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Samir Adly, Mohamed Ait Mansour, Laura Scrimali

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Sensitivity analysis of solutions to a class of quasi-variational inequalities

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Sunto. – *Si propone un risultato di sensitività delle soluzioni di disequazioni quasi-variazionali finito-dimensionali del tipo:*

$$(QVI) \quad u \in K(u), \langle C(u), v-u\rangle \geq 0, \quad \forall v \in K(u),$$

in presenza di perturbazioni dell'operatore C e dell'insieme convesso K. In particolare, si prova la continuità Hölderiana degli insiemi delle soluzioni dei problemi perturbati intorno al problema iniziale. I risultati illustrati possono essere estesi anche al caso infinito-dimensionale.

Summary. – *We provide a sensitivity result for the solutions to the following finite-dimensional quasi-variational inequality*

$$(QVI) \quad u \in K(u), \langle C(u), v-u\rangle \geq 0, \quad \forall v \in K(u),$$

when both the operator C and the convex K are perturbed. In particular, we prove the Hölder continuity of the solution sets of the problems perturbed around the original problem. All the results may be extended to infinite-dimensional (*QVI*).

Sensitivity analysis plays a central role in variational inequality theory, since it arises in many different applied problems, which range from transportation theory to physics, from economics to finance.

For this reason, we were motivated to study the Hölder continuity of the solution sets of the parametric finite-dimensional (QVI), measuring the τ-Hausdorff distance (see Attouch and Wets [2]) between the solution sets of the problems perturbed around the original problem.

Let E be a convex and compact subset of $\mathbb{R}^m_+$, $C : E \to \mathbb{R}^m_+$, $\rho : E \to \mathbb{R}^l_+$, $K_\rho : E \rightrightarrows \mathbb{R}^m_+$ a set-valued map with convex and closed values and let A denote an opportune $l \times m-$matrix (l and m are two given integers, $m > l$). We consider the following quasi-variational inequality:

$$(QVI) \quad u \in K_\rho(u), \langle C(u), v-u\rangle \geq 0, \quad \forall v \in K_\rho(u),$$

where

$$K_\rho(u) := \{v \in E : Av = \rho(u)\}$$

(see [4] for a survey on (QVI)).

In order to state the parametric (QVI), we assume that the operator C is subject to change, which can be seen in a general perturbation form by involving a parameter μ, where μ belongs to a subset of a finite-dimensional space Λ, whose norm is denoted by $\|\cdot\|$. Thus, we consider the family of operators $C(\cdot,\mu)_\mu$ defined from E into $\mathbb{R}^m_+$.

The perturbation of constraints will be done with respect to the map ρ, whereas the matrix A will be fixed. Specifically, we consider a kind of small perturbation of the map ρ as follows: ρ will be perturbed by a parameter λ, element of a subset M of a given Euclidean subspace, whose norm is also denoted by $\|\cdot\|$. For any μ and λ, $\mathcal{V}(\mu)$ and $\mathcal{V}(\lambda)$ will denote a neighborhood of μ and λ, respectively. The reference value of the parameters are given by $\bar{\mu}$ and $\bar{\lambda}$.

We also suppose that ρ satisfies a Hölder continuity assumption:

(h_0) ρ is Hölder continuous, i.e., for some $L_1, L_2 > 0$ and $\xi, \xi' \in]0,1[$,

$$|\rho_\lambda(u) - \rho_{\lambda'}(v)| \le L_1\|\lambda - \lambda'\|^{\xi'} + L_2|u - v|^{\xi}$$

$$\forall u, v \in E,\ \forall \lambda, \lambda' \in \mathcal{V}(\bar{\lambda}).$$

The family $\{\rho_\lambda\}_{\lambda\in\mathcal{V}}$ is considered as a perturbation of the initial ρ. Hereinafter, for the sake of simplicity, the constraints $K_{\rho_\lambda}(u)$ will be denoted by $K_\lambda(u)$, for every $\lambda \in \mathcal{V}(\bar{\lambda})$ and $u \in E$. Therefore, the corresponding perturbed problem can be stated as follows

$$(QVI_{\mu,\lambda}) \quad \exists u(\mu,\lambda) \in K_\lambda(u(\mu,\lambda)) : \quad \langle C(u(\mu,\lambda),\mu), v - u(\mu,\lambda)\rangle \ge 0,$$

$$\forall v \in K_\lambda(u(\mu,\lambda)).$$

We suppose that the solution set of $(QVI_{\mu,\lambda})$ is nonempty and focus our attention on the local behavior of the set-valued map $S : \Lambda \times M \rightrightarrows E$ which associates the set $S(\mu,\lambda)$ with each pair (μ,λ). Our aim is to prove the Hölder continuity of the solution map S around the reference value of the parameters $(\bar{\mu},\bar{\lambda})$.

First, we apply the result of Walkup and Wets (1969) [7], which can be seen as a particular case of famous Hoffman's Lemma, to derive the following:

LEMMA 1. – *Let A be an $l \times m$-matrix, d_1 and d_2 be given vectors in $\mathbb{R}^l$. Let S_i denote the solution set of the linear equality $Ax = d_i$, $i = 1, 2$. Then, there exists $\theta = \theta(A) > 0$ such that for each $x_1 \in S_1$ there exists $x_2 \in S_2$ satisfying*

$$|x_1 - x_2| \le \theta|d_1 - d_2|.$$

In the subsequent proposition we show that for Hölder continuous map ρ, we can dispose of a Hölder-type behavior of the map $(u, \rho) \mapsto K_\rho(u)$, which is a stronger property than the well-known Aubin Lipschitz property, or the Pseudo-Lipschitz property as referred in the literature (see [3]).

PROPOSITION 1. – *Let us assume that* (h_0) *holds. Then, there exist* $k_1, k_2 > 0$ *such that* $\forall \lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$ *and* $\forall u, v \in E$ *one has:*

$$K_\lambda(u) \subset K_{\lambda'}(v) + (k_1\|\lambda - \lambda'\|^{\xi'} + k_2|u - v|^{\xi})\mathbb{B}_m, \tag{1}$$

where $\mathbb{B}_m$ *stands for the closed unit ball of* $\mathbb{R}^m$.

REMARK 1. – *We suppose that* v *are nontrivial and norm-bounded from below, namely for some* $f_0 > 0$ *and for all* $v \in E^* := E\backslash\{0\}$, *it results that* $\|v\| \geq f_0$.

Now, we are able to state our main result.

THEOREM 1. – *Let us assume that* (h_0) *holds and the following conditions are satisfied:*

(h_1) *for some* $m > 0$ *and* $a \geq 2$, C *is uniformly* (a, m)*-strongly monotone, i.e.,*

$$\langle C(u, \mu) - C(v, \mu), u - v\rangle \geq m|u - v|^a, \ \forall u, v \in E, \ \forall \mu \in \mathcal{V}(\bar{\mu});$$

(h_2) *for some* $b_0 > 0$, C *is uniformly* b_0*-bounded, i.e., for all* $\mu \in \mathcal{V}(\bar{\mu})$ *and all* $u \in E$ *one has* $|C(u, \mu)| \leq b_0$;

(h_3) *for some* $\gamma \in]0, 1[$ *and* $c > 0$, $\mu \mapsto C(., \mu)$ *is uniformly (in* u*)* (γ, c)*-Hölder, i.e., for all* $u \in E$ *and all* $\mu, \mu' \in \mathcal{V}(\bar{\mu})$,

$$|C(u, \mu) - C(u, \mu')| \leq c\|\mu - \mu'\|^\gamma;$$

(h_4) $m > (2k_2)^\beta b$, where $\beta = \frac{a}{\xi}$ and $b = \frac{b_0}{f_0^{\beta-1}}$,

where f_0 *is the minimal value of* v *as in Remark* 1 *and* k_2 *is given in Proposition* 1. *Then, the solution map* S *is Hölder continuous around* $(\bar{\mu}, \bar{\lambda})$*, i.e., there exist* $c_1, c_2 > 0, d_1, d_2 \in]0, 1[$ *such that for some* $\tau > 0$

$$haus_\tau(S(\mu, \lambda), S(\mu', \lambda')) \leq c_1\|\mu - \mu'\|^{d_1} + c_2\|\lambda - \lambda'\|^{d_2}, \tag{2}$$

for all $\mu, \mu' \in \mathcal{V}(\bar{\mu}), \lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$.

It is worth noting that:

1. estimate (2) means that:

$$S(\mu, \lambda)_\tau \subset S(\mu', \lambda') + (c_1\|\mu - \mu'\|^{d_1} + c_2\|\lambda - \lambda'\|^{d_2})\mathbb{B}_m. \tag{3}$$

2. For any sequences $\{\mu_n\}_n$ and $\{\lambda_n\}_n$ converging to $\bar{\mu}$ and $\bar{\lambda}$, respectively, denoting $S(\mu_n, \lambda_n)$ by S_n and $S(\bar{\mu}, \bar{\lambda})$ by $\bar{S}$, (2) implies that:

$$\lim_{n\to+\infty} \text{haus}_\tau (S_n, \bar{S}) = 0. \tag{4}$$

Equivalently, for any $\tau > \tau_0$ ($\tau_0 > 0$ arbitrarily chosen) and any $\varepsilon > 0$, for n large enough the following conditions hold:

$$\bar{S}_\tau \subset S_n + \varepsilon\mathbb{B}_m \text{ and } (S_n)_\tau \subset \bar{S} + \varepsilon\mathbb{B}_m. \tag{5}$$

3. If E is a finite-dimensional space, (4) forces $\{S_n\}_n$ to converge to S (the strong stability result) in the sense of Painlevé-Kuratowski, i.e.,

$$\limsup_{n\to+\infty} S_n \subset \bar{S} \text{ and } \bar{S} \subset \liminf_{n\to+\infty} S_n. \tag{6}$$

In other words, (6) implies:

- *upper stability, i.e.,* whenever a sequence $\{u_n\}_n$ of solutions to $(QVI)_{(\mu_n,\lambda_n)}$ converges to some point, say $\bar{u}$, then $\bar{u}$ is necessarily a solution to $(QVI)_{\bar{\mu},\bar{\lambda}}$;
- *lower stability, i.e.,* every solution $\bar{u}$ to $(QVI)_{\bar{\mu},\bar{\lambda}}$ can be approximated by a sequence, say $\{u_n\}_n$, of solutions to $(QVI)_{(\mu_n,\lambda_n)}$.

The above results, which improve and extend some existing results in the literature (see [1, 3, 5, 6]), may be generalized to infinite-dimensional quasi-variational inequalities assigned in a time-dependent setting.

REMARK 2. – *As a practical application of the sensitivity result, we may refer to the traffic network equilibrium problem. In this case, the operator C represents the path cost operator depending on the distribution of flows through the network, and ρ denotes the travel demand affected by the equilibrium flow u.*

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Laco, University of Limoges, 123, Avenue A. Thomas, 87000 Cedex France
e-mail: samir.adly@unilim.fr mohamed.ait-mansour@unilim.fr

DMI, Università di Catania, Viale A. Doria 6, 95125 Catania, Italy
e-mail: scrimali@dmi.unict.it

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