(0))^{\frac{a-1}{a}}, \tag{A15}$$

so that, in this region, (A14) becomes

$$\text{(A16)}\quad \frac{1}{2}\frac{1}{a-1}H\big(r_i^2(T)\big)^{\frac{a-1}{a}}-\frac{1}{2}\frac{1}{a-1}H\big(r_i^2(0)\big)^{\frac{a-1}{a}}$$
$$\le 4L\sup_\mu Q(\boldsymbol{X}^n(T);\mu,1)+4L\sup_\mu Q(\boldsymbol{X}^n(0);\mu,1)$$
$$+12LC_{22}\int_0^T\Big(\sup_\mu Q(\boldsymbol{X}^n(t);\mu,1)\Big)^2dt.$$

Define region II as the region in which $0\le x^2+y^2<L^2-\delta^2$. If $r_i(T)$ belongs to region I and $r_i(0)$ to region II, we can use that

$$\frac{1}{2}P\big(r_i^2(T)\big)-\frac{1}{2}P(L^2-\delta^2)\le\frac{1}{2}P\big(r_i^2(T)\big)-\frac{1}{2}P\big(r_i^2(0)\big),$$

to obtain (using (A15) for the l.h.s. and (A14) for the r.h.s.):

$$\text{(A17)}\quad \frac{1}{2}\frac{1}{a-1}H\big(r_i^2(T)\big)^{\frac{a-1}{a}}-\frac{1}{2}\frac{1}{a-1}\left[\frac{1}{\delta^{2a}}\right]^{\frac{a-1}{a}}$$
$$\le 4L\sup_\mu Q(\boldsymbol{X}^n(T);\mu,1)+4L\sup_\mu Q(\boldsymbol{X}^n(0);\mu,1)$$
$$+12LC_{22}\int_0^T\left(\sup_\mu Q(\boldsymbol{X}^n(t);\mu,1)\right)^2dt.$$

Finally, if $r_i(T)$ belongs to region II (wherever $r_i(0)$ is), we have:

$$\text{(A18)}\qquad \frac{1}{2}\frac{1}{a-1}H\big(r_i^2(T)\big)^{\frac{a-1}{a}}\le\frac{1}{2}\frac{1}{a-1}\left[\frac{1}{\delta^{2a}}\right]^{\frac{a-1}{a}}.$$

Combining Eqs. (A16), (A17), and (A18), in order to obtain a condition which holds for any $r_i(T)$ and $r_i(0)$ inside the cylinder, we get:

$$\text{(A19)}\quad \frac{1}{2}\frac{1}{a-1}H\big(r_i^2(T)\big)^{\frac{a-1}{a}}\le\frac{1}{2}\frac{1}{a-1}H\big(r_i^2(0)\big)^{\frac{a-1}{a}}+\frac{1}{2}\frac{1}{a-1}\left[\frac{1}{\delta^{2a}}\right]^{\frac{a-1}{a}}+$$
$$+4L\sup_\mu Q(\boldsymbol{X}^n(T);\mu,1)+4L\sup_\mu Q(\boldsymbol{X}^n(0);\mu,1)$$
$$+12LC_{22}\int_0^T\left(\sup_\mu Q(\boldsymbol{X}^n(t);\mu,1)\right)^2dt.$$

For what concerns the Lipschitz function $\mathcal{L}\big(\boldsymbol{r}_i^n, \boldsymbol{r}_i^{n-1}\big)$ that appears in (25), we have

$$\mathcal{L}\big(\boldsymbol{r}_i^n, \boldsymbol{r}_i^{n-1}\big) \leq \max_{r \in \mathcal{A}_i^n} \frac{dH(r^2)}{dr},$$

where $\mathcal{A}_i^n$ denotes the closed interval of extremes $|\boldsymbol{r}_i^n - (\boldsymbol{r}_i^n \cdot \boldsymbol{v})\boldsymbol{v}|$ and $|\boldsymbol{r}_i^{n-1} - (\boldsymbol{r}_i^{n-1} \cdot \boldsymbol{v})\boldsymbol{v}|$ (recall $\boldsymbol{v}$ is the symmetry axis of the cylinder). Consequently for $dH/dr$ we can apply the same argument used for $H$, hielding

$$\text{(A20)} \quad \frac{dH(r^2)}{dr} = \frac{d}{dr}\left[\frac{h(r^2)}{(L^2 - r^2)^a}\right] = 2ar\big(L^2 - r^2\big)^{-a-1} h(r^2) + \frac{1}{(L^2 - r^2)^a}\frac{dh(r^2)}{dr} \leq C_{23}[H(r^2)]^{\frac{a+1}{a}}.$$

By (A19) we have:

$$\text{(A21)} \quad \mathcal{L}\big(\boldsymbol{r}_i^n, \boldsymbol{r}_i^{n-1}\big) \leq \max_{r \in \mathcal{A}_i^n} \frac{dH(r^2)}{dr} \leq C_{23} \max_{r \in \mathcal{A}_i^n} [H(r^2)]^{\frac{a+1}{a}}$$
$$\leq C_{23} \sum_{\ell = n-1}^{n} \Bigg[\Bigg\{ H(r_i^2(0))^{\frac{a-1}{a}} + 8(a-1)L \sup_{\mu} Q(\boldsymbol{X}^\ell(t); \mu, 1)$$
$$+ \left[\frac{1}{\delta^{2a}}\right]^{\frac{a-1}{a}} + 8(a-1)L \sup_{\mu} Q(\boldsymbol{X}^\ell(0); \mu, 1) +$$
$$+ 24(a-1)\, L C_{22} \int_0^t \big(\sup_{\mu} Q(\boldsymbol{X}^\ell(s); \mu, 1)\big)^2 ds \Bigg\}^{\left(\frac{a}{a-1}\right)}\Bigg]^{\left(\frac{a+1}{a}\right)}.$$

Since $\boldsymbol{X} \in \Xi^{(2)}$ we have,

$$\text{(A22)} \qquad \big(L^2 - r_i^2\big) > \frac{1}{[A \log^5 (e + |\boldsymbol{r}_i \cdot \boldsymbol{v}|)]^{\left(\frac{1}{a-1}\right)}}$$

which implies

$$\text{(A23)} \qquad \big(L^2 - r_i^2\big) > \frac{1}{[A \log^5 (e + n)]^{\left(\frac{1}{a-1}\right)}} \qquad \forall i \in I_n.$$

Therefore

$$\text{(A24)} \qquad [H(r_i^2(0))]^{\frac{a-1}{a}} < A \log^5 (e + n) \qquad \forall i \in I_n.$$

By using (A19), (A21), (40), and (A24), the bound (26) follows.

The estimate (27) is a consequence of (26), as it follows from (A8) and (40).

**b)** We prove that if (A22) is satisfied at the initial time it will be satisfied at any successive time. By the form of the magnetic field (5), from (A22) we get

$$H\left(r_i^2\right) < [A \log^5 (e + |\boldsymbol{r}_i \cdot \boldsymbol{v}|)]^{\left(\frac{a}{a-1}\right)} \tag{A25}$$

which implies that at time $t$ it results, from (A19) and (40),

$$H(r_i^2(t)) \leq \Bigg\{ A \log^5 (e + |\boldsymbol{r}_i \cdot \boldsymbol{v}|) + \frac{1}{\delta^{2a-2}} + 24(a-1)LC_{24}t\,[t^2 + \log (e + |r_i \cdot \boldsymbol{v}|)]^2 \Bigg\}^{\frac{a}{a-1}}. \tag{A26}$$

Hence:

$$H\left(r_i^2(t)\right) \leq \left\{A_t \log^5 (e + |\boldsymbol{r}_i \cdot \boldsymbol{v}|)\right\}^{\frac{a}{a-1}} \tag{A27}$$

which is still a condition like (A25). It is immediate from (A27) to get

$$\left(L^2 - r_i^2(t)\right) > \frac{1}{[A_t \log^5 (e + |\boldsymbol{r}_i \cdot \boldsymbol{v}|)]^{\left(\frac{1}{a-1}\right)}}$$

which proves that $\boldsymbol{X}(t) \in \Xi^{(2)}$. □

## Appendix B.

*Proof of Proposition* 3.1.

A notation warning: in the sequel we shall denote by $C$ a generic positive constant whose numerical value may change from line to line, possibly depending on the interaction $\phi$ and on the initial state $\boldsymbol{X}$ of the system.

We introduce a mollified version of $Q(\boldsymbol{X};\mu,R)$ by defining

$$W(\boldsymbol{X};\mu,R) = \sum_i f_i^{\mu,R} \left\{ \frac{\boldsymbol{v}_i^2}{2} + \frac{1}{2} \sum_{j:j\neq i} \phi(\boldsymbol{r}_i - \boldsymbol{r}_j) + 1 \right\} \tag{B1}$$

where

$$f_i^{\mu,R} = f\left(\frac{|\boldsymbol{r}_i \cdot \boldsymbol{v} - \mu|}{R}\right) \tag{B2}$$

and $f \in C^\infty(\mathbb{R}_+)$ is not increasing and satisfies: $f(x) = 1$ for $x \in [0,1]$, $f(x) = 0$ for

$x \geq 2$ and $|f'(x)| \leq 2$. Clearly:

$$Q(\boldsymbol{X};\mu,R) \leq W(\boldsymbol{X};\mu,R) \leq Q(\boldsymbol{X};\mu,2R). \tag{B3}$$

For $0 \leq s \leq t$, we define

$$R_n(t,s) = \log(e+n) + \int_0^t d\tau\ V_n(\tau) + \int_s^t d\tau\ V_n(\tau) \tag{B4}$$

(note that $R_n(t,t) = R_n(t)$ and $R_n(t,0) \leq 2R_n(t)$) and compute

$$\partial_s W(\boldsymbol{X}^n(s);\mu,R_n(t,s)) = \sum_i [\kappa_i(t,s)\varepsilon_i(s) + f_i^{\mu,R_n(t,s)}\dot{\varepsilon}_i(s)] \tag{B5}$$

where, denoting by $\sigma_i^\mu(s)$ the sign of $\boldsymbol{r}_i(s)\cdot\boldsymbol{v}-\mu$,

$$\kappa_i(t,s) = f'\left(\frac{|\boldsymbol{r}_i(s)\cdot\boldsymbol{v}-\mu|}{R_n(t,s)}\right)\left[\frac{\sigma_i^\mu(s)\boldsymbol{v}_i(s)\cdot\boldsymbol{v}}{R_n(t,s)} - \frac{\partial_s R_n(t,s)}{R_n(t,s)^2}|\boldsymbol{r}_i(s)\cdot\boldsymbol{v}-\mu|\right],$$

$$\varepsilon_i(s) = \frac{\boldsymbol{v}_i(s)^2}{2} + \frac{1}{2}\sum_{j:j\neq i}\phi(\boldsymbol{r}_i(s)-\boldsymbol{r}_j(s)) + 1$$

and, to simplify notation, we have omitted the explicit dependence on $n$ of $\boldsymbol{r}_i$, $\boldsymbol{v}_i$, $\kappa_i$ and $\varepsilon_i$.

By the properties of $f$, noticing that $\partial_s R_n(t,s) = -V_n(s)$, and $|\boldsymbol{v}_i(s)| \leq V_n(s)$, then $\kappa_i(t,s) \leq 0$. On the other hand, from the equations of motion it follows that:

$$\dot{\varepsilon}_i(s) = -\sum_{j:j\neq i}\nabla\phi(\boldsymbol{r}_i(s)-\boldsymbol{r}_j(s))\cdot\frac{\boldsymbol{v}_i(s)+\boldsymbol{v}_j(s)}{2}.$$

Notice that $\boldsymbol{v}_i(s)\cdot[\boldsymbol{v}_i(s)\wedge\boldsymbol{H}(\boldsymbol{r}_i(s))] = 0$, so the magnetic field does not enter into the proof of this proposition. Then, by (B5) and using the fact that $\nabla\phi$ is odd,

$$\partial_s W(\boldsymbol{X}^n(s);\mu,R_n(t,s)) \leq -\frac{1}{2}\sum_{i\neq j}(f_i^{\mu,R_n(t,s)} - f_j^{\mu,R_n(t,s)}) \times\nabla\phi\big(\boldsymbol{r}_i(s)-\boldsymbol{r}_j(s)\big)\cdot\boldsymbol{v}_i(s). \tag{B6}$$

Moreover:

$$|f_i^{\mu,R} - f_j^{\mu,R}| \leq 2\frac{|\boldsymbol{r}_i-\boldsymbol{r}_j|}{R}[\chi_i(\mu,2R)+\chi_j(\mu,2R)]. \tag{B7}$$

From (3) we have

$$|\boldsymbol{x}|\cdot|\nabla\phi(\boldsymbol{x})| \leq C(\phi(\boldsymbol{x}) + e^{-\frac{\gamma}{2}|\boldsymbol{x}|}). \tag{B8}$$

Hence by (B8) and (B7) we have, putting $r_{ij}(s) = |\boldsymbol{r}_i(s) - \boldsymbol{r}_j(s)|$,

$$\text{(B9)} \qquad \partial_s W(\boldsymbol{X}^n(s); \mu, R_n(t,s)) \leq -C \frac{\partial_s R_n(t,s)}{R_n(t,s)} \sum_{i \neq j} [e^{-\frac{\gamma}{2} r_{ij}(s)} + \phi(r_{ij}(s))]$$
$$\times \{\chi_i(\mu, 2R_n(t,s)) + \chi_j(\mu, 2R_n(t,s))\} =$$
$$= -2C \frac{\partial_s R_n(t,s)}{R_n(t,s)} \sum_{i \neq j} [e^{-\frac{\gamma}{2} r_{ij}(s)} + \phi(r_{ij}(s))]\, \chi_i(\mu, 2R_n(t,s)).$$

Clearly

$$\sum_{i \neq j} \phi(r_{ij}(s)) \chi_i(\mu, 2R_n(t,s)) \leq 2W(\boldsymbol{X}^n(s); \mu, 2R_n(t,s))$$

whereas

$$\text{(B10)} \quad \sum_{i \neq j} e^{-\frac{\gamma}{2} r_{ij}(s)} \chi_i(\mu, 2R_n(t,s))$$
$$\leq \sum_{m=0}^{+\infty} e^{-m\frac{\gamma}{2}} \sum_{i \neq j} \chi_i(\mu, 2R_n(t,s))\, \chi(m \leq r_{ij}(s) \leq m+1).$$

Observe that, since the potential (3) is non negative and it is strictly positive at the origin, there exist two constants $B_1 > 0$ and $B_2 \geq 0$ depending only on $\phi$, such that for each finite configuration of particles $\{\boldsymbol{r}_1, \dots, \boldsymbol{r}_N\}$ inside the cylinder it results:

$$\text{(B11)} \qquad \frac{1}{2} \sum_{i \neq j} \phi(\boldsymbol{r}_i - \boldsymbol{r}_j) \geq B_1 \sum_k N_k^2 - B_2 N,$$

where $N_k$ is the number of particles in the set $\{k \leq \boldsymbol{r} \cdot \boldsymbol{v} \leq k+1\}$, $k \in \mathbb{Z}$.

Let us examine, in (B10), the term

$$\text{(B12)} \quad \sum_{i \neq j} \chi_i(\mu, 2R_n(t,s)) \chi(m \leq r_{ij}(s) \leq m+1)$$
$$= \sum_{i \neq j} \chi_i(\mu, 2R_n(t,s)) \chi(m \leq r_{ij}(s) \leq m+1) \sum_{\substack{k,k' \\ m \leq |k-k'| \leq m+1}} \chi_k(\boldsymbol{r}_i(s)) \chi_{k'}(\boldsymbol{r}_j(s))$$

denoting by

$$\chi_k(\boldsymbol{r}_i(s)) = \chi(k \leq \boldsymbol{r}_i(s) \cdot \boldsymbol{v} \leq k+1) \qquad k \in \mathbb{Z}.$$

Consequently, the number of particles of the $k$-th cell is

$$N_k = \sum_i \chi_k(\boldsymbol{r}_i(s)).$$

In this way we can write

$$\text{r.h.s. of (B12)} \le \sum_{\substack{k,k' \\ |k-\mu|\le 2R_n(t,s) \\ m\le|k-k'|\le m+1}} N_k N_{k'} \le \sum_{\substack{k,k' \\ |k-\mu|\le 2R_n(t,s) \\ m\le|k-k'|\le m+1}} \left(N_k^2 + N_{k'}^2\right)$$

$$= \sum_{\substack{k \\ |k-\mu|\le 2R_n}} \sum_{\substack{k' \\ m\le|k-k'|\le m+1}} N_k^2 + \sum_{\substack{k \\ |k-\mu|\le 2R_n}} \sum_{\substack{k' \\ m\le|k-k'|\le m+1}} N_{k'}^2$$

$$\le 5 \sum_{\substack{k \\ |k-\mu|\le 2R_n}} N_k^2 + \sum_{\substack{k \\ |k-\mu|\le 2R_n}} (N_{k-m-1}^2 + N_{k-m}^2 + N_{k+m}^2 + N_{k+m+1}^2)$$

$$\le 9C \sup_\mu W(\boldsymbol{X}^n(s); \mu, 2R_n(t,s))$$

where we have used (B11) and the fact that $N_{k+m}$ is simply the number of particles in the «shifted» cell $k+m$.

Recalling (B10), we have

$$\sum_{i\neq j} e^{-\frac{\gamma}{2}r_{ij}(s)}\chi_i(\mu, 2R_n(t,s)) \le 9C \sup_\mu W(\boldsymbol{X}^n(s);\mu, 2R_n(t,s)) \sum_{m=0}^{+\infty} e^{-m\frac{\gamma}{2}}$$

$$= \frac{9C}{1-e^{-\frac{\gamma}{2}}} \sup_\mu W(\boldsymbol{X}^n(s);\mu, 2R_n(t,s)).$$

Going back to (B6) we get

$$\text{(B13)} \qquad \partial_s W(\boldsymbol{X}^n(s);\mu, R_n(t,s)) \le -C\frac{\partial_s R_n(t,s)}{R_n(t,s)} \sup_\mu W(\boldsymbol{X}^n(s);\mu, 2R_n(t,s)).$$

Notice a «covering» property of $W$, that is

$$W(\boldsymbol{X};\mu,2R) \le \sum_{|k|\le 2} W(\boldsymbol{X};\mu+2kR, R).$$

Then we have

$$\text{(B14)} \qquad \sup_\mu W(\boldsymbol{X};\mu,2R) \le 5W(\boldsymbol{X};R),$$

where we define

$$W(\boldsymbol{X};R) = \sup_\mu W(\boldsymbol{X};\mu,R).$$

Hence (B13) becomes

$$\text{(B15)} \qquad \partial_s W(\boldsymbol{X}^n(s);\mu, R_n(t,s)) \le -C\frac{\partial_s R_n(t,s)}{R_n(t,s)} W(\boldsymbol{X}^n(s); R_n(t,s))$$

from which, by integrating and taking the supremum on $\mu$,

$$W(\boldsymbol{X}^n(s); R_n(t,s)) \leq W(\boldsymbol{X}^n(0); R_n(t,0)) - C\int_0^s d\tau \, \frac{\partial_s R_n(t,\tau)}{R_n(t,\tau)} W(\boldsymbol{X}^n(\tau); R_n(t,\tau))$$

which gives, by Gronwall's lemma, remembering that $C > 0$ and $\partial_s R_n(t,\tau) = -V_n(\tau)$,

$$W(\boldsymbol{X}^n(s); R_n(t,s)) \leq [W(\boldsymbol{X}^n(0); R_n(t,0))] \left(\frac{R_n(t,0)}{R_n(t,s)}\right)^C . \tag{B16}$$

Putting $s = t$ and using that $R_n(t,0) \leq 2R_n(t,t) = 2R_n(t)$, we get

$$W(\boldsymbol{X}^n(t); R_n(t)) \leq CW(\boldsymbol{X}^n(0); R_n(t)).$$

Finally, by (B3) and (7) we have

$$\begin{aligned} Q(\boldsymbol{X}^n(t); \mu, R_n(t)) \leq & CW(\boldsymbol{X}^n(0); R_n(t)) \\ \leq & C \sup_\mu Q(\boldsymbol{X}^n(0); \mu, 2R_n(t)) \\ \leq & CQ(\boldsymbol{X}) 4R_n(t) \end{aligned}$$

which ends the proof of the Proposition. □

*Proof that $\boldsymbol{X}(t) \in \Xi^{(1)}$.*

We want to prove now that, if $Q(\boldsymbol{X}) < +\infty$, then $Q(\boldsymbol{X}(t)) < +\infty$. We shall prove in fact that:

$$Q(\boldsymbol{X}(t); \mu, R) \leq C[R + 1 + t^2] \tag{B17}$$

which clearly gives $Q(\boldsymbol{X}(t)) < +\infty$ and it gives also its correct quadratic growth in time. By (B3) it is enough to prove (B17) with $Q(\boldsymbol{X}(t); \mu, R)$ replaced by $W(\boldsymbol{X}(t); \mu, R)$.

Given $\beta \geq 1$, $\mu \in \mathbb{R}$ and $R > \log(e + |\mu|)$ let

$$n_0 = n_0(\mu, R, t) = Int\,[\beta e^{2R+t}]. \tag{B18}$$

Clearly $\log(e + n_0) \geq R$ so that, by (B3), (18), (19), and (21),

$$\begin{aligned} W(\boldsymbol{X}^{n_0}(t); \mu, R) &\leq Q(\boldsymbol{X}^{n_0}(t); \mu, 2R_{n_0}(t)) \\ &\leq CR_{n_0}(t) \leq C[\log(e + n_0) + t^2] \\ &\leq C[R + 1 + t^2]. \end{aligned} \tag{B19}$$

On the other hand:

$$W(\boldsymbol{X}(t);\mu,R) \le W(\boldsymbol{X}^{n_0}(t);\mu,R) + \sum_{n>n_0} |W(\boldsymbol{X}^n(t);\mu,R) - W(\boldsymbol{X}^{n-1}(t);\mu,R)| \tag{B20}$$

and the sum on the r.h.s. of (B20), by the choice (B18) of $n_0$, converges uniformly with respect to $\mu \in \mathbb{R}$, $R > \log(e+|\mu|)$ and $t \ge 0$, so it is bounded by a constant (this can be seen using the same techniques of Proposition 3.1). □

*Proof of Proposition* 3.2.

For any $t \ge 0$ let $N_n(\mu,t)$ be the number of all the particles $i \in I_n$ such that $\boldsymbol{r}_i^n(t) \in \Omega(\mu, R_n(t))$ and $|\boldsymbol{v}_i^n(t)| > b_n(t)$, with

$$b_n(t) = C_{25}\sqrt{\log(e+n)} + \frac{V_n(t)}{2}, \tag{B21}$$

where $C_{25} > 0$ will be fixed later and $V_n(t)$ is defined in (20). Then:

$$Q(\boldsymbol{X}^n(t);\mu,R_n(t)) > \frac{b_n(t)^2}{2} N_n(\mu,t)$$

so that, by Proposition 3.1 and using Definitions (19) and (20),

$$N_n(\mu,t) < 8C_3 \frac{\log(e+n) + tV_n(t)}{\left(2C_{25}\sqrt{\log(e+n)} + V_n(t)\right)^2}$$

from which, after neglecting some positive terms,

$$N_n(\mu,t) < \frac{2C_3}{C_{25}^2} + \frac{8C_3 t}{2C_{25} + V_n(t)}. \tag{B22}$$

We now choose $C_{25} = 2\sqrt{2C_3}$; by (B22), if $V_n(t) \ge 32C_3 t$ then $N_n(\mu,t) < 1/2$, i.e. $N_n(\mu,t) = 0$. The above argument is independent of $\mu$, so that $V_n(t) \ge 32C_3 t$ actually implies $|\boldsymbol{v}_i^n(t)| \le b_n(t)$ for all $i \in I_n$. Since $b_n(t)$ is not decreasing, we have in fact $V_n(t) \le b_n(t)$ when $V_n(t) \ge 32C_3 t$. Thus

$$V_n(t) \le b_n(t) + 32C_3 t \qquad \forall t \ge 0$$

and, by (21) it follows:

$$V_n(t) \le 2C_{25}\sqrt{\log(e+n)} + 64C_3 t \qquad \forall t \ge 0,$$

that is inequality (21). □

## REFERENCES


[1] R. ALEXANDER, *Time evolution for infinitely many hard spheres*, Commun. Math. Phys., **49** (1976), 217-232.

[2] C. BAHN - Y.M. PARK - H.J. YOO, *Nonequilibrium dynamics of infinite particle systems with infinite range interactions*, J. Math. Phys., **40** (1999), 4337-4358.

[3] P. BUTTÀ - E. CAGLIOTI - C. MARCHIORO, *On the long time behavior of infinitely extended systems of particles interacting via a Kac potential*, J. Stat. Phys., **108** (2002), 317-339.

[4] P. BUTTÀ - E. CAGLIOTI - C. MARCHIORO, *On the motion of a charged particle interacting with an infinitely extended system*, Commun. Math. Phys., **233** (2003), 545-569.

[5] E. CAGLIOTI - C. MARCHIORO - M. PULVIRENTI, *Non-equilibrium dynamics of three-dimensional infinite particle systems*, Commun. Math. Phys., **215** (2000), 25-43.

[6] E. CAGLIOTI - C. MARCHIORO, *On the long time behavior of a particle in an infinitely extended system in one dimension*, J. Stat. Phys., **106** (2002), 663-680.

[7] P. CALDERONI - S. CAPRINO, *Time evolution of infinitely many particles: an existence theorem*, J. Stat. Phys., **28** (1982), 815-833.

[8] R.L. DOBRUSHIN - J. Fritz, *Non-equilibrium dynamics of one-dimensional infinite particle systems with a hard-core interaction*, Commun. Math. Phys., **55** (1977), 275-292.

[9] A. FRIEDMAN, *Stochastic differential equations and applications*, vol. I, Academic Press New York (1975).

[10] J. FRITZ - R.L. DOBRUSHIN, *Non-equilibrium dynamics of two-dimensional infinite particle systems with a singular interaction*, Commun. Math. Phys., **57** (1977), 67-81.

[11] O.E. LANFORD, *Classical Mechanics of one-dimensional systems with infinitely many particles. I An existence theorem*, Commun. Math. Phys., **9** (1968), 176-191.

[12] O.E. Lanford, *Classical Mechanics of one-dimensional systems with infinitely many particles. II Kinetic Theory*, Commun. Math. Phys. **11** (1969), 257-292.

[13] O.E. LANFORD, *Time evolution of large classical systems*, Moser Ed. Lect. Notes in Phys., Springer **38** (1975).

[14] C. MARCHIORO - A. PELLEGRINOTTI - E. PRESUTTI, *Existence of time evolution in υ-dimensional Statistical Mechanics*, Commun. Math. Phys., **40** (1975), 175-185.

[15] C. MARCHIORO - A. PELLEGRINOTTI - M. PULVIRENTI, *Remarks on the existence of non-equilibrium dynamics*, Coll. Math. Soc. Janos Bolyai, Proceedings Esztergom Summer School, **27** (1978), 733-746.

[16] C. MARCHIORO - M. PULVIRENTI, *Time evolution of infinite one-dimensional Coulomb Systems*, J. Stat. Phys., **27** (1982), 809-822.

[17] E. PRESUTTI - M. PULVIRENTI - B. TIROZZI, *Time evolution of infinite classical systems with singular, long-range, two-body interaction*, Commun. Math. Phys., **47** (1976), 81-95.

[18] M. PULVIRENTI, *On the time evolution of the states of infinitely extended particle systems*, J. Stat. Phys., **27** (1982), 693-713.

[19] D. RUELLE, *Statistical Mechanics. Rigorous Results*, W.A. Benjamin, Inc., New York-Amsterdam 1969.

[20] R. SIGMUND-SCHULTZE, *On non-equilibrium dynamics of multidimensional infinite particle systems in the translation invariant case*, Commun. Math. Phys., **100** (1985), 245-265.

[21] Y. SINAI, *Construction of the dynamics for one-dimensional systems of Statistical Mechanics*, Sov. Theor. Math. Phys., **12** (1973), 487-501.
[22] Y. SINAI, *The construction of the cluster dynamics of dynamical systems in Statistical Mechanics*, Vest. Moskow Univ. Sez. I Math. Mech., **29** (1974), 153-176.

P. Buttà - G. Cavallaro - C. Marchioro: Dipartimento di Matematica,
Università di Roma «La Sapienza», P.le Aldo Moro 2, 00185 Roma (Italy)
e-mail: butta@mat.uniroma1.it, cavallar@mat.uniroma1.it, marchior@mat.uniroma1.it

S. Caprino: Dipartimento di Matematica, Università di Roma «Tor Vergata»,
Via della Ricerca Scientifica 1, 00133 Roma (Italy)
e-mail: caprino@mat.uniroma2.it

*Pervenuta in Redazione*
*il 19 dicembre 2003*