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Sunto. – *In questa nota viene fornita una trattazione geometrica della teoria dell'omologia di intersezione.*

Summary. – *We give, here, a geometric treatment of intersection homology theory.*

Introduction.

In [2] M. Goreski and R. MacPherson developed a theory of intersection of homology cycles on a pseudomanifold X, generalization of the Poincaré-Lefschetz theory. In their paper the authors attached to an oriented stratified pseudomanifold X a collection of groups $\{IH_n^{\bar{p}}(X)\}_{n\geq 0}$, called intersection homology groups, by using cycles and homologies, as in classical simplicial theory, such that their supports meet the strata of X according to the perversity $\bar{p}$.

In this note we give a different geometrical approach to this theory by showing that the objects may be defined starting from singular geometric cycles.

More in detail, given a stratified pseudomanifold X, we construct (sections 2. and 3.) a collection of groups $\{H_n^{\bar{p}}(X)\}_{n\geq 0}$, called $\bar{p}$-transverse homology groups of X, and we prove (section 4.) that for each $n \geq 0$ the group $H_n^{\bar{p}}(X)$ coincides, up to an isomorphism, with the homology intersection group $IH_n^{\bar{p}}(X)$.

We think that our different approach could make easier to investigate some important questions and to outcome new results.

For example, the groups $H_n^{\bar{p}}(X)$ have functorial properties, if one defines appropriate maps between stratified pseudomanifolds, and they result invariant with respect to a suitable definition of homotopy. The above investigation and their relative results will be object of a later paper.

For the reader's convenience we include in this paper a section that provides the definitions of stratified pseudomanifold and homology intersection groups.

1. – Preliminaries.

A stratified pseudomanifold X of dimension m is a geometric m-cycle (i.e. the closure of the union of the m-simplices in any triangulation of X, and each (m-1)-

simplex is a face of exactly two m-simplices), with a filtration by subpolyhedra

$$X = X_m \supseteq X_{m-1} = X_{m-2} \supseteq ... \supseteq X_1 \supseteq X_0$$

such that for each point $x \in X_i - X_{i-1}$ there is a filtered space

$$Y = Y_m \supseteq Y_{m-1} \supseteq ... \supseteq Y_1 \supseteq Y_0 = \text{a point}$$

and a map $Y \times D^i \to X$ which for each j, takes $Y_j \times D^i$ PL-homeomorphically to a neighborhood of x in X_j. (Here, D^i is the PL-disk of dimension i).

If $X_i - X_{i-1}$ is not empty, it is a PL manifold of dimension i, called the i-dimensional stratum of the stratification.

A perversity $\bar{p}$ is a sequence of integers $(p_j)_{j\geq 2}$ such that $p_2 = 0$ and $p_{j+1} = p_j$ or $p_{j+1} = p_j + 1$.

A compact polyhedron $P \subseteq X$ of dimension $\leq h$ is said to be $(\bar{p}, h)$-allowable to X if dim $(P \cap X_i) \leq i + h - m + p_{m-i}$ for each $i \leq m - 2$.

Let $IC^{\bar{p}}(X)$ be the chain complex of simplicial chains whose support is $(\bar{p}, .)$-allowable to X, for each $n \geq 0$ the nth homology group of $IC^{\bar{p}}(X)$, denoted by $IH_n^{\bar{p}}(X)$, is the homology intersection group introduced by Goreski and MacPherson in [2].

2. – Transverse cycles and transverse homology.

Let X be a stratified pseudomanifold of dimension m, and let $\bar{p}$ a perversity.

A $\bar{p}$-transverse n-cycle without boundary of X is a pair (C, f), where C is an oriented geometric n-cycle without boundary and $f : C \to X$ is a simplicial map such that the polyhedron $f(C)$ is $(\bar{p}, n)$-allowable to X, that is

$$\dim (f(C) \cap X_i) \leq i + n - m + p_{m-i}$$

for each polyhedron X_i of the filtration of X which meets $f(C)$.

A $\bar{p}$-transverse n-cycle with boundary of X is a pair (C, f), where C is an oriented geometric n-cycle with boundary ∂C, and $f : C \to X$ is a simplicial map such that $f(C)$ is $(\bar{p}, n)$-allowable to X and $f(\partial C)$ is $(\bar{p}, n-1)$-allowable to X, that is

$$\dim (f(C) \cap X_i) \leq i + n - m + p_{m-i}$$

for each polyhedron X_i of the filtration of X which meets $f(C)$, and

$$\dim (f(\partial C) \cap X_i) \leq i + n - m - 1 + p_{m-i}$$

for each polyhedron X_i of the filtration of X which meets $f(\partial C)$.

Clearly, given a $\bar{p}$-transverse (n+1)-cycle with boundary (C, f) of X, the singular n-cycle $\partial(C, f) = \left(\partial C, f/_{\partial C}\right)$ is a $\bar{p}$-transverse n-cycle without boundary of X, called boundary of (C, f).

REMARK 2.1. – *If* $\bar{p} = (0, 1, 2, ..., j, j+1, ...)$ *every singular n-cycle is a* $\bar{p}$-*transverse n-cycle.*

REMARK 2.2. – *Let* (C, f) *be a* $\bar{p}$-*transverse n-cycle of* X, *then if* $h \geq n$ *each h-cycle* (C', f') *is a* $\bar{p}$-*transverse h-cycle provided that* $f'(C') \subseteq f(C)$.

Two $\bar{p}$-transverse n-cycles without boundary of X (C, f) and (C', f') are said $\bar{p}$-transverse homologous if there exists a $\bar{p}$-transverse (n+1)-cycle with boundary (W, F) such that

$$1)\ \partial W = C \,\dot{\cup} - C'$$
$$2)\ F/_C = f, \quad F/_{C'} = f'$$

where $-C'$ is the cycle obtained from C' by reversing the orientation.

(W, F) is said a $\bar{p}$-transverse homology between (C, f) and (C', f').

Let (C, f) be a $\bar{p}$-transverse n-cycle of X, consider the map $F : C \times I \to X$ defined by $F(y, t) = f(y)$ for each $t \in I$. By Remark 2.2 it follows that $(C \times I, F)$ is a $\bar{p}$-transverse (n+1)-cycle of X.

The gluing property of the geometric cycles with boundary can be extended to $\bar{p}$-transverse cycles with boundary as follows.

Let (C', f') and (C'', f'') be two $\bar{p}$-transverse (n+1)-cycles with boundary, and suppose that C is an oriented geometric n-cycle contained, up to a PL homeomorphism, in $\partial C'$ and in $\partial C''$, and such that $f'/_C = f''/_C$. Consider the oriented geometric (n+1)-cycle W obtained by gluing C' and C'' by a PL map $C \to C$ orientation reversing, and $f : W \to X$ the unique map which extends f' and f'' to W (usually denoted by $f' \cup f''$). We have that $f(W) = f'(C') \cup f''(C'')$, and hence $\dim f(W) = \max \dim (f'(C'), f''(C''))$. So (W, f) is a $\bar{p}$-transverse (n+1)-cycle of X.

As immediate consequence of the above we have that the $\bar{p}$-transverse homology relation is an equivalence relation.

PROPOSITION 2.3. – *Let* x *be a point of a stratified pseudomanifold* X *of dimension* m. *If* $x \notin X_i$, *for each subpolyhedron* X_i *of the filtration of* X *such that* $i < m - n - p_{m-i}$, *then an n-cycle* (C, c_x), *where* c_x *is the constant map to* x, *is a* $\bar{p}$-*transverse n-cycle. Furthermore two* $\bar{p}$-*transverse n-cycles* (C, c_x) *and* $(C', c_{x'})$ *are* $\bar{p}$-*transverse homologous.*

PROOF. – (C, c_x) is a $\bar{p}$-transverse n-cycle because

$$\dim (c_x(C) \cap X_i) = \dim (x \cap X_i) = 0; \quad 0 \leq i + n - m + p_{m-i} \text{ if } i \geq m - n - p_{m-i}.$$

Two $\bar{p}$-transverse n-cycles are $\bar{p}$-transverse homologous because (C, c_x) and $(-C', c_{x'})$ are the complete boundary of the $\bar{p}$-transverse (n+1)-cycle (W, F) where W is the disjoint union of the cone on C with the cone on $-C'$, and F is the map which carries the first cone to x and the second cone to x'. ∎

PROPOSITION 2.4. – *Let (C, f) be a $\bar{p}$-transverse n-cycle of a stratified pseudomanifold X of dimension m, and let $\bar{y}$ be a point of an n-dimensional simplex Δ of C. Then (C, f) is $\bar{p}$-transverse homologous to a $\bar{p}$-transverse cycle (C, f') where f' is a map which agrees with f on $C - \mathring{\Delta}$ and carries an n-simplex $\Delta' \subset \mathring{\Delta}$ $(\bar{y} \in \Delta')$ on $f(\bar{y})$.*

PROOF. – Let $H : \Delta \times I \to \Delta$ be a homotopy between the identity and a map $g : \Delta \to \Delta$ such that $g/_{\dot{\Delta}} = id_{\dot{\Delta}}$, $g(\Delta') = \bar{y}$.

Let $W = C \times I$, $F : W \to X$ the map defined by

$$F(y,t) = \begin{cases} f(y) & \text{if } \ y \notin \mathring{\Delta} \\ f \circ H(y,t) & \text{if } \ y \in \Delta \end{cases}$$

By Remark 2.2 (W, F) is a $\bar{p}$-transverse (n+1)-cycle of X because $F(W) = f(C)$. So, let $f' = F/_{C \times \{1\}}$, the $\bar{p}$-transverse n-cycle (C, f') is $\bar{p}$-transverse homologous to (C, f) and it satisfies the required conditions. ■

By Prop. 2.3 the $\bar{p}$-transverse n-cycles (C, c_x) belong to the same $\bar{p}$-transverse homology class. Every representative of such a class is called 0-homologous. A fundamental characterization of these n-cycles is given by the following

PROPOSITION 2.5. – *A $\bar{p}$-transverse n-cycle of X is* 0*-homologous if and only if it is a boundary.*

PROOF. – Let (C, f) be a $\bar{p}$-transverse n-cycle 0-homologous and let (W, F) be a $\bar{p}$-transverse homology between (C, f) and (C_0, c_x).

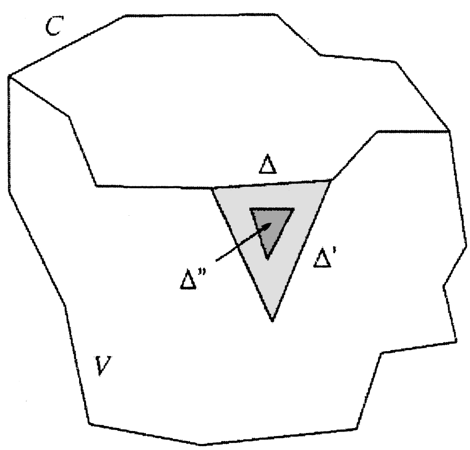

Fig. 1.

Consider the cycle (W', F') where W' is obtained by gluing W to a cone L on C_0, and F' coincides with F on W and with the constant map to x on L. By Remark 2.2 (W', F') is a $\bar{p}$-transverse cycle because $F'(W') = F(W)$. Moreover the boundary of (W', F') is (C, f).

Conversely, assume (C, f) is the boundary of the $\bar{p}$-transverse cycle (V, G). Let Δ be a top dimensional simplex of C. Suppose $f(\Delta) \subseteq X_i$ and $f(\Delta) \nsubseteq X_{i-1}$. If Δ is a face of an (n+1)-simplex Δ' of V, then $G(\Delta') \nsubseteq X_{i-1}$. Let Δ'' be an (n+1)-simplex of V contained in $\overset{\circ}{\Delta}{}'$. (See Fig. 1).

By Prop. 2.4 we can suppose that, up to $\bar{p}$-transverse homology, G is constant on Δ'', that is $G(\Delta'') = \overline{x} \in X_i$, $i \geq m - n - p_{m-i}$. The cycle $\left(\partial\Delta'', G/_{\partial\Delta''}\right)$ is a $\bar{p}$-transverse cycle 0-homologous. The pair $(V - \overset{\circ}{\Delta}{}'', G/)$ is a $\bar{p}$-transverse homology between (C, f) and $\left(\partial\Delta'', G/_{\partial\Delta''}\right)$. Then (C, f) is 0-homologous. ■

3. – The transverse homology groups.

Let X be a stratified pseudomanifold of dimension m, and let $\bar{p}$ a perversity.

Denote by $H_n^{\bar{p}}(X)$ $(n \geq 0)$ the set of $\bar{p}$-transverse homology classes of $\bar{p}$-transverse n-cycles without boundary of X.

Now given two $\bar{p}$-transverse n-cycles (C, f), (C', f') of X, it is easy to see that $(C \dot{\cup} C', f \cup f')$ is a $\bar{p}$-transverse n-cycle of X and that its $\bar{p}$-transverse homology class depends only on the $\bar{p}$-transverse homology class of (C, f) and (C', f').

Then it makes sense to define:

$$[(C, f)] + [(C', f')] = [(C \dot{\cup} C', f \cup f')]$$

PROPOSITION 3.1. – $\left(H_n^{\bar{p}}(X), +\right)$ *is an abelian group for each* $n \geq 0$.

PROOF. – The associativity and commutativity property of the addition are evident.

The zero element of $(H_n^{\bar{p}}(X), +)$ is the class of the n-cycles 0-homologous.

For, the sum of a $\bar{p}$-transverse n-cycle (C, f) with a 0-homologous n-cycle (C_0, c_x) is $\bar{p}$-transverse homologous to (C, f), and a $\bar{p}$-transverse homology between them is given by the cycle (W, f) where W is the disjoint union of the cylinder $C \times I$ on C with the cone L on $-C_0$, and $F : W \to X$ is the constant map to x in L, while $F/_{C \times I}$ is the map defined by $F(y, t) = f(y)$ for each $t \in I$.

From Prop. 2.5 it follows that the inverse element of the class of (C, f) is given by the class of $(-C, f)$. ■

4. – The main theorem.

In this section we prove that the $\bar{p}$-transverse homology groups $H_n^{\bar{p}}(X)$ are isomorphic to the intersection homology groups $IH_n^{\bar{p}}(X)$ introduced by Goreski and MacPherson in [2].

Let $\xi = \sum m_h \sigma_h$ be a simplicial n-cycle of X. Consider the geometric realization of ξ, that is the singular geometric cycle (C_ξ, f_ξ) where C_ξ is the geometric cycle obtained by taking, for each h, m_h copies Δ_h^n of standard n-simplex Δ^n if $m_h > 0$, $-m_h$ copies of $-\Delta^n$ if $m_h < 0$, and gluing two copies $\Delta_{h'}^n$ and $\Delta_{h''}^n$ along an $(n-1)$-face with opposite orientations, and f_ξ is a simplicial map such that $f_\xi(\Delta_h^n) = \sigma_h$. Since $f_\xi(C_\xi)$ coincides with the support $|\xi|$ of the simplicial cycle ξ, we have that, if, and only if, ξ lies in $IC_n^{\bar{p}}(X)$, then (C_ξ, f_ξ) is a $\bar{p}$-transverse cycle of X. Moreover, if ξ is a boundary, then (C_ξ, f_ξ) is also a boundary.

Then, if $n \leq \dim X$, it makes sense to consider the map $\Psi : IH_n^{\bar{p}}(X) \to H_n^{\bar{p}}(X)$ defined by

$$\Psi([\xi]) = [(C_\xi, f_\xi)]$$

It is easy to see that Ψ is a homomorphism.

In order to prove that Ψ is an isomorphism we need the following lemma:

LEMMA 4.1. – *Let (C, f) be a $\bar{p}$-transverse n-cycle of X. If $\dim f(C) < n$, then (C, f) is 0-homologous.*

PROOF. – Since $\dim f(C) < n$, we have that $H_n(f(C)) = 0$. So (C, f) is the boundary of a singular geometric cycle (C', F) of $f(C_n)$. Being (C, f) a $\bar{p}$-transverse cycle and $F(C') \subseteq f(C)$, by Remark 2.2 (C', F) is a $\bar{p}$-transverse cycle of X. The assert follows by Prop. 2.5. ■

From the above lemma it follows:

REMARK 4.2. – $H_n^{\bar{p}}(X) = 0$ *for each* $n > \dim X$.

LEMMA 4.3. – *Let (C, f) be a $\bar{p}$-transverse n-cycle of X. If $\dim f(C) = n$, then there exists a $\bar{p}$-transverse n-cycle (C', f') $\bar{p}$-transverse homologous to (C, f) such that f' is injective on each top dimensional simplex of C'.*

PROOF. – Let P' be the subpolyhedron of C consisting of n-simplices on which f is one to one, with their faces, and let P'' be the subpolyhedron consisting of the remaining n-simplices and their faces. Let $Q = P' \cap P''$, we observe that each (n-1)-simplex of $P' - Q$ (or $P'' - Q$) is a face of two n-simplices of P' (or P'') whereas each (n-1)-simplex of Q is a face of an n-simplex of P' and a face of an n-simplex of P''. The polyhedron P' (or P'') is not in general a cycle just because Q is not in general a cycle.

Briefly, the idea of the proof is that of obtaining, starting from $C = P' \cup P''$, by appropriate identifications on Q, two $\bar{p}$-transverse cycles without boundary (C', f') and (C'', f'') such that (C'', f'') is 0-homologous and (C', f') is $\bar{p}$-transverse homologous to (C, f).

Step 1 Construction of (C', f') and (C'', f'').

Let $\sigma = (V_1, ..., V_n)$ be an (n-1)-simplex of Q and let $\tau = (V_0, V_1, ..., V_n)$ the n-simplex of P'' such that $\sigma \prec \tau$. There exists an unique (n-1)-face $\sigma_1 \neq \sigma$ of τ such that $f(\sigma_1) = f(\sigma)$. If $\sigma_1 = \left(V_0, ..., V_{i-1,} V_{i+1}, ..., V_n\right)$ is contained in Q we identify σ to σ_1 by the orientation-reversing simplicial isomorphism φ defined by putting $\varphi(V_r) = V_s \Leftrightarrow f(V_r) = f(V_s)$. If σ_1 is not contained in Q, it is a face of another simplex $\tau_1 \neq \tau$ of P'' and there exists an unique (n-1)-face $\sigma_2 \neq \sigma_1$ of τ_1 such that $f(\sigma_1) = f(\sigma_2)$. If σ_2 is contained in Q we identify σ_2 to σ_1 as before, otherwise we can iterate the proceeding and we obtain a sequence of simplices $\sigma \prec \tau \succ \sigma_1 \prec \tau_1 \succ \sigma_2 ... \succ \sigma_h \prec \tau_h ....$ We show that the simplices of the previous sequence are distinct.

Suppose by induction that the assert is true for the sequence $\sigma \prec \tau \succ \sigma_1 \prec \tau_1 \succ \sigma_2 ... \prec \tau_{l-1} \succ \sigma_l$, then we have

1) $\tau_l \neq \tau_{l-1}$ by construction;
2) $\tau_l \neq \tau_i$ if $i < l-1$ because otherwise $\sigma_l, \sigma_i, \sigma_{i+1}$ should be distinct faces of $\tau_l = \tau_i$ with $f(\sigma_l) = f(\sigma_i) = f(\sigma_{i+1})$;
3) $\sigma_{l+1} \neq \sigma_l$ by construction;
4) $\sigma_{l+1} \neq \sigma_i$ if $i < l$ because otherwise $\sigma_i = \sigma_{l+1}$ should be a face of distinct simplices $\tau_{i-1}, \tau_i, \tau_l$.

From the above and from the compactness of C it follows that there exists an (n-1)-simplex $\sigma_r \subset Q$ such that $f(\sigma_r) = f(\sigma)$. We identify as before σ to σ_r.

By proceeding in this manner for each (n-1)-simplex of Q we obtain, by appropriate subdivision, two geometric n-cycles C' and C'' without boundary. Finally we denote by $f' : C' \to X$ and $f'' : C'' \to X$ the PL maps naturally induced by f, and so we obtain the singular cycles (C', f'), (C'', f''). These cycles are $\bar{p}$-transverse cycles because $f'(C') \subseteq f(C)$, $f''(C'') \subseteq f(C)$ and (C, f) is a $\bar{p}$-transverse cycle. Observe that f' is one to one on each n-simplex of C', while the map f'' is not one to one on each n-simplex of C'', so $\dim f''(C'') < n$ and hence (C'', f'') is 0-homologous by Lemma 4.1.

Step 2 (C', f') is $\bar{p}$-transverse homologous to (C, f).

Let (D, g) be a $\bar{p}$-transverse (n+1)-cycle whose boundary is (C'', f'') and let M_p be the simplicial mapping cylinder of the quotient map $p : C \to C' \cup C''$. By gluing M_p to the geometric cycle D along C'' we obtain the geometric cycle $W = M_p \cup D$ such that $\partial W = C \cup -C'$.

Let $F : W \to X$ the PL map defined by

$$F(y,t) = f(y) \text{ if } t \neq 0$$
$$F(p(y), 0) = f(y)$$
$$F(y) = g(y) \text{ if } y \in D$$

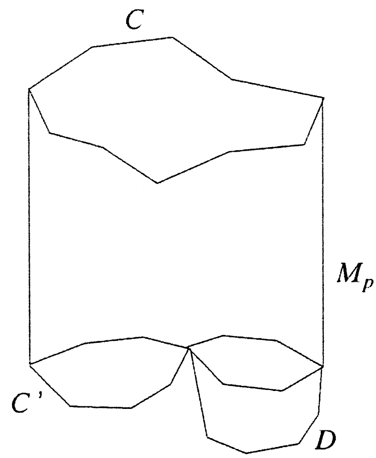

Fig. 2.

Being $F(W) \subseteq f(C) \cup g(D)$, the singular (n+1)-cycle (W, F) is a $\bar{p}$-transverse (n+1)-cycle of X. Furthermore we have $F/_C = f$, $F/_{C'} = f'$ and hence (C', f') is $\bar{p}$-transverse homologous to (C, f). ■

THEOREM 4.4. – $\Psi : IH_n^{\bar{p}}(X) \to H_n^{\bar{p}}(X)$ *is an isomorphism.*

PROOF. –
Ψ is onto.
Let (C, f) be a $\bar{p}$-transverse n-cycle of X.
If $\dim f(C) < n$, by Lemma 4.1 (C, f) is 0-homologous, and hence $[(C, f)]$ is the image of the zero element of $IH_n^{\bar{p}}(X)$.
Suppose $\dim f(C) = n$. From Lemma 4.3 (C, f) is $\bar{p}$-transverse homologous to an n-cycle (C', f') where f' is injective on each top dimensional simplex τ_n of C'. So (C', f') is the geometric realization of the simplicial cycle $\sum_{\tau_n \in C'} f'(\tau_n)$.

Ψ is 1-1.
Let $[\xi]$ an element of $IH_n^{\bar{p}}(X)$ such that $\Psi([\xi]) = [(C_\xi, f_\xi)] = 0$ and let (W, F) be a $\bar{p}$-transverse (n+1)-cycle of X whose boundary is (C_ξ, f_ξ). We denote by S the set of (n+1)-simplices of W on which F is one to one. We will prove that $\xi = \partial \sum_{\tau_{n+1} \in S} F(\tau_{n+1})$.

Suppose $\xi = \sum m_h \sigma_h$, and let Δ_h^n be an n-symplex of C_ξ such that $f_\xi(\Delta_h^n) = \sigma_h$. We need to show that, if $m_h \neq 0$, there exists an (n+1)-symplex $\bar{\tau} \in S$ such that $\sigma_h = F(\Delta_h^n) \prec F(\bar{\tau})$.
Let $\Delta_h^n \prec \tau$.

If $\tau \in S$, it is the required $\bar{\tau}$.

If not, there exists an unique face σ' of τ such that $F(\sigma') = -\sigma_h$. σ' does not lie in $C_\xi = \partial W$ (because otherwise $m_h = 0$), and hence there exists an unique (n+1)-symplex $\tau' \neq \tau$ of W such that $\sigma' \prec \tau'$. If $\tau' \in S$, it is the required $\bar{\tau}$. If not, we repeat the previous procedure starting from $\sigma' \prec \tau'$.

Arguing as in proof of Lemma 4.1, the above procedure has a finite number of steps, and it determines $\bar{\tau} \in S$.

Observe that, if $S = \emptyset$, then $\xi = 0$. ■

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