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Mean Values of Convexly Arranged Numbers and Monotone Rearrangements in Reverse Integral Inequalities.

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Sunto. – *Si studiano medie di funzioni con valori sulla frontiera di un insieme convesso bidimensionale. Come applicazione si prova che disuguaglianze integrali inverse implicano esattamente le stesse disuguaglianze per il riordinamento monotono. Si ottengono quindi versioni ottimali del classico lemma di Gehring, del teorema di Gurov-Reshetnyak e del teorema di Muckenhoupt.*

Summary. – *We analyse mean values of functions with values in the boundary of a convex two-dimensional set. As an application, reverse integral inequalities imply exactly the same inequalities for the monotone rearrangement. Sharp versions of the classical Gehring lemma, the Gurov-Resetnyak theorem and the Muckenhoupt theorem are obtained.*

Introduction

We are interested in reverse (integral) inequalities. For $f : [0,1] \to [0,\infty[$ these are inequalities like reverse Hölder inequalities, oscillation inequalities and Muckenhoupt inequalities:

$$RH(c,q,\mathcal{I}) : \quad \fint_I f^q \le c\left(\fint_I f\right)^q \quad \text{for all } I \in \mathcal{I},$$

$$OS(c,q,\mathcal{I}) : \quad \fint_I |f - f_I|^q \le c\left(\fint_I f\right)^q \quad \text{for all } I \in \mathcal{I},$$

$$MU(c,q,\mathcal{I}) : \quad \fint_I f\left(\fint_I f^{-\frac{1}{q-1}}\right)^{q-1} \le c \quad \text{for all } I \in \mathcal{I}.$$

Here, $1 < q < \infty$, for oscillation inequalities also $q = 1$ is reasonable, $\mathcal{I}$ is the system of compact intervals in $[0,1]$, and $f_I = \fint_I f$ denotes the mean value. For a function $f : [0,1]^N \to [0,\infty[$ we investigate such inequalities on N-dimensional intervals (i.e. boxes parallel to the axes), on cubes parallel to the axes, and on other systems of subsets of $[0,1]^N$. The system of all intervals and of all cubes in $[0,1]^N$ we denote by $\mathcal{Q}$ respectively $\mathcal{W}$.

Many generalizations of above inequalities appear in the literature: More general convex functions are used than $f \mapsto f^q$, additional terms and other measures appear, the domain of integration may increase from the left-hand side to the right-hand side, and other variants. Reverse inequalities are important in different fields like PDE, variational problems, quasiregular mappings and mapping properties of operators. For the general background we refer to the books [Gia83, BoIw83, Gia93, GaRu85, IwMa01].

The Gehring lemma, Gurov-Reshetnyak theorem, Muckenhoupt theorem and generalizations are vital for the applications of reverse inequalities. The famous Gehring lemma states that an L^q-integrable function satisfying reverse Hölder inequalities $RH(c,q,\mathcal{I})$ also satisfies reverse Hölder inequalities $RH(\tilde{c},\tilde{q},\mathcal{I})$ with an enlarged exponent and constant. In particular, this implies a better integrability than assumed a priori. The Gurov-Reshetnyak and Muckenhoupt theorems are similar results for the other reverse inequalities.

In addition to early qualitative versions of these useful theorems it has been a goal to prove quantitative versions and to get as much information as possible from reverse inequalities. Thus [Iwa82, Boj85] explicitely ask for the sharp bound for the improvement of the exponent in the Gehring lemma and some first asymptotic answers are given in [Boj85, Wik90]. For monotone functions the sharp bound is calculated in [ApSb90] for reverse Hölder inequalities (and in [Kor92a, Kor92b] for the other reverse inequalities). The result for monotone functions is important since the arbitrary situation reduces to the monotone situation [FrMo85, Sbo86]. The arguments in [FrMo85, Sbo86] use maximal functions which unfortunately lead to a loss of information; in particular, no sharp bound can be obtained. But in [Kor92a] it is proved that monotone functions are in fact extremal in the set of all functions on $[0,1]$ in the sence that

$$\sup \frac{\displaystyle\fint_I f^q}{\left(\displaystyle\fint_I f\right)^q} \geq \sup \frac{\displaystyle\fint_I f^{*q}}{\left(\displaystyle\fint_I f^*\right)^q}, \tag{1}$$

where f^* denotes the decreasing rearrangement of f and the suprema are taken over reasonable $I \in \mathcal{I}$. This extremal property gives a good reduction of reverse Hölder inequalities $RH(c,q,\mathcal{I})$ to the monotone situation and proves sharp bounds. Similar extremal properties and good reductions to the monotone situation were found for $RH(c,q,\mathcal{Q})$, $OS(c,q,\mathcal{I})$, $MU(c,q,\mathcal{I})$ in [Kin94, Kor92a, Kor92b], but could not be found in the same way for $RH(c,q,\mathcal{W})$,

$OS(c, q, \mathcal{Q})$, $OS(c, q, \mathcal{W})$, $MU(c, q, \mathcal{Q})$, $MU(c, q, \mathcal{W})$ (because of fundamental inherent reasons at least for reverse inequalities on cubes $\mathcal{W}$, as we think).

We develop a unified, geometrical approach to the Gehring lemma, the Gurov-Resetnyak theorem and Muckenhoupt theorem with improved and sharp statements. In Corollary 5.3 we are able to prove the most simple form of extremal property: *reverse inequalities for f on $\mathcal{I}$, $\mathcal{Q}$, and on some systems similar to $\mathcal{W}$ imply exactly the same reverse inequalities for the decreasing rearrangement f^* on $\mathcal{I}$.* This (global) extremal property implies the above extremal property (1) which only makes a statement about the suprema. In the same way all other known «good» reductions to the monotone situation are improved and all known sharp bounds are recovered. For the first time an extremal property and a good reduction to the monotone situation is proved for a cube-like system and for $OS(c, q, \mathcal{Q})$, $MU(c, q, \mathcal{Q})$. Unfortunately, for cubes the extremal property does not hold in the same simple form. Some counterexamples in Section 3 demonstrate that the cubes are not the natural system for this question. Natural are systems with a kind of one-dimensional order. With the help of a space-filling curve such a system $\mathcal{P}$ can be defined in $[0,1]^n$ which is similar to the system of cubes in the sence that the ratio of the smallest cube containing P and the largest cube in P is bounded for sets P in $\mathcal{P}$.

The extremal property stated above does not depend on the special form of the reverse inequalities and can be easily generalized in different directions. It is a new kind of extremal property in the theory of real valued functions where extremal properties of monotone functions have a long history, see [Kol89].

Our method is different from previous approaches to reverse inequalities and approaches in the theory of real valued functions. In particular, we do not use any covering lemma. The lack of sharp higher-dimensional covering lemmata is an obstacle to get good reductions to the monotone situation. We do not use the usual maximal functions either. Instead two-dimensional geometrical results turn out to be stronger tools which are also more natural. Since covering lemmata and maximal functions are important in many other problems, our new point of view may be of independent interest and may have further applications, different from the ones treated here.

The main idea is to analyse two-dimensional mean value sets like

$$Mf := \left\{ \left(\fint_I f, \fint_I f^q \right);\ I \in \mathcal{I} \right\}.$$

Sets of this kind have some striking properties. We show that *the upper boundary of Mf is the graph of a Lipschitz function*, while the upper boundary of a set in the plane is not a graph in general. Furthermore, the corresponding mean value set Mf^* of the decreasing rearrangement has the most remarkable extremal property: *the upper boundary of Mf is bounded from below by the* *upper boundary of* Mf^*. For the definition of upper boundary and for the exact statements see Theorem 5.1.

Figure 1 depicts the situation for an example f, its decreasing rearrangement f^* and $q = 3$. While mean value sets for monotone functions always look similar to the plotted Mf^* (two concave boundary parts), mean value sets of nonmonotone functions need not be as «well-behaved» as the Mf here; an example with a hole is in Section 1.

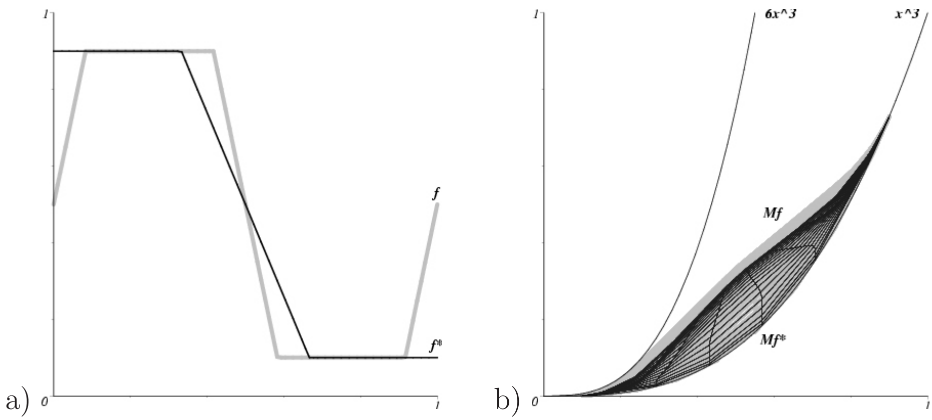

Fig. 1. – a) f and its decreasing rearrangement f^*; b) computer plot of Mf (grey area) and Mf^* (black lattice area).

The mean value set Mf is associated to reverse Hölder inequalities in a natural way and allows an obvious geometric reformulation: f satisfies $RH(c, q, \mathcal{I})$ if and only if the upper boundary of Mf is bounded from above by the graph of cx^q (and ordinary Hölder inequality implies that Mf is bounded from below by the graph of x^q). Thus from the above extremal property we can directly conclude that f^* satisfies the same reverse Hölder inequalities $RH(c, q, \mathcal{I})$ as f. In the figure it can be seen that f and hence f^* satisfy $RH(6, 3, \mathcal{I})$.

What we stated for Mf is just a special case of the more general result below. Functions with values in the boundary of a convex two-dimensional set A are the natural setting. In Section 1 we develop the concepts and arguments to examine the mean values of such a function h when it is defined on $[0, 1]$. The main Theorems 1.2 and 1.3 show that the mean value set of h on $\mathcal{I}$ can fail to be convex only in a very special way which is extremal in case of the monotone rearrangement: *each cave of the mean value set is visible from the boundary and it is included in the cave associated with the monotone rearrangement.* Here, a *cave* of a subset M of A is a connected component of $A \setminus M$ adjacent to the boundary ∂A, and a cave is *visible* from the boundary, if each cave point can be seen from each point of the entry (precise statements in Section 1). The key to the theorems is a kind of winding argument in the proof of Theorem 1.2. This indicates that a two-dimensional setting is natural. We stress that two-dimensional arguments are also essential in the special case of Mf above, although the statement refers to graphs of functions then.

For a function h on $[0,1]^N$ and mean values on $\mathcal{Q}$ the same method works (Section 2), but surprisingly for mean values on $\mathcal{W}$ an analogous statement does not hold (Section 3). This is unsatisfactory since in applications reverse inequalities are often proved with test-functions supported by balls or cubes, and then the possible line/plane degeneration in $\mathcal{Q}$ is unnatural. As a way out, we prove results on systems which are similar to $\mathcal{W}$ in the sense that they only allow point degenerations like $\mathcal{W}$ and which are constructed with space-filling curves (Section 4). In Section 5 we apply the results with $h := (f, f^q)$ to reverse Hölder inequalities, with $h := (f, |f - y|^q)$ to oscillation inequalities, and with $h := (f, f^{-1/(q-1)})$ to Muckenhoupt inequalities. We will conclude Section 5 with a detailed comparison of our results to results in the literature and we will discuss the relation to some results concerning maximal functions.

1. – Mean values on intervals

Let A be a nonempty, open, convex, bounded set in $\mathbb{C}$. Denote by ∂A the boundary and for $x \neq y$ in ∂A by $\partial A[x, y]$, $\partial A[x, y[$, $\partial A]x, y]$, $\partial A]x, y[$ boundary segments from x to y traced counterclockwise including the endpoints or not as indicated by the notation. We assume a decomposition $\partial A = B \cup C$ with boundary points $x \neq y$ satisfying $B = \partial A[x, y]$ and $C = \partial A]y, x[$. We define an order on B by $b_1 \leq b_2 :\Leftrightarrow \partial A[b_1, b_2] \subset B$. Subsets of B are called sets of convexly arranged numbers. We consider (Lebesgue) measurable functions $f : I_0 \to B$ on a compact interval I_0 of positive length. For a measurable $T \subset I_0$ with measure $|T| > 0$ we define the **mean value** of f on T by $f_T := \fint_T f := \frac{1}{|T|} \int_T f(t)dt$. For a system $\mathcal{T}$ of sets with positive measure we define the **mean value set** of f on $\mathcal{T}$ by $f_{\mathcal{T}} := \{f_T : T \in \mathcal{T}\}$. We are mainly interested in the system $\mathcal{I}$ of all subintervals of I_0 of positive measure. The set $f_{\mathcal{I}}$ we call the **interval mean value set** of f. The set of the mean values on *all* measurable subsets of I_0 of positive measure is simply the convex hull of the range of f. It is interesting to study in what way $f_{\mathcal{I}}$ differs from this convex hull.

We first consider step functions $f : I_0 \to B$ with a decomposition $I_0 = I_1 \cup .. \cup I_n$ in intervals of positive length numbered in increasing order of the real line and f constant on each interior. We abbreviate $I_{i..i+j} := I_i \cup .. \cup I_{i+j}$ and $\mathcal{I}_{i..i+j}$ for the system of all subintervals of $I_{i..i+j}$ with positive length. The interval mean value set of step functions can be computed with the recursion formula

$$f_{\mathcal{I}_{1..n}} = f_{\mathcal{I}_{1..n-1}} \cup f_{\mathcal{I}_{2..n}} \cup Q_{1..n}. \tag{2}$$

Here $Q_{1..n}$ is a convex quadrangle (possibly degenerated to a line or point) with vertices $f_{I_{2..n-1}}, f_{I_{1..n-1}}, f_{I_{2..n}}, f_{I_{1..n}}$. In order to prove this, we remark that the set $\{F(I) := \left(\int_I f, |I|\right) : I_{2..n-1} \subset I \subset I_{1..n}\}$ is a parallelogram in $\mathbb{C} \times]0, \infty[$ with vertices $F(I_{2..n-1})$, $F(I_{1..n-1})$, $F(I_{2..n})$, $F(I_{1..n})$. The central projection $p : \mathbb{C} \times]0, \infty[\to \mathbb{C}, p(z, r) := z/r$ maps weakly monotonically parameterized line segments onto weakly monotonically parameterized line segments, and convex sets onto convex sets. The mean value set on all intervals I with $I_{2..n-1} \subset I \subset I_{1..n}$ is $Q_{1..n}$ and has the stated properties, since it is the parallelogram projection. All other intervals of $\mathcal{I}_{1..n}$ are in $\mathcal{I}_{1..n-1}$ or $\mathcal{I}_{2..n}$.

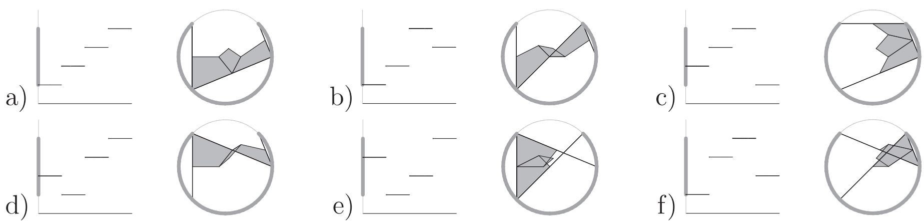

Fig. 2. – a)-f) Symbolized f and its interval mean value set $f_{\mathcal{I}}$.

Using the recursion formula, the interval mean value set of step functions can be reduced to interval mean value sets of step functions with two values (here the interval mean value set is just the line segment between the two values) and to special quadrangles. In Figure 2 we computed the interval mean value sets of some step functions with four values in the dark grey unit circle arc B. The functions only differ in the arrangement of their values. On each left side we show a symbolized version of the function by identifying B with the dark grey interval, and on the right we plot the interval mean value set. Although there are only four values, some of the interval mean value sets are quite complicated due to quadrangle intersection. For instance, $f_{\mathcal{I}}$ need not be simply connected, as demonstrated in figure 2e. However, all interval mean value sets share certain properties (cf. below).

Since B is ordered, we can define that f is **decreasing** on $I \subset I_0$ if $f(t_1) \geq f(t_2)$ for all $t_1 \leq t_2$ in I. We define «increasing» analogously. A monotone function decreases on I or increases on I. Measurable functions $f, g : I_0 \to B$ are equimeasurable and g is a **rearrangement** of f if the measure of the inverse images of boundary segments in B coincide, i.e. $|f^{-1}(\partial A[b_1, b_2])| = |g^{-1}(\partial A[b_1, b_2])|$ for all $b_1 \leq b_2$ in B. As in the case of real valued functions (cf. [BeSh88, Kap. 2]) every measurable function $f : I_0 = [\alpha, \beta] \to B$ has a decreasing, right-continuous rearrangement $f^* : I_0 \to B$ defined by

$$f^*(t) := \inf\{b \in B : \ \alpha + |\{s \in I_0 : f(s) > b\}| \leq t\}.$$

Any other decreasing rearrangement can only differ at the countably many points of discontinuity of f^*. So the **decreasing rearrangement** is unique in a measure theoretical sense. Sometimes we use the shifted $f^* : [0, \beta - \alpha] \to B$, $f^*(t) := \inf\{b \in B : \ |\{s \in I_0 : f(s) > b\}| \leq t\}$ as the decreasing rearrangement. Note that the decreasing rearrangement depends on the decomposition $\partial A = B \cup C$ with $B \supset f(I_0)$. If B is «from 10 o' clock to 7 o' clock» instead of «from 2 o' clock to 10 o' clock», the function in figure 4.2c is monotone and the function in 2a is not.

We call a function $f : I_0 \to B$ a **Riemann function** if left and right limits exist at all points of I_0 (in the endpoints of I_0 only the meaningful one). It is well known that monotone functions, step functions, continuous functions and uniform limits of Riemann functions are Riemann functions. Every Riemann function can be realized as a uniform limit of step functions.

To describe the shape of interval mean value sets, we need the following two concepts. For a set M in the closure $\overline{A}$ we define the C-**cave** of M in A by

$$\mathrm{H}\,(A, C, M) := \big\{a \in A : \ \exists c \in C \ \ \exists \text{path from } a \text{ to } c \text{ in } \big(A \cup \{c\}\big) \setminus M\big\}.$$

We call a set $H \subset A$ **visible** from $\overline{C}$, if for all $h \in H$ and all $c \in \overline{C}$ the segment $[h, c[$ is a subset of H. Whenever A and C are fixed, we abbreviate $\mathrm{H}\,M := \mathrm{H}\,(A, C, M)$ and just speak of the cave of M and of visibility. In the above examples the cave $\mathrm{H}f_{\mathcal{I}}$ is the area which can be reached from the light grey arc C without crossing the interval mean value set. Note that all caves are visible and the cave for the monotone rearrangement in 1a is the largest (cf. below).

We need a notion of **convergence of sequences of sets**. Two nonempty sets $M, N \subset \mathbb{C}$ have Hausdorff distance less than ε, if M is in the neighborhood $U_\varepsilon(N) := \{z \in \mathbb{C} : |z - y| < \varepsilon \text{ for some } y \in N\}$ and N in $U_\varepsilon(M)$. It is well known that this can be quantified by a distance $d(M, N)$, which is a pseudo-metric on the nonempty subsets of $\mathbb{C}$ and a complete metric on the compact nonempty subsets of $\mathbb{C}$. For a sequence of sets in $\mathbb{C}$ convergence is defined with this Hausdorff distance. We also use this concept for convergence of interval sequences.

We collect some properties of «caves» and «visibility», which do not depend on an interval mean value set situation.

LEMMA 1.1. – *Let $M, N, M_n, N_n, M_\lambda$ be subsets of $\overline{A}$ and H_λ subsets of A for all $n \in \mathbb{N}$ and λ in an index set Λ.*

1) *The inclusion $M \subset N$ implies the reverse cave inclusion* $\mathrm{H}\,M \supset \mathrm{H}\,N$.
2) *Visibility of all H_λ implies visibility of the intersection $\cap_{\lambda \in \Lambda} H_\lambda$.*
3) *If* $(\mathrm{H}\,M) \setminus N$ *is visible, then* $\mathrm{H}\,(M \cup N) = (\mathrm{H}\,M) \setminus N$.

4) *If all caves* $\mathrm{H}\,M_\lambda$ *are visible and* $C \setminus (\cup_{\lambda\in\Lambda} M_\lambda) \neq \emptyset$, *then* $\mathrm{H}\,(\cup_{\lambda\in\Lambda} M_\lambda) = \cap_{\lambda\in\Lambda} \mathrm{H}\,M_\lambda$ *and this set is again visible.*

5) If $M \subset \operatorname{conv} B$ *and if* $\mathrm{H}\,M$ *is visible, then a segment between two points outside the cave does not destroy the visibility; i.e. for* $x, y \in (\operatorname{conv} B) \setminus \mathrm{H}\,M$ *the possibly smaller cave* $\mathrm{H}\,(M \cup [x,y])$ *is again visible.*

6) *If* M_n *are closed sets converging to* M *and all caves* $\mathrm{H}\,M_n$ *are visible, then* $\mathrm{H}\,M$ *is visible.*

7) *If* M_n, N_n *are closed sets converging to* M, N *and* $\mathrm{H}\,M_n \subset \mathrm{H}\,N_n$ *for all* n, *then* $\mathrm{H}\,M \subset \mathrm{H}\,N$.

Note that the convergence of M_n to M does not imply the convergence of the caves $\mathrm{H}\,M_n$ to $\mathrm{H}\,M$, e.g. take $M_n := [-1, -1/n] \cup [1/n, 1]$ in the unit disk.

PROOF. – 1) For $h \in \mathrm{H}\,N \subset A$ we find an element $c \in C$ and a path from h to c in $(A \cup \{c\}) \setminus N \subset (A \cup \{c\}) \setminus M$, so $h \in \mathrm{H}\,M$.

2) Visibility implies $[h, c[\, \subset H_\lambda$ for $h \in \cap H_\lambda$, $c \in \overline{C}$ and all λ. So $[h, c[\, \subset \cap H_\lambda$ and the intersection is visible.

3) «$\subset$» follows from $\mathrm{H}\,(M \cup N) \subset \mathrm{H}\,M \cap \mathrm{H}\,N \subset \mathrm{H}\,M \cap (A \setminus N) = \mathrm{H}\,M \setminus N$. «$\supset$» follows, since visibility implies for $h \in (\mathrm{H}\,M) \setminus N$ existence of an element $c \in \overline{C}$ with $[h, c[\, \subset (\mathrm{H}\,M) \setminus N \subset A \setminus (M \cup N)$. So $h \in \mathrm{H}\,(M \cup N)$.

4) «$\subset$» follows from statement 1. For «$\supset$» we observe that statement 2. implies the visibility of the cave intersection. So with $h \in \cap \mathrm{H}\,M_\lambda$ and $c \in C \setminus (\cup M_\lambda)$ we get $[h, c[\, \subset \cap \mathrm{H}\,M_\lambda \subset \cap (A \setminus M_\lambda) = A \setminus (\cup M_\lambda)$. In particular, $[h, c]$ is the range of a path from h to $c \in C$ in $(A \cup \{c\}) \setminus (\cup M_\lambda)$ and so $h \in \mathrm{H}\,(\cup M_\lambda)$.

5) Without restriction let the $\overline{C}$-shadows of x and y, defined by $S_x := \{a \in A :\ \exists\, c \in \overline{C}$ with $x \in [a, c]\}$ and analogously S_y, be subsets of M. (Otherwise take $\tilde{M} := M \cup S_x \cup S_y$ and $\mathrm{H}\,\tilde{M} = \mathrm{H}\,M$ follows from visibility of $\mathrm{H}\,M$.) For $h \in \mathrm{H}\,(M \cup [x,y]) \subset \mathrm{H}\,M$, $c \in \overline{C}$ and a path w from h to an element $r \in C$ in $(A \cup \{r\}) \setminus (M \cup [x,y])$ the segment $[h, c[$ is a subset of $\mathrm{H}\,M \subset A \setminus M$ and of $A \setminus [x,y]$ (otherwise w runs through $S_x \cup S_y \subset M$). Hence $[h, c[$ is a subset of $A \setminus (M \cup [x,y])$ and so of $\mathrm{H}\,(M \cup [x,y])$.

6) Assuming that $\mathrm{H}\,M$ is *not* visible, we find $h \in \mathrm{H}\,M$, $m \in M$ and $c \in \overline{C}$ with $m \in\,]h, c[$. By definition there is a path from an element $\tilde{c} \in C$ to h in $(A \cup \{\tilde{c}\}) \setminus M$. Let W be the range of this path and r be the minimal distance between W and M. Both sets are compact, so $r > 0$. Let s be the distance between m and the cone $U_{r/2}(h) * c := \{a \in A : \exists y \in U_{r/2}(h)$ with $a \in [y, c[\}$. Since m is an inner point of $U_{r/2}(h) * c$, the distance s is positive as well. Due to the M_n-convergence, for sufficiently large n there is no point of M_n in $U_{r/2}(W)$ (in particular, h and also $U_{r/2}(h)$ are in $\mathrm{H}\,M_n$) and there is a point of M_n in $U_s(m)$. This contradicts the assumed visibility of $\mathrm{H}\,M_n$.

7) For $h \in \mathrm{H}\,M$ we find a path from an element $c \in C$ to h in $(A \cup \{c\}) \setminus M$.

Let W be the range of this path and $r > 0$ the minimal distance between W and M. Then $U_r(W) \subset A \setminus M$. The M_n-convergence implies $U_{r/2}(W) \subset A \setminus M_n$ for sufficiently large n. So we conclude $U_{r/2}(W) \subset \mathrm{H}M_n \subset \mathrm{H}N_n \subset A \setminus N_n$. The N_n-convergence then yields $U_{r/4}(W) \subset A \setminus N$ and hence $h \in W \subset \mathrm{H}N$. □

THEOREM 1.2 (Cave visibility). – *If $f : I_0 \to B$ is a Riemann function with convexly arranged range, then the cave $\mathrm{H}\overline{f_{\mathcal{I}}}$ of the interval mean value closure is visible. If $f : I_0 \to B$ is a continuous or a step function, then $\mathrm{H}f_{\mathcal{I}} = \mathrm{H}\overline{f_{\mathcal{I}}}$ is visible.*

PROOF. – By $f_{|I|\to 0} := \cap_{r>0}\mathrm{clos}\{f_I : 0 < |I| < r\}$, we define the accumulation set of mean values for vanishing length. Obviously $\overline{f_{\mathcal{I}}} = f_{\mathcal{I}} \cup f_{|I|\to 0}$. We will prove:

If $f : I_0 \to B$ is measurable and $\mathrm{H}f_{|I|\to 0}$ is visible, then $\mathrm{H}\overline{f_{\mathcal{I}}}$ is visible.

Then the theorem is reduced to an observation about the accumulation set. For Riemann functions the accumulation set consists of boundary points (= accumulation set, when the intervals shrink to points with coinciding left and right limit) and of boundary-point-connecting line segments (= accumulation set, when the intervals shrink to points with different left and right limits). So by Lemma 1.1(5) the cave $\mathrm{H}f_{|I|\to 0}$ and hence $\mathrm{H}\overline{f_{\mathcal{I}}}$ is visible for a Riemann function f. In particular, for continuous functions we have $A \cap f_{\mathcal{I}} = A \cap \overline{f_{\mathcal{I}}}$ and so $\mathrm{H}f_{\mathcal{I}} = \mathrm{H}\overline{f_{\mathcal{I}}}$. For a step function and an interval $I \in \mathcal{I}$ with length not exceeding the length l of the shortest interval of constancy, another interval $J \in \mathcal{I}$ exists with length at least l. Thus $f_{\mathcal{I}} = \overline{f_{\mathcal{I}}}$ holds for step functions.

Let $\mathrm{H}f_{|I|\to 0}$ be visible. We assume that $\mathrm{H}\overline{f_{\mathcal{I}}}$ is *not* visible and will construct a contradiction. We choose a closed sector $Y \subset \overline{A}$, determind by boundary points $c_1 \neq c_2 \in C$ and a cave point $h \in \mathrm{H}\overline{f_{\mathcal{I}}}$, such that the interior of Y is in the cave and an interval mean value is in $]h, c_1[\cup]h, c_2[\subset \partial Y$. Such a sector exists: Since $\mathrm{H}\overline{f_{\mathcal{I}}}$ is not visible, there are $k \in \mathrm{H}\overline{f_{\mathcal{I}}}$, a path u from a boundary point $d \in C$ to k in $\{d\} \cup \mathrm{H}\overline{f_{\mathcal{I}}}$ and a boundary point $e \in C$, from which k is not visible. Since $\mathrm{H}\overline{f_{\mathcal{I}}}$ is open, we can take a sector completely in $\mathrm{H}\overline{f_{\mathcal{I}}}$ determind by boundary points $d_1, d_2 \in C$ and a cave point $u(t)$. By increasing t and shifting d_1 or d_2 towards e the sector changes and there is a time when $\overline{f_{\mathcal{I}}}$ is just tangent to the sector (by continuity). A tangent point can only be in $f_{\mathcal{I}}$, since a tangent point in $f_{|I|\to 0}$ (cave visibility assumed) would imply existence of an element in the empty set $f_{|I|\to 0} \cap \mathrm{range}(u)$. This proves the claimed existence of Y, h, c_1, c_2.

Let $I = [t_0, t_1] \in \mathcal{I}$ be an interval with $f_I \in]h, c_1[\cup]h, c_2[$ and with minimal distance between h and all such f_J. Since $\overline{f_{\mathcal{I}}} = f_{\mathcal{I}} \cup f_{|I|\to 0}$ is closed and no point of $f_{|I|\to 0}$ is in $]h, c_1[\cup]h, c_2[$, the interval I exists. The functions

$$w : [t_0, t_1[\to f_{\mathcal{I}}, \quad w(t) := f_{[t,t_1]} \quad \text{and} \quad v :]t_0, t_1] \to f_{\mathcal{I}}, \quad v(t) := f_{[t_0,t]}$$

are continuous and the values are in $f_{\mathcal{I}}$. The function w starts in f_I and v ends there. Both are not constant, for otherwise f_I would be an element of $f_{|I|\to 0}$ (cf. above). Since $f_I = f_{[t_0,t_1]}$ is a linear combination of $f_{[t_0,t]}$ and $f_{[t,t_1]}$, the functions are «running f_I-opposite», i.e. for all $t \in \,]t_0, t_1[$ the mean value f_I is in the segment $[w(t), v(t)]$ and if $w(t) = f_I$, then also $v(t) = f_I$ and vice versa.

The lines through c_1, h and through c_2, h divide A in four sectors. The closure of one is Y, we call the closure of the opposite one Z, the closure of the remaining sector adjacent to $]c_1, h[$ we call X_1 and the closure of the other one X_2. We assume without restriction that $f_I \in \,]c_1, h[$. The properties of w, v imply the existence of $s_0 \in \,]t_0, t_1[$ with $w(s_0) \in X_1 \setminus [f_I, h]$. The only possibilities for the opposite point $v(s_0)$ are the interior of Y (contradiction to the fact that the interior of Y is in $\mathrm{H}\overline{f_{\mathcal{I}}}$), the segment $]f_I, h]$ (contradiction to the choice of I), or X_2. In the last case $v|_{[s_0,t_1]}$ is a path from X_2 to $f_I \in X_1$. Since it can not cross the interior of Y or h, there must be a time s_1 when $v(s_1)$ is in the interior of Z. Then the opposite point $w(s_1)$ is in the interior of Y, which is the final contradiction. □

THEOREM 1.3 (Extremal property of monotone rearrangements). – *For every Riemann function $f : I_0 \to B$ with convexly arranged range $\mathrm{H}\overline{f_{\mathcal{I}}} \subset \mathrm{H}\overline{f^*_{\mathcal{I}}}$ holds. If f is in addition continuous or a step function, then also $\mathrm{H}f_{\mathcal{I}} \subset \mathrm{H}f^*_{\mathcal{I}}$.*

PROOF. – We will prove the following statement by induction using cave visibility in an essential way:

$\mathcal{A}(n) :\Leftrightarrow$ For a step function $f : I_0 \to B, f = b_1 \mathbb{1}_{I_1} + .. + b_n \mathbb{1}_{I_n}$ with $b_1 \leq .. \leq b_n$ in B and a decomposition $I_0 = I_1 \cup .. \cup I_n$ in intervals of positive length, the cave inclusion $\mathrm{H}f_{\mathcal{I}} \subset \mathrm{H}f^*_{\mathcal{I}}$ is true.

Then the statement for a Riemann function f follows by choosing step functions f_n converging uniformly to f. The f^*_n also uniformly converge to f^*. The convergence of the interval mean value sets $(f_n)_{\mathcal{I}}$ to $\overline{f_{\mathcal{I}}}$ and of $(f^*_n)_{\mathcal{I}}$ to $\overline{f^*_{\mathcal{I}}}$ follows, since

$$|(f_n)_I - f_I| \leq \fint_I |f_n - f| \leq \sup_{t \in I_0} |f_n(t) - f(t)| \to 0 \quad \text{for all } I \in \mathcal{I}$$

implies for $\varepsilon > 0$ and sufficiently large n, that $(f_n)_{\mathcal{I}} \subset U_\varepsilon(f_{\mathcal{I}}) \subset U_\varepsilon(\overline{f_{\mathcal{I}}})$ and $\overline{f_{\mathcal{I}}} \subset U_{\varepsilon/2}(f_{\mathcal{I}}) \subset U_\varepsilon((f_n)_{\mathcal{I}})$. So Lemma 1.1(7) implies the theorem for all Riemann functions if it can be proved for step functions.

The **induction starts** with $\mathcal{A}(1)$ and $\mathcal{A}(2)$. For $n = 1$ we have $f^* = f$, hence $\mathrm{H}f_{\mathcal{I}} = \mathrm{H}f^*_{\mathcal{I}}$ is trivial. For $n = 2$ the interval mean value set is the line segment connecting the two values, so $f_{\mathcal{I}} = [b_1, b_2] = f^*_{\mathcal{I}}$ and $\mathrm{H}f_{\mathcal{I}} = \mathrm{H}f^*_{\mathcal{I}}$. For the **induction step** $\mathcal{A}(n-1) \Rightarrow \mathcal{A}(n)$ with $n \geq 3$ we consider a step function $f : I_0 = [\alpha, \beta] \to B$, $f = b_1 \mathbb{1}_{I_1} + .. + b_n \mathbb{1}_{I_n}$ with $b_1 \leq .. \leq b_n$ and a decomposition $I_0 = I_1 \cup .. \cup I_n$. We get step functions with $n-1$ values by forgetting the smallest or the largest value: Let $J_0 := [\alpha + |I_1|, \beta]$, $K_0 := [\alpha, \beta - |I_n|]$ and

$$g : J_0 \to B, g := b_2 \mathbb{1}_{J_2} + .. + b_n \mathbb{1}_{J_n} \text{ with } J_i := \begin{cases} I_i + |I_1|, & \text{if } I_i < I_1 \\ I_i, & \text{if } I_i > I_1 \end{cases},$$

$$h : K_0 \to B, h := b_1 \mathbb{1}_{K_1} + .. + b_{n-1} \mathbb{1}_{K_{n-1}} \text{ with } K_i := \begin{cases} I_i, & \text{if } I_i < I_n \\ I_i - |I_n|, & \text{if } I_i > I_n \end{cases}.$$

We denote by $\mathcal{I}, \mathcal{J}, \mathcal{K}$ the systems of all subintervals of I_0, J_0, K_0 of positive length. The assumption $\mathcal{A}(n-1)$ for g and h implies

$$\text{(3)} \qquad \mathrm{H}\, g_{\mathcal{J}} \subset \mathrm{H}\, g^*_{\mathcal{J}} \quad \text{and} \quad \mathrm{H}\, h_{\mathcal{K}} \subset \mathrm{H}\, h^*_{\mathcal{K}}.$$

We will prove

$$\text{(4)} \qquad \mathrm{H} f_{\mathcal{I}} \subset \mathrm{H}\, g_{\mathcal{J}} \cap \mathrm{H}\, h_{\mathcal{K}} \cap \mathrm{H}\, S$$

$$\text{(5)} \qquad \mathrm{H} f^*_{\mathcal{I}} = \mathrm{H}\, g^*_{\mathcal{J}} \cap \mathrm{H}\, h^*_{\mathcal{K}} \cap \mathrm{H}\, S.$$

Here $S := \{a \in A : \exists c \in \partial A[b_n, b_1] \text{ with } f_{I_0} \in [a, c]\}$ is the shadow of f_{I_0} with $\partial A[b_n, b_1]$ as a light source. Combining 4, 3, 5 we get $\mathrm{H} f_{\mathcal{I}} \subset \mathrm{H} f^*_{\mathcal{I}}$ as desired.

To prove **formula** 4 we first show $\mathrm{H} f_{\mathcal{I}} \subset \mathrm{H}\, g_{\mathcal{J}}$ (and analogously $\mathrm{H} f_{\mathcal{I}} \subset \mathrm{H}\, h_{\mathcal{K}}$). For each interval mean value m of g we find an interval mean value $\tilde{m}$ of f, which is in the segment $[m, b_1]$ (definition of g). The visibility of $\mathrm{H} f_{\mathcal{I}}$ implies that m cannot belong to $\mathrm{H} f_{\mathcal{I}}$ as $\tilde{m}$ cannot. So $g_{\mathcal{J}} \cap \mathrm{H} f_{\mathcal{I}} = \emptyset$ and $\mathrm{H} f_{\mathcal{I}} \subset \mathrm{H}\, g_{\mathcal{J}}$. To complete the proof of 4 we show $\mathrm{H} f_{\mathcal{I}} \subset \mathrm{H}\, S$. The total mean value f_{I_0} is not in the cave $\mathrm{H} f_{\mathcal{I}}$. Visibility implies that no point of the shadow S is in $\mathrm{H} f_{\mathcal{I}}$ either. So $S \subset A \setminus \mathrm{H} f$ and $\mathrm{H} f_{\mathcal{I}} \subset \mathrm{H}\, S$.

We get **formula 5** if we prove that for decreasing functions the reverse inclusion «$\supset$» in formula 4 is true. Supposing that the step function f is decreasing, then the intervals $I_1, .., I_n$ are numbered from left to right. We use the recursion formula 2 with the present notations here and get

$$f_{\mathcal{I}} = f_{\mathcal{I}_{1..n}} = f_{I_{2..n}} \cup f_{I_{1..n-1}} \cup Q_{1..n} = g_{\mathcal{J}} \cup h_{\mathcal{K}} \cup Q_{1..n}.$$

The lines extending the quadrangle edges $[f_{I_{2..n}}, f_{I_{1..n}}]$ and $[f_{I_{1..n-1}}, f_{I_{1..n}}]$ meet the smallest, respectively largest value b_1, b_n, hence $Q_{1..n} \subset S$ and

$$f_{\mathcal{I}} \subset g_{\mathcal{J}} \cup h_{\mathcal{K}} \cup S.$$

Lemma 1.1(1) implies the reverse cave inclusion and Lemma 1.1(4) with the visibility of $\mathrm{H}\, g_{\mathcal{J}}, \mathrm{H}\, h_{\mathcal{K}}, \mathrm{H}\, S$ implies that the cave of the right-hand side is equal to the intersection of the caves

$$\mathrm{H} f_{\mathcal{I}} \supset \mathrm{H}\,(g_{\mathcal{J}} \cup h_{\mathcal{K}} \cup S) = \mathrm{H}\, g_{\mathcal{J}} \cap \mathrm{H}\, h_{\mathcal{K}} \cap \mathrm{H}\, S.$$

So we have proved formula 5. □

The monotone functions play a special role for interval mean value sets. Here these sets can be easily described.

THEOREM 1.4 (Mean value sets of monotone functions). – *For a monotone function $f : I_0 = [a, \beta] \to B$ with convexly arranged range the mean value set is borderd by the curves $t \mapsto f_{[a,t]}$ and $t \mapsto f_{[t,\beta]}$ and the «generalized values» of f:*

$$\partial f_{\mathcal{I}} = \{f_{[a,t]} : t \in I_0\} \cup \{f_{[t,\beta]} : t \in I_0\} \cup \bigcup_{t \in I_0} [f(t^-), f(t^+)] \tag{6}$$

Here, we define $f(a^-) := f_{[a,a]} := f(a^+)$ and $f(\beta^+) := f_{[\beta,\beta]} := f(\beta^-)$ to make the terms on the right-hand side meaningful for $t = a$ and $t = \beta$.

PROOF. – For monotone step functions the statement can be proved by induction and recursion formula (2). We just remark that the segments $[f_{I_{1..n-1}}, f_{I_{1..n}}]$ and $[f_{I_{2..n}}, f_{I_{1..n}}]$ are edges of the quadrangle, which are not in $f_{\mathcal{I}_{1..n-1}}$ and $f_{\mathcal{I}_{2..n}}$ and whose linear extensions meet the smallest, respectively largest value of f.

For an arbitrary monotone function $f : I_0 \to B$ we choose a sequence of monotone step functions f_n converging uniformly to f. Then the right-hand side of (6) for f_n converges to the corresponding set for f. Also the sets $(f_n)_{\mathcal{I}}$ converge to $f_{\mathcal{I}}$. For a sequence of sets M_n converging to M the boundary of M may be smaller than the limit of ∂M_n. But here the visibility of $\mathrm{H}\,(f_n)_{\mathcal{I}}$ and $\mathrm{H} f_{\mathcal{I}}$ can be used to prove that no such effect is possible and that $\partial (f_n)_{\mathcal{I}}$ converges to $\partial f_{\mathcal{I}}$. □

REMARK 1.5 (measurable functions). – The cave visibility holds for a larger class of functions $f : I_0 = [a, \beta] \to B$ than Riemann functions. For the argument in Theorem 1.2 we only need the existence of all left and right interval mean value limits instead of all left and right limits of f, i.e. we need for all t_0 in the interior of I_0 elements $b_1, b_2 \in B$ with

$$\lim_{r \searrow 0} \sup_{I \subset [t_0 - r, t_0]} \left| \fint_I f - b_1 \right| = 0 \quad \text{and} \quad \lim_{r \searrow 0} \sup_{I \subset [t_0, t_0 + r]} \left| \fint_I f - b_2 \right| = 0.$$

By an approximation argument (which we omit here) more than one accumulation value can be allowed at the endpoints a, β. We do not know how to prove (or disprove) cave visibility if there is an inner point where f has more than one left or right accumulation value. However, we conjecture that Theorem 1.2 is true for all measurable functions. This would be a remarkable geometric statement for the behavior near non-Lebesgue points. The conjecture would also imply the extremal property of monotone rearrangements (Theorem 1.3) for all measurable functions. For the applications we have in mind we only need the continuous versions of Theorem 1.2 and 1.3, since then regularization will always be possible.

REMARK 1.6 (weighted Lebesgue measures). – Instead of the Lebesgue measure we can take a weighted Lebesgue measure μ on I_0 with a nonnegative weight function $w : I_0 \to [0, \infty[$ such that $\mu(A) = \int_A w(t)dt$ for every Lebesgue measurable set A. The system $\mathcal{I}(\mu)$ of all subintervals $I \subset I_0$ with $\mu(I) > 0$ may be smaller than the system $\mathcal{I}$ of all $I \subset I_0$ of positive length $|I| > 0$. For $I \in \mathcal{I}(\mu)$ and a μ-integrable function $f : I_0 = [\alpha, \beta] \to B$ we define the μ-mean value $f_{I,\mu} := \fint_I f d\mu := \frac{1}{\mu(I)} \int_I f d\mu$. Let $f_{\mathcal{I}(\mu)}$ be the corresponding interval μ-mean value set. The decreasing μ-rearrangement of f is the function $f^{*\mu} : [0, \mu(I_0)] \to B$, $f^{*\mu}(t) := \inf\{b \in B : \ \mu(\{s \in I_0 : f(s) > b\}) \leq t\}$.

An inspection of the proofs shows that Theorem 1.2 and 1.3 holds for weighted Lebesgue measures, too. We just remark that for weighted Lebesgue measures the paths w, v in the proof of 1.2 are continuous again. This is not true for more general measures on I_0, and indeed 1.2 and 1.3 need not be true for arbitrary measures. For example: let A be the unit disk, B the lower boundary semicircle, $f : [0,1] \to B$ defined by $f(t) = 1$ for $t > 0$ and $f(0) = -1$ and $\mu := \mathcal{L} + \delta_0$ the sum of the Lebesgue measure and the point measure in 0. Then $\overline{f_{\mathcal{I}(\mu)}} = f_{\mathcal{I}(\mu)} = [-1, 0] \cup \{1\}$ with nonvisible cave $\mathrm{H} f_{\mathcal{I}(\mu)}$.

Forgetting the smallest and largest value in the proof of Theorem 1.3 can also be realized by setting the weight function equal to zero in the matching intervals. This argument is required for higher-dimensional intervals (cf. below), when a «glueing the gap» like in the proof of Theorem 1.3 is no longer possible.

2. – Mean values on N-dimensional intervals

Assume the same conditions on A, B, C as in Section 1 and let $Q_0 \subset \mathbb{R}^N$ be an N-dimensional interval, i.e. there are $a_1, .., a_N, b_1, .., b_N$ with $Q_0 = [a_1, b_1] \times .. \times [a_N, b_N]$ and $a_i < b_i$ for $i = 1, .., n$. We consider Lebesgue measurable functions $f : Q_0 \to B$ and mean values on N-dimensional subintervals. Unlike the one-dimensional case we cannot prove the main results for the Lebesgue measure without considering weighted Lebesgue measures at the same time. So we assume a weighted Lebesgue measure w on Q_0, defined by a Lebesgue measurable weight function $w : Q_0 \to [0, \infty[$ (we do not expect confusion and use the same symbol for the measure and the weight) with $w(A) = \int_A w$ for all Lebesgue measurable $A \subset Q_0$ and with $0 < w(Q_0) < \infty$. Let $\mathcal{Q}(Q_0, w)$ be the system of all N-dimensional subintervals $Q \subset Q_0$ with $w(Q) > 0$. By

$$f_{\mathcal{Q}(Q_0,w)} := \left\{ f_{Q,w} := \frac{1}{w(Q)} \int_Q fw : \ Q \in \mathcal{Q}(Q_0, w) \right\}$$

we define the **N-dimensional interval w-mean value set**. The decreasing rearrangement $f^{*w} : [0, w(Q_0)] \to B$ is defined by

$$f^{*w}(t) := \inf\{b \in B : \ w\{s \in Q_0 : \ f(s) > b\} \leq t\}.$$

The function f (or analogously the weight w) is called a step function, if $Q_0 = Q_1 \cup .. \cup Q_n$ and f is constant on the interior of the intervals Q_i for $i = 1..n$.

THEOREM 2.1 (Cave visibility). – *If $f : Q_0 \to B$ and $w : Q_0 \to [0, \infty[$ are step functions, then the cave* $\mathrm{H}\overline{f_{\mathcal{Q}(Q_0,w)}}$ *of the closure of the N-dimensional interval w-mean value set is visible.*

For Lebesgue measure on Q_0, i.e. $w \equiv 1$, we abbreviate $\mathcal{Q} := \mathcal{Q}(Q_0, w)$. Like in the one-dimensional case we have $\mathrm{H}\overline{f_{\mathcal{Q}}} = \mathrm{H}f_{\mathcal{Q}}$ for continuous or step functions. Since uniform convergence $f_n \to f$ implies the set convergence $\overline{(f_n)_{\mathcal{Q}}} \to \overline{f_{\mathcal{Q}}}$, Lemma 1.1 yields:

COROLLARY 2.2. – *If $f : Q_0 \to B$ is a uniform limit of step functions, then* $\mathrm{H}\,\overline{f_{\mathcal{Q}}}$ *is visible. If f is a continuous or a step function, then* $\mathrm{H}\, f_{\mathcal{Q}}$ *is visible.*

PROOF OF 2.1 Let $f_{w(Q)\to 0} := \cap_{r>0}\overline{\{f_{Q,w} : \ Q \in \mathcal{Q}(Q_0, w), w(Q) \leq r\}}$ the accumulation set of mean values $f_{Q,w}$ with $w(Q) \to 0$. Obviously $\overline{f_{\mathcal{Q}(Q_0,w)}} = f_{w(Q)\to 0} \cup f_{\mathcal{Q}(Q_0,w)}$. The proof is based on the following fact:

> If $f : Q_0 \to B$, $w : Q_0 \to [0, \infty[$ are measurable and $\mathrm{H}f_{w(Q)\to 0}$ is visible, then $\mathrm{H}\overline{f_{\mathcal{Q}(Q_0,w)}}$ is visible, too.

Since the proof of this fact is almost the same as in Theorem 1.2, we omit most details. It is essential that the construction of opposite paths is possible again. Therefore choose Y, c_1, c_2, h in the same way as in the proof of Theorem 1.2 and let $Q = [a_1, b_1] \times .. \times [a_N, b_N] \in \mathcal{Q}(Q_0, w)$ with $f_{Q,w} \in]c_1, h[\cup]c_2, h[$ and of minimal distance to h. We abbreviate $P := [a_2, b_2] \times .. \times [a_N, b_N]$ and assume without restriction that $w([t, b_1] \times P) > 0$ and $w([a_1, t] \times P) > 0$ for all $t \in]a_1, b_1[$; otherwise just forget the zero side parts. Define then the opposite paths

$$u : [a_1, b_1[\to \overline{A}, \ u(t) := f_{[t,b_1]\times P}, \quad v :]a_1, b_1] \to \overline{A}, \ v(t) := f_{[a_1,t]\times P}.$$

and argue in the same manner as in Theorem 1.2.

It remains to be proved that for step functions f, w the cave of the accumulation set $f_{w(Q)\to 0}$ is visible. We concentrate on the model case:

> If $f : [0, n]^2 \to B$, $w : [0, n]^2 \to [0, \infty[$ are step functions, constant on the squares $(i, k) +]0, 1[^2$ for $i, k = 0, .., n-1$, then $\mathrm{H}f_{\mu(Q)\to 0}$ is visible.

This can easily be generalized to general step functions and to situations where f, w have different constancy intervals. For dimensions $N > 2$ an induction yields the proof. Since no argument is needed which does not appear in the proof (= induction step from $N = 1$ to $N = 2$), we omit this induction.

Since w may vanish on some squares, there are different possibilities for a sequence of $Q_m \in \mathcal{Q}(Q_0, w)$ to converge to a set Q with $w(Q) = 0$. The accumulation set $f_{w(Q)\to 0}$ is always a union of sets we will consider in 1.-7. below and the cave visibility follows from the cave visibility of the sets in 1.-7. and from Lemma 1.1. We will not distinguish functions which have only different values on the line segments $\{i\} \times [0, n]$ and $[0, n] \times \{k\}$ with $i, k = 0, .., n$. We call a square $W = [i-1, i] \times [k-1, k]$ with $w(W) = 0$ a zero square.

1) Let $x \in\,]0, n]$. Let $M(x^-, x, 0, n)$ be the accumulation set of sequences f_{Q_m} with $Q_m \in \mathcal{Q}([0, x] \times [0, n], w)$ converging to a subset of $\{x\} \times [0, n]$. For m sufficiently large, Q_m is in $[i, x] \times [0, n]$ with the largest integer $i < x$, and hence f, w are constant in the first variable. Thus $M(x^-, x, 0, n)$ is the closure of the one-dimensional interval mean value set of $f(x^-, \cdot)$ with weight $w(x^-, \cdot)$:

$$M(x^-, x, 0, n) = \overline{f(x^-, \cdot)_{\mathcal{I}([0,n], w(x^-, \cdot))}}.$$

Theorem 1.2 and Remark 1.6 imply cave visibility for $M(x^-, x, 0, n)$. Analogously we treat $M(x, x^+, 0, n)$, $M(0, n, y^-, y)$, $M(0, n, y, y^+)$.

2) Let $x \in\,]0, n[$. Let $M(x^-, x^+, 0, n)$ be the accumulation set of sequences f_{Q_m} with $Q_m \in \mathcal{Q}([0, n] \times [0, n], w)$ converging to a subset of $\{x\} \times [0, n]$. We get

$$M(x^-, x^+, 0, n) = M(x^-, x, 0, n) \cup M(x, x^+, 0, n) \cup N.$$

Here N denotes the closure of the union of all $[f(x^-, \cdot)_I, f(x^+, \cdot)_I]$ with I in $\mathcal{I}([0, n], w(x^-, \cdot))$ and $\mathcal{I}([0, n], w(x^+, \cdot))$. So N consists of line segments starting in $M(x^-, x, 0, n)$ and ending in $M(x, x^+, 0, n)$. Part 1 and Lemma 1.1 imply the cave visibility for $M(x^-, x^+, 0, n)$. Analogously we argue for $M(0, n, y^-, y^+)$.

3) Let $0 < i$ and $Q := [i, j] \times [k, l]$ be a nondegenerated rectangle with $w(Q) = 0$ and a nonzero square on its left side. Let $M(i^-, j, k, l)$ be the accumulation set of sequences f_{Q_m} with $Q_m \in \mathcal{Q}([0, j] \times [k, l], w)$ converging to a nondegenerated rectangle in Q. For sufficiently large index the nonzero part of Q_m is in $[i-1, i] \times [k, l]$ and f, w are constant in the first variable. We get

$$M(i^-, j, k, l) = \overline{f(i^-, \cdot)_{\mathcal{I}([k,l], w(i^-, \cdot))}}$$

and with Theorem 1.2 and Remark 1.6 the cave visibility of $M(i^-, j, k, l)$. Analogously we deal with $M(i, j^+, k, l)$, $M(i, j, k^-, l)$, $M(i, j, k, l^+)$.

4) Let $0 < i, j < n$ and $Q := [i, j] \times [k, l]$ be a nondegenerated rectangle with $w(Q) = 0$ and nonzero squares on its left and right side. Let $M(i^-, j^+, k, l)$ be the accumulation set of sequences f_{Q_m} with $Q_m \in \mathcal{Q}([0, n] \times [k, l], w)$ converging to a nondegenerated rectangle in Q. We get

$$M(i^-, j^+, k, l) = M(i^-, j, k, l) \cup M(i, j^+, k, l) \cup N.$$

Here N is the closure of the union of $[f(i^-,\cdot)_I, f(j^+,\cdot)_I]$ with I in $\mathcal{I}([k,l],w(i^-,\cdot))$ and $\mathcal{I}([k,l],w(j^+,\cdot))$. So N consists of line segments starting in $M(i^-,j,k,l)$ and ending in $M(i,j^+,k,l)$. Part 3 and Lemma 1.1 imply cave visibility for $M(i^-,j^+,k,l)$. Analogously we see visibility for $M(i,j,k^-,l^+)$.

5) Let $0 < i$, $0 < k$ and $Q := [i,j] \times [k,l]$ be a nondegenerated rectangle with $w(Q) = 0$ and nonzero squares on its left and lower side. Let $M(i^-,j,k^-,l)$ be the accumulation set of sequences f_{Q_m} with $Q_m \in \mathcal{Q}([0,j] \times [0,l], w)$ converging to a nondegenerated rectangle in Q. We get

$$M(i^-,j,k^-,l) = M(i^-,j,k,l) \cup M(i,j,k^-,l) \cup N.$$

Here N denotes the closure of the union of $[f(i^-,\cdot)_I, f(\cdot,k^-)_J]$ with $k \in I$, $I \in \mathcal{I}([k,l],w(i^-,\cdot))$, $i \in J$, $J \in \mathcal{I}([i,j],w(\cdot,k^-))$; notice that the quadratically vanishing edge part of Q_m in $[i-1,i] \times [k-1,k]$ is not relevant compared to the linearly vanishing side parts in $[i-1,i] \times [k,l]$ and $[i,j] \times [k-1,k]$. So N consists of line segments starting in $M(i^-,j,k,l)$ and ending in $M(i,j,k^-,l)$. Part 3 and Lemma 1.1 imply cave visibility for $M(i^-,j,k^-,l)$. Analogously we treat $M(i^-,j,k,l^+)$, $M(i,j^+,k^-,l)$, $M(i,j^+,k,l^+)$.

6) Let $0 < i, j < n$, $0 < k$ and $Q := [i,j] \times [k,l]$ be a nondegenerated rectangle with $w(Q) = 0$ and nonzero squares on its left, right and lower side. Let $M(i^-,j^+,k^-,l)$ be the accumulation set of sequences f_{Q_m} with $Q_m \in \mathcal{Q}([0,n] \times [0,l], w)$ converging to a nondegenerated rectangle in Q. We get

$$M(i^-,j^+,k^-,l) = M(i^-,j^+,k,l) \cup M(i,j^+,k^-,l) \cup M(i^-,j,k^-,l) \cup N.$$

Here N is the closure of the union of some $[z, f(\cdot,k^-)_{[i,j]}]$ with $z \in M(i^-,j^+,k,l)$. Hence N consists of line segments from $M(i^-,j^+,k,l)$ to $M(i^-,j,k^-,l)$. Part 4, 5 and Lemma 1.1 imply cave visibility for $M(i^-,j^+,k^-,l)$. Analogously we argue for $M(i^-,j^+,k,l^+)$, $M(i^-,j,k^-,l^+)$, $M(i,j^+,k^-,l^+)$.

7) Finally, let $0 < i$, $j < n$, $0 < k$, $l < n$ and $Q := [i,j] \times [k,l]$ be a nondegenerated rectangle with $w(Q) = 0$ and nonzero squares on all its sides. Let $M(i^-,j^+,k^-,l^+)$ be the accumulation set of sequences f_{Q_m} with $Q_m \in \mathcal{Q}([0,n] \times [0,n], w)$ converging to a nondegenerated rectangle in Q. We get

$$\begin{aligned} M(i^-,j^+,k^-,l^+) = {} & M(i^-,j^+,k^-,l) \cup M(i^-,j^+,k,l^+) \\ & \cup M(i^-,j,k^-,l^+) \cup M(i,j^+,k^-,l^+) \cup N. \end{aligned}$$

Here N is the convex hull of the vertices $f(i^-,\cdot)_{[k,l]}$, $f(j^+,\cdot)_{[k,l]}$, $f(\cdot,k^-)_{[i,j]}$, $f(\cdot,l^+)_{[i,j]}$. Since all vertices are in one of the four sets, part 6 and Lemma 1.1 imply cave visibility for $M(i^-,j^+,k^-,l^+)$. □

THEOREM 2.3 (Extremal property of monotone rearrangement). – *If* $f : Q_0 \to B$ *and* $w : Q_0 \to [0,\infty[$ *are step functions, if* $f^{*w} : [0, w(Q_0)] \to B$ *is the decreasing* w*-rearrangement of* f*, if* $\mathcal{Q}(w)$ *is the set of all* N*-dimensional sub-*

intervals $Q \subset Q_0$ *with* $w(Q) > 0$ *and if* $\mathcal{I}(w)$ *is the set of all subintervals* $I \subset [0, w(Q_0)]$ *with* $|I| > 0$*, then we have the cave inclusion* $\mathrm{H}\overline{f_{\mathcal{Q}(w)}} \subset \mathrm{H}\overline{f^{*w}_{\mathcal{I}(w)}}$.

If w is the Lebesgue measure on Q_0 and if we abbreviate $\mathcal{Q} := \mathcal{Q}(w)$, $\mathcal{I} := \mathcal{I}(w)$ and $f^* := f^{*w}$, uniform convergence then implies readily.

COROLLARY 2.4. – *If* $f : Q_0 \to B$ *is a uniform limit of step functions, then* $\mathrm{H}\,\overline{f_{\mathcal{Q}}} \subset \mathrm{H}\,\overline{f^*_{\mathcal{I}}}$. *If* f *is a continuous or a step function, then* $\mathrm{H}\,f_{\mathcal{Q}} \subset \mathrm{H}\,f^*_{\mathcal{I}}$.

PROOF of 2.3. – Without restriction we assume for f and w the same intervals of constancy. We will prove the claim by induction on the number of w-nonzero intervals:

> $\mathcal{A}(n) :\Leftrightarrow$ If $Q_1, .., Q_n$ be N-dimensional subintervals of Q_0 with nonintersecting interior, $w = w_1 \mathbb{1}_{Q_1} + .. + w_n \mathbb{1}_{Q_n}$ has positive values $w_1, .., w_n$ on $Q_1, .., Q_n$ and vanishes on $Q_0 \setminus (Q_1 \cup .. \cup Q_n)$, and f has values $b_1 \geq .. \geq b_n$ in B on $Q_1, .., Q_n$, then $\mathrm{H}\overline{f_{\mathcal{Q}(w)}} \subset \mathrm{H}\overline{f^{*w}_{\mathcal{I}(w)}}$.

For such f and w the decreasing w-rearrangement is $f^{*w} = b_1 \mathbb{1}_{I_1} + .. + b_n \mathbb{1}_{I_n}$ with $I_1 = [0, w_1|Q_1|]$ and $I_i := [\, w_1|Q_1| + .. + w_{i-1}|Q_{i-1}|,\ w_1|Q_1| + .. + w_i|Q_i|\,]$ for $i = 2, .., n$.

The **induction starts** with $\mathcal{A}(1)$ and $\mathcal{A}(2)$. For $n = 1$ the sets $f_{\mathcal{Q}(w)}$ and $f^{*w}_{\mathcal{I}(w)}$ are just $\{b_1\}$. For $n = 2$ they are the line segment $[b_1, b_2]$. In both cases we have $\mathrm{H}\overline{f_{\mathcal{Q}(w)}} = \mathrm{H}\overline{f^{*w}_{\mathcal{I}(w)}}$. To prove the **induction step** $\mathcal{A}(n-1) \Rightarrow \mathcal{A}(n)$ for $n \geq 3$ we consider intervals $Q_1, .., Q_n$ with a nonintersecting interior, $w = w_1 \mathbb{1}_{Q_1} + .. + w_n \mathbb{1}_{Q_n}$ with $w_i > 0$ and f with constant values $b_1 \geq .. \geq b_n$ in B on $Q_1, .., Q_n$. We use the weight functions $v := w_1 \mathbb{1}_{Q_1} + .. + w_{n-1} \mathbb{1}_{Q_{n-1}}$ and $u := w_2 \mathbb{1}_{Q_2} + .. + w_n \mathbb{1}_{Q_n}$ to forget the largest and the smallest values of f. $\mathcal{A}(n-1)$ implies $\mathrm{H}\overline{f_{\mathcal{Q}(v)}} \subset \mathrm{H}\overline{f^{*v}_{\mathcal{I}(v)}}$ and $\mathrm{H}\overline{f_{\mathcal{Q}(u)}} \subset \mathrm{H}\overline{f^{*u}_{\mathcal{I}(u)}}$. With the visibility of $\mathrm{H}\overline{f_{\mathcal{Q}(w)}}$ (Theorem 2.1), argueing as in Theorem 1.3, we get $\mathrm{H}\overline{f_{\mathcal{Q}(w)}} \subset \mathrm{H}\overline{f_{\mathcal{Q}(v)}}$, $\mathrm{H}\overline{f_{\mathcal{Q}(w)}} \subset \mathrm{H}\overline{f_{\mathcal{Q}(u)}}$, $\mathrm{H}\overline{f_{\mathcal{Q}(w)}} \subset \mathrm{H}\,S$, where S is again the $\partial A[b_n, b_1]$-shadow of the total mean value $f_{Q_0,w}$. Following the arguments in Theorem 1.3, we finally obtain $\mathrm{H}\overline{f_{\mathcal{Q}(w)}} \subset \mathrm{H}\overline{f^{*w}_{\mathcal{I}(w)}}$. □

3. – Mean values on cubes, counterexamples

It would be interesting to know Theorem 2.1 and 2.3 for N-dimensional cube mean value sets instead of N-dimensional interval mean value sets. Unfortunately, the cube situation is much more complicated. Even worse, formulated verbatim as above, the theorems are wrong for cubes.

For example, let $b_1 = \mathrm{e}^{0.75\pi \mathrm{i}}$, $b_2 = \mathrm{e}^{1.5\pi \mathrm{i}}$, $b_3 = \mathrm{e}^{0.25\pi \mathrm{i}}$, B be the unit circle arc from b_1 counterclockwise to b_3, $f : [0,3]^2 \to B$ be the step function which is suggested in figure 3a and $\mathcal{W}$ be the set of all nondegenerateted squares $I \times J \subset [0,3]^2$. Figure 3b shows the square mean value set $f_{\mathcal{W}}$. To be more exact, the plotted lines are a subset $f_{\mathcal{W}_m}$ with the set $\mathcal{W}_m$ of all squares having at least one vertice in $\{(3i/m, 3j/m),\ i,j = 0,..,m\}$. It is easy to compute $f_{\mathcal{W}_m}$ with a computer program. The sets $f_{\mathcal{W}_m}$ approximate $f_{\mathcal{W}}$ (the plot is for $m = 9$). Figure 3c presents the mean value set $f^*_{\mathcal{I}}$ of the decreasing rearrangement $f^* : [0,9] \to B$. Obviously, the cave $\mathrm{H} f_{\mathcal{W}}$ is not visible and the inclusion $\mathrm{H} f_{\mathcal{W}} \subset \mathrm{H} f^*_{\mathcal{I}}$ is wrong. So Theorem 2.1 and 2.3 are not true when $f_{\mathcal{Q}}$ is replaced by $f_{\mathcal{W}}$.

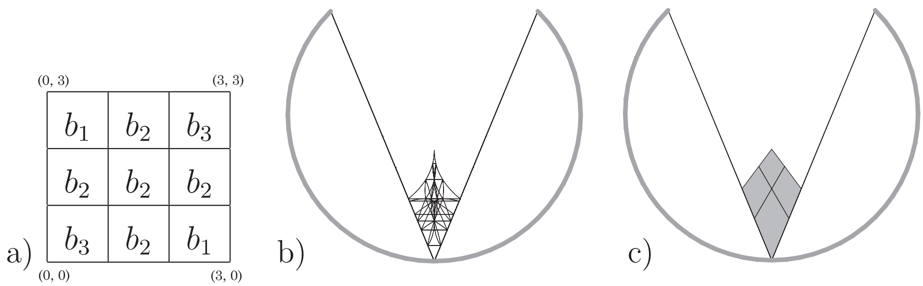

Fig. 3. – a) Where f equals b_i on $[0,3]^2$; b) $f_{\mathcal{W}}$; c) $f^*_{\mathcal{I}}$.

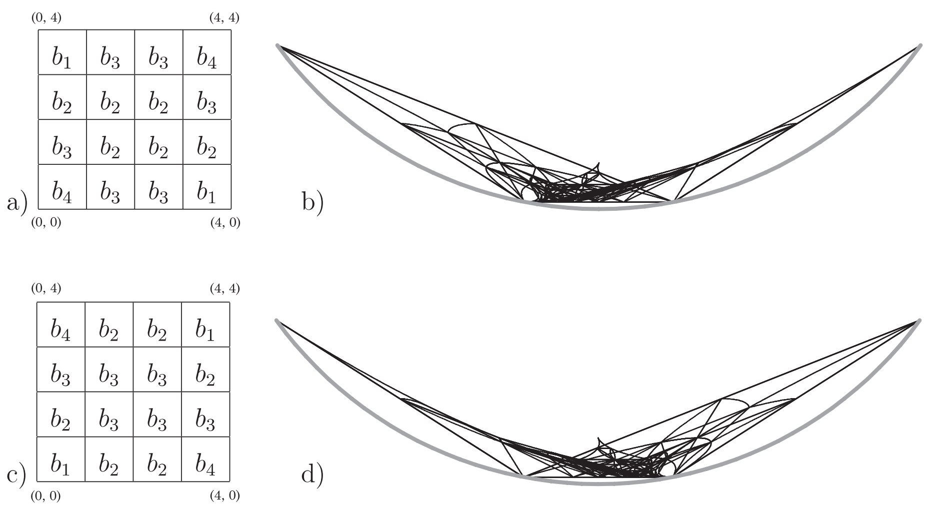

Fig. 4. – a) Where f equals b_i on $[0,4]^2$. b) $f_{\mathcal{W}}$. c) Where g equals b_i on $[0,4]^2$. d) $g_{\mathcal{W}}$

The following example demonstrates another aspect. Let $b_1 = \mathrm{e}^{1.2\pi\mathrm{i}}$, $b_2 = \mathrm{e}^{1.44\pi\mathrm{i}}$, $b_3 = \mathrm{e}^{1.56\pi\mathrm{i}}$, $b_4 = \mathrm{e}^{1.8\pi\mathrm{i}}$, B be the unit circle arc from b_1 counterclockwise to b_4, f and

$g : [0,4]^2 \to B$ be the functions suggested in figure 4a and 4c. The figures 4b and 4d show the square mean value sets $f_{\mathcal{W}}$, $g_{\mathcal{W}}$ (or to be exact, an approximation of these sets) for the set $\mathcal{W}$ of all nondegenerated squares $I \times J \subset [0,4]^2$. In both cases the cusp point in the middle is the total mean value $f_{[0,4]^2}$. Notice that, locally at the cusp point, $f_{\mathcal{W}}$ is on the left and $g_{\mathcal{W}}$ on the right side of the vertical line through the cusp point. The resolution may hide this effect a bit; anyway, the effect can be proved by elementary calculations. We can conclude that *no rearrangement of f and g exists with a largest cave*. (If we would assume h to be a rearrangement of f, g with $\mathrm{H} f_{\mathcal{W}} \cup \mathrm{H}\, g_{\mathcal{W}} \subset \mathrm{H}\, h_{\mathcal{W}}$, then a small disk around $f_{[0,4]^2}$ would exist in $\mathrm{H}\, h_{\mathcal{W}}$. Since the total mean value $f_{[0,4]^2}$ is in the mean value set of all rearrangements and since the mean value set is always path connected, there would be a contradiction.)

For a N-dimensional cube W_0 and every (reasonable) function $f : W_0 \to B$ we conjecture a weakened extremal property of monotone rearrangement like.

CONJECTURE 3.1. – $\mathrm{H}\overline{f_{\mathcal{W}}} \subset \mathrm{H}\,(\overline{f^*_{\mathcal{I}_1}} \cup \overline{f^*_{\mathcal{I}_2}})$ *holds for the sets* $\mathcal{I}_1, \mathcal{I}_2$ *of all non-degenerated intervals in* $[0, |W_0|/2]$, $[|W_0|/2, |W_0|]$.

The validity of this conjecture would be enough for the applications we have in mind. However, since cubes cannot be divided in two subcubes, the essential construction of opposite paths used in Theorem 1.2 or 2.1 is no longer possible. This is the reason why the methods developed in the sections above fail in the cube situation.

4. – Mean values on curves

We generalize the one-dimensional results of Section 1 to functions on curves. By using space filling curves this includes higher-dimensional results.

Let I_0 be a compact interval, P_0 a set, d a metric on P_0 and μ an (outer) measure on P_0 with $\mu(P_0) > 0$. We call a function $p : I_0 \to P_0$ **measure preserving** if the inverse image $p^{-1}(P)$ of every μ-measurable set $P \subset P_0$ is Lebesgue measurable with $|p^{-1}(P)| = \mu(P)$. Furthermore, p is defined as **almost injective** if the set $\{t \in I_0 : p^{-1}(\{p(t)\}) \neq \{t\}\}$ has Lebesgue measure zero. P_0 (with metric d and measure μ) is called a **curve interval** by $p : I_0 \to P_0$ if p is continuous, surjective, almost injective, measure preserving and if all images of subintervals of I_0 are μ-measurable. For curve intervals we define the set $p(\mathcal{I}) := \{p(I) : I \text{ subinterval of } I_0 \text{ with } |I| > 0\}$ of all curve subintervals.

THEOREM 4.1 (Cave visibility and extremal property on curves). – *Let P_0 with metric d and measure μ be a curve interval by $p : I_0 \to P_0$. For a continuous function $f : P_0 \to B$ with convexly arranged range the cave $\mathrm{H} f_{p(\mathcal{I})}$ of the curve interval mean values is visible and it is a subset of the cave $\mathrm{H} f^*_{\mathcal{I}}$ of the interval mean values of the monotone μ-rearrangement $f^* : I_0 \to B$.*

PROOF. – $f \circ p : I_0 \to B$ is a continuous function with convexly arranged range. Theorem 1.2 and 1.3 show visibility of $\mathrm{H}\,(f \circ p)_{\mathcal{I}}$ and its inclusion into the cave $\mathrm{H}\,(f \circ p)^*_{\mathcal{I}}$. Since p is measure preserving and almost injective, we have $\mu(p(I)) = |p^{-1}(p(I))| = |I|$ for every nondegenerated subintervals $I \subset I_0$ and $\mu\{x \in p(I) : f(x) > b\} = |p^{-1}\{x \in p(I) : f(x) > b\}| = |\{t \in I : f \circ p(t) > b\}|$ for $b \in B$. So the distribution functions $b \mapsto \mu\{x \in p(I) : f(x) > b\}$ and $b \to |\{t \in I : f \circ p(t) > b\}|$ coincide. Since integrals only depend on the distribution function, the mean values $(f \circ p)_I = \frac{1}{|I|} \int\limits_I f \circ p$ and $f_{p(I)} = \frac{1}{\mu(p(I))} \int\limits_{p(I)} f \, d\mu$ are the same. The identity of the distribution functions also proves $(f \circ p)^* = f^*$. So we have

$$\mathrm{H} f_{p(\mathcal{I})} = \mathrm{H}\,(f \circ p)_{\mathcal{I}} \subset \mathrm{H}\,(f \circ p)^*_{\mathcal{I}} = \mathrm{H} f^*_{\mathcal{I}},$$

which proves the statement. □

As an example let P_0 be the unit circle $\mathbb{S}^1$ with the standard metric (as a subset of $\mathbb{C}$) and with the one-dimensional Hausdorff measure. $\mathbb{S}^1$ is a curve interval by $p : [0, 2\pi] \to \mathbb{S}^1, p(t) := \mathrm{e}^{\mathbb{i}t}$. For functions $f : \mathbb{S} \to B$ with convexly arranged range the theorem deals with the mean values on $p(\mathcal{I})$, which is the set of all nondegenerated arcs not having $1 = \mathrm{e}^{\mathbb{i}0}$ as an «inner» point. For the (in this context more interesting) set $\mathcal{S}$ of all nondegenerates arcs we get cave visibility and the extremal property, too: choose $\tilde{p} : [0, 2\pi] \to \mathbb{S}^1, \tilde{p}(t) = \mathrm{e}^{\mathbb{i}t + \pi}$ to get $\mathcal{S} = p(\mathcal{I}) \cup \tilde{p}(\mathcal{I})$ and with Lemma 1.1 $\mathrm{H} f_{\mathcal{S}} = \mathrm{H} f_{p(\mathcal{I})} \cap \mathrm{H} f_{\tilde{p}\mathcal{I}} \subset \mathrm{H} f^*_{\mathcal{I}}$ plus the visibility of $\mathrm{H} f_{\mathcal{S}}$.

Space filling curves yield interesting N-dimensional results. Such curves produce curve interval systems, which allow only point degeneration and not line/plane degeneration like N-dimensional intervals. For example let $\mathbb{h}_N : [0, 1] \to [0, 1]^N$ be the N-dimensional Hilbert curve, cf. [Sag94]. It is well known that $[0, 1]^N$ is a curve interval by $\mathbb{h}_N$. The range of each dyadic interval $[k/2^{Nn}, (k + 1)/2^{Nn}] \subset [0, 1]$ is a cube in $\mathcal{P}$. Every other set in $\mathcal{P}$ is a certain union of at most contably many such cubes. These unions may have a complicate boundary, but $\mathcal{P}$ is similar to the cube system $\mathcal{W}$ and only allows point degeneration in the following sense: For every $P \in \mathcal{P}$ a cube $W \supset P$ and a cube $V \subset P$ with $1 \leq |W|/|V| \leq 4^N$ exist (the first inequality is trivial, the second is a consequence of the construction of $\mathbb{h}_N$).

5. – Applications to reverse inequalities

We apply our method to functions satisfying reverse inequalities like reverse Hölder inequalities, oscillation inequalities or Muckenhoupt inequalities. For $c > 1$ and $q > 1$ let $RH(c, q, \mathcal{I})$ be the set of all continuous $f : [0, 1] \to [0, \infty[$ satisfying reverse Hölder inequalities $⨍_I f^q \leq c\big(⨍_I f\big)^q$ for all $I \in \mathcal{I}$. For $0 < c < 2$

and $q \geq 1$ let $OS(c, q, \mathcal{I})$ be the set of all continuous $f : [0,1] \to [0, \infty[$ satisfying oscillation inequalities (Gurov-Reshetnyak condition) $⨍_I |f - f_I|^q \leq c\big(⨍_I f\big)^q$ for all $I \in \mathcal{I}$. In the most important case we abbreviate $OS(c, \mathcal{I}) := OS(c, 1, \mathcal{I})$. For $c > 0$ and $q > 1$ let $MU(c, q, \mathcal{I})$ be the set of all continuous $f : [0,1] \to [0, \infty[$ satisfying Muckenhoupt inequalities $⨍_I f\big(⨍_I f^{-1/(q-1)}\big)^{q-1} \leq c$ for all $I \in \mathcal{I}$. For $N \geq 2$ we define in the same way the sets $RH(c, q, \mathcal{Q})$, $RH(c, q, \mathcal{P})$, $OS(c, q, \mathcal{Q})$, $OS(c, q, \mathcal{P})$, $MU(c, q, \mathcal{Q})$, $MU(c, q, \mathcal{P})$ of continuous functions $f : [0,1]^N \to [0, \infty[$ satisfying corresponding inequalities. Here $\mathcal{Q}$ is the system of the N-dimensional intervals and $\mathcal{P} := \mathbb{h}_N(\mathcal{I})$ the system of N-dim. Hilbert intervals (see Section 4). The restriction to continuous functions is made to simplify the considerations. It is no serious restriction, since for discontinuous functions regularization yields the same reverse inequalities for an approximating sequence of continuous functions (at least for reverse Hölder inequalities, see [BoIw83]).

For the present purpose we suppose more generally (negative values allowed), continuous $f : [0,1]^N \to \mathbb{R}$. Let $k : [\inf f, \sup f] \to [0, \infty[$ be a convex Lipschitz function. This function k takes the role of $x \mapsto x^q$ for reverse Hölder, $x \mapsto |x - y|^q$ for oscillation inequalities and $x \mapsto x^{-1/(q-1)}$ for Muckenhoupt inequalities. Since k is fixed in every context, we suppress k-dependence in notations. The decreasing rearrangement $f^* : [0,1] \to \mathbb{R}$ is defined by $f^*(t) := \inf\{y \in \mathbb{R} : |\{s \in [0,1]^N : f(s) > y\}| \leq t\}$. Obviously, $h := (f, k \circ f) : [0,1]^N \to \operatorname{graph}(k)$ is a function with convexly arranged range and the decreasing rearrangement of h, as defined in Section 1 and 2, is $h^* = (f^*, k \circ f^*)$. In the previous sections we have analysed **mean value sets** like

$$M(f, \mathcal{I}) := h_{\mathcal{I}} = \left\{ \left(⨍_I f, ⨍_I k \circ f \right) : I \in \mathcal{I} \right\}$$

or $M(f, \mathcal{Q}), M(f, \mathcal{W}), M(f, \mathcal{P})$. For such sets we define the **upper boundary**

$$B(f, \mathcal{I}) := \partial M(f, \mathcal{I}) \cap \partial E(f, \mathcal{I}) \subset \,]\inf f, \sup f[\times [0, \infty[$$

as the joint boundary of $M(f, \mathcal{I})$ and of the unbounded connected component $E(f, \mathcal{I})$ of $(]\inf f, \sup f[\times [0, \infty[) \setminus M(f, \mathcal{I})$. Note that it is in no way obvious that the upper boundary is the graph of a function. The following theorem is a summary of the previous results for the special situation considered here.

THEOREM 5.1 (Caves in the graph situation). – 1. *For a monotone, continuous $f : [0,1] \to \mathbb{R}$ the upper boundary Bf is the union of the ranges of the paths $w :]0,1] \to \mathbb{C}$ and $v := [0,1[\to \mathbb{C}$, defined as*

$$w(t) := \left(⨍_0^t f, ⨍_0^t k \circ f \right) \text{ and } v(t) := \left(⨍_t^1 f, ⨍_t^1 k \circ f \right).$$

2. *For every continuous function* $f : [0,1] \to \mathbb{R}$ *there is a Lipschitz function* $b_f :]\inf f, \sup f[\to \mathbb{R}$ *that describes the upper boundary:* $B(f, \mathcal{I}) = \mathrm{graph}(b_f)$. *This Lipschitz function is bounded from below by the corresponding Lipschitz function for the decreasing rearrangement:*

$$b_f(t) \geq b_{f^*}(t) \text{ for all } t \in]\inf f, \sup f[.$$

3. *The same is true for continuous* $f : [0,1]^N \to \mathbb{R}$ *and for the upper boundary* $B(f, \mathcal{Q})$ *or* $B(f, \mathcal{P})$*, but not in this form for* $B(f, \mathcal{W})$.

PROOF. – Apart from trivial cases (k linear or f constant) this is a summary of the Theorems 1.2, 1.3, 1.4, 2.2, 2.4, 4.1 for the graph situation. For that we choose the open, convex set A bordered by the graph B of k and the line segment $C :=](\inf f, k(\inf f)), (\sup f, k(\sup f))[$. The Lipschitz property is a consequence of the visibility which implies in fact also an estimate of the Lipschitz constant in terms of $\inf f$, $\sup f$ and k. The counterexamples in Section 3 are discontinuous functions, but smoothing these examples yields the negative results for $B(f, \mathcal{W})$. □

For the terms in reverse inequalities in particular we can state

COROLLARY 5.2. – *Let* $f : [0,1] \to [0, \infty[$ *be continuous and* $t \in]0,1[$.
1a) *An interval* $I \in \mathcal{I}$ *exists with* $⨍_I f = ⨍_0^t f^*$ *and* $⨍_I f^q \geq ⨍_0^t f^{*q}$.
1b) *An interval* $I \in \mathcal{I}$ *exists with* $⨍_I f = ⨍_t^1 f^*$ *and* $⨍_I f^q \geq ⨍_t^1 f^{*q}$.
2a) *An interval* $I \in \mathcal{I}$ *exists with* $⨍_I f = ⨍_0^t f^*$ *and* $⨍_I |f - f_I|^q \geq ⨍_0^t |f^* - f^*_{[0,t]}|^q$.
2b) *An interval* $I \in \mathcal{I}$ *exists with* $⨍_I f = ⨍_t^1 f^*$ *and* $⨍_I |f - f_I|^q \geq ⨍_t^1 |f^* - f^*_{[t,1]}|^q$.
3a) *An interval* $I \in \mathcal{I}$ *exists with* $⨍_I f = ⨍_0^t f^*$ *and* $⨍_I f^{-1/(q-1)} \geq ⨍_0^t (f^*)^{-1/(q-1)}$.
3b) *An interval* $I \in \mathcal{I}$ *exists with* $⨍_I f = ⨍_t^1 f^*$ *and* $⨍_I f^{-1/(q-1)} \geq ⨍_t^1 (f^*)^{-1/(q-1)}$.

All these statements are optimal, since for monotone f *the equality* $⨍_I f = ⨍_0 t f^*$ *implies* $⨍_I f^q \leq ⨍_0^t f^{*q}$, $⨍_I |f - f_I|^q \leq ⨍_0^t |f^* - f^*_{[0,t]}|^q$, $⨍_I f^{-1/(q-1)} \leq ⨍_0^t (f^*)^{-1/(q-1)}$ *and the equality* $⨍_I f = ⨍_t^1 f^*$ *implies the corresponding inequalities.*

For continuous $f : [0,1]^N \to [0, \infty[$ *similarly* $Q \in \mathcal{Q}$ *and* $P \in \mathcal{P}$ *exist with analogous properties.*

PROOF. – For every t with $⨍_0^t f^*$ and $⨍_t^1 f^*$ in $]\inf f, \sup f[$ the statements 1a and 1b follow directly from Theorem 5.1 for $k(x) = x^q$. In the «borderline» cases, e.g. $⨍_0^t f^* = \sup f$, the decreasing rearrangement f^* is constant on $[0,t]$ and continuity of f implies the existence of the stated I. Statements 3a and 3b are proved in the same way with $k(x) = x^{-1/(q-1)}$. To prove 2a we consider $k_t(x) := |x - f^*_{[0,t]}|^q$ for every $t \in]0,1[$. There is an interval $I \in \mathcal{I}$ with $⨍_I f = ⨍_0^t f^*$ and $⨍_I |f - f_I|^q = ⨍_I |f - f^*_{[0,t]}|^q \geq ⨍_0^t |f - f^*_{[0,t]}|^q$. Statement 2b follows analogously. □

COROLLARY 5.3 [Monotone rearrangement in reverse inequalities]. – *Reverse inequalities for a function imply exactly the same for its decreasing rearrangement:*

1) $f \in RH(c,q,\mathcal{I})$, $RH(c,q,\mathcal{Q})$ *or* $\mathrm{RH}(c,q,\mathcal{P})$ *implies* $f^* \in RH(c,q,\mathcal{I})$.
2) $f \in OS(c,q,\mathcal{I})$, $OS(c,q,\mathcal{Q})$ *or* $OS(c,q,\mathcal{P})$ *implies* $f^* \in OS(c,q,\mathcal{I})$.
3) $f \in MU(c,q,\mathcal{I})$, $MU(c,q,\mathcal{Q})$ *or* $MU(c,q,\mathcal{P})$ *implies* $f^* \in MU(c,q,\mathcal{I})$.

Thus reverse inequalities on $\mathcal{I}$, $\mathcal{Q}$, $\mathcal{P}$ reduce to the easier monotone situation. For decreasing power functions $f(x) = x^{-1/p}$ it need just an integration to check for which p reverse inequalities $⨍_0^t (x^{-1/p})^q \leq c\big(⨍_0^t x^{-1/p}\big)^q$ are true. Elementary calculations like this yield the below equalities below defining the sharp bounds. It has been proved that these bounds are the sharp bounds indeed for all monotone functions satisfying reverse Hölder inequalities [ApSb90], Muckenhoupt inequalities [Kor92a], or oscillation inequalities with $q=1$ [Kor92b]. This allow us to state the following sharp versions of important, classical results about reverse inequalities. Note that these theorems for continuous functions imply higher integrability for general (measurable) functions satisfying the respective hypothesis.

THEOREM 5.4 (Sharp version of the Gehring lemma). – *For a function* $f \in RH(c,q,\mathcal{I})$, $RH(c,q,\mathcal{Q})$ *or* $RH(c,q,\mathcal{P})$ *and* $p \in [q,Q[$ *a constant* $\tilde{c}$ *exists with* $f^* \in RH(\tilde{c},p,\mathcal{I})$. *The sharp bound Q is determined by*

$$\frac{Q-1}{Q}\left(\frac{Q}{Q-q}\right)^{\frac{1}{q}} = c.$$

THEOREM 5.5 (Sharp version of the Gurov-Reshetnyak theorem). – *For a function* $f \in OS(c,\mathcal{I})$, $OS(c,\mathcal{Q})$ *or* $OS(c,\mathcal{P})$ *and* $p \in [1,Q[$ *a constant* $\tilde{c}$ *exists with* $⨍_0^t f^* \leq \tilde{c}t^{-1/p}$ *for all* $t \in]0,1]$. *The sharp bound Q is determined by*

$$\left(\frac{Q-1}{Q}\right)^Q \frac{1}{Q-1} = \frac{c}{2}.$$

THEOREM 5.6 (Sharp version of the Muckenhoupt theorem). – *For a function* $f \in MU(c,q,\mathcal{I})$, $MU(c,q,\mathcal{Q})$ *or* $MU(c,q,\mathcal{P})$ *and* $p \in]Q,q]$ *a constant* $\tilde{c}$ *exists with* $f^* \in MU(\tilde{c},p,\mathcal{I})$. *The sharp bound Q is determined by*

$$\frac{q-Q}{q-1}(cQ)^{\frac{1}{q-1}} = 1.$$

REMARK 5.7 (Comparison with other results). – The **Gehring lemma** was first proved in [Geh73]. Many proofs, generalisations and applications have been found. Detailed references are in [Iwa98]. In [FrMo85] and [Sbo86] a reduction to monotone functions is used. Since the resulting reverse Hölder inequalities for f^* have an enlarged constant, the proof cannot yield sharp results. The question of sharp bounds in the Gehring lemma is posed e.g. in [Boj85, Iwa82]. In [Boj85, Wik90] some asymptotic answers are given. In [ApSb90] the sharp bound for monotone functions is calculated, see also [Nan90]. Our approach gives some additional information which is not contained in [ApSb90]: Only the inequalities on the «boundary» intervals $[0, t]$ and $[t, 1]$ (and not arbitrary intervals $I \subset [0, 1]$) are relevant in the reverse inequalities for monotone functions (Theorem 5.1 or Corollary 5.2). Further calculations show that the upper boundary part $\{(⨍_0^t f, ⨍_0^t f^q); t \in]0, 1]\}$ with $f(x) = x^{-1/Q}$ and Q as in Theorem 5.4 is part of the graph of cx^q. This makes the limit case more clear. Korenovskij proves in [Kor92a] that for $f \in RH(c, q, \mathcal{I})$ the same sharp bound as in the monotone situation is true. His main tool is the inequality

$$\sup \frac{⨍_I f^q}{\left(⨍_I f\right)^q} \geq \sup \frac{⨍_I f^{*q}}{\left(⨍_I f^*\right)^q}, \tag{7}$$

where the suprema are taken over all $I \in \mathcal{I}$ with $⨍_I f > 0$. This is an extremal property of the decreasing rearrangement. The extremal property in Theorem 5.1 is stronger and has a geometric interpretion. Furthermore the theorem includes additional informations like the Lipschitz continuity. In [Fio96] the sharp bound found by Koronovskij is used for one-dimensional variational problems. Kinnunen proves in [Kin94] with another method that for $f \in RH(c, q, \mathcal{Q})$ the same sharp bound as in the monotone situation is true. Again, we recover this result with additional information. For reverse Hölder inequalities on $\mathcal{W}$ or similar systems a Gehring lemma but no sharp bound is known. So, our contributions on $\mathcal{P}$ is the first sharp result. Variants of the counterexamples in Section 3 are a first negative results for $\mathcal{W}$: for reverse Hölder inequalities on $\mathcal{W}$ the decreasing rearrangement f^* need not satisfy the same reverse Hölder inequalities. We do not know if for $\mathcal{W}$ the same sharp bound as in the monotone situation is true or not. The counterexamples do not disprove this and Conjecture 3 would be enough to prove it.

The **Gurov-Reshetnyak theorem** was first proved in [GuRe76]. We also refer to [Iwa82]. In [Kor92b] a sharp bound is found for $OS(c, \mathcal{I})$. We recover this result with additional information. The results for $OS(c, \mathcal{Q})$ and $OS(c, \mathcal{P})$ are the first sharp, higher-dimensional results.

The **Muckenhoupt theorem** was first proved in [Muc72]. We also refer to [GaRu85]. Korenovskii proves in [Kor92a] with a variant of (7) that for $f \in MU(c, q, \mathcal{I})$ the same sharp bound as for monotone functions is true and he calculates the sharp bound in this easier situation. We recover the reduction of $MU(c, q, \mathcal{I})$ to the monotone situation with additional information. For higher-dimensional Muckenhoupt inequalities our results on $\mathcal{Q}$ and $\mathcal{P}$ are the first sharp ones.

For all mentioned sharp results [Kor92a, Kor92b, Kin94] a sharp, one-dimensional version of the rising sun lemma of Riesz is used. In the proof of this **covering lemma** special properties of open, one-dimensional sets are needed. There is no sharp higher-dimensional analogous statement. Our method is the first approach to reverse inequalities without any covering lemma. Our proof unifies the approach to reverse inequalities and is based on a new **extremal property** of the decreasing rearrangement. For other extremal properties of the decreasing rearrangement we refer to [Kol89].

In many approaches to reverse inequalities maximal functions like the **Hardy-Littlewood maximal function** $\mathcal{M}f(t) := \sup_{I\in\mathcal{I}(t)} ⨍_I |f|$ appear, where $\mathcal{I}(t)$ is the system of all intervals in $\mathcal{I}$ containing t. Maximal functions are used in a lot of other problems as well [BeSh88, Ste93]. Inequalities connecting the decreasing rearrangement of the maximal function with f^* are essential for the usefulness of maximal functions. For example, the Herz theorem, see [BeSh88, Theorem 3.3.8], states the existence of a constant c with

$$(8) \qquad ⨍_0^t f^* \leq c(\mathcal{M}f)^*(t) \quad \text{for all } t \in]0,1].$$

We think that modified maximal functions can be useful. We take the suprema with respect to intervals giving a prescribed mean value instead of intervals containing a given point. This stresses the range of f instead of its domain. Thus Theorem 5.1 with the convex function $k(x) = |x|$ suggests the «maximal function» $Bf(t) := \sup\{⨍_I |f|; I \in \mathcal{I}, ⨍_I f = ⨍_0^t f^*\}$. The graph of Bf is the upper boundary part of the mean value set with first coordinate larger than $⨍_0^1 f$; there is no such geometric interpretation for $\mathcal{M}f$. Theorem 5.1 yields

$$(9) \qquad ⨍_0^t |f^*| \leq Bf(t) \quad \text{for all } t \in]0,1].$$

This is comparable to (8), but here the optimal constant 1 appears. We are not aware of any optimal version of (8) and of any result for $\mathcal{M}f$ comparable to the Lipschitz property of Bf. As Bf is a better tool for reverse Hölder inequalities than $\mathcal{M}f$, we expect improvements also in other problems when $\mathcal{M}f$ is substituted by Bf.

For another prominent maximal function, the **sharp maximal function** $f^\sharp(t) := \sup_{I\in\mathcal{I}(t)} ⨍_I |f - f_I|$, the Fefferman-Stein inequality plays a similar role as

(8) for $\mathcal{M}f$. With our notations the following version can be found in [BeSh88, Theorem 5.7.3]: a constant $c > 0$ exists with

$$(10) \qquad \left(\fint_0^t |f|^*\right) - |f|^*(t) \le c(f^\sharp)^*(t) \quad \text{for all } t \in \,]0, 1/6].$$

The modified maximal functions $S_1 f(t) := \sup\{\fint_I |f - f_I|; I \in \mathcal{I}, \fint_I f = \fint_0^t f^*\}$ and $S_2 f$, defined analogously with $\fint_t^1 f^*$ instead of $\fint_0^t f^*$, are comparable to the sharp maximal function and Theorem 5.1 yields a comparable result to the Fefferman-Stein inequality:

$$(11) \quad \fint_0^t |f^* - f^*_{[0,t]}| \le S_1 f(t) \quad \text{and} \quad \fint_t^1 |f^* - f^*_{[t,1]}| \le S_2 f(t) \quad \text{for all } t \in \,]0, 1[$$

Note that here, in contrast to (10), the optimal constant 1 appears and the inequalities are true for the whole interval. Our results for oscillation inequalities can be formulated with $S_1 f$ and $S_2 f$, and they are stronger than results for oscillation inequalities proved with $f^\sharp$. Thus we expect improvements also in other problems, when $S_1 f$ and $S_2 f$ instead of $f^\sharp$ are used.

Closely connected to the sharp maximal function is the **BMO**-«norm», defined by $\|f\|_{BMO} := \sup_{I \in \mathcal{I}} \fint_I |f - f_I|$ for functions on $[0, 1]$. Functions of bounded mean oscillation, i.e. $\|f\|_{BMO} < \infty$, are of great importance, cf. [Ste93]. In [Kle85] the above mentioned sharp, one-dimensional version of the rising sun lemma of Riesz is proved and used to yield

$$(12) \qquad \|f^*\|_{BMO} \le \|f\|_{BMO}.$$

By taking the suprema in (11) we recover this result. Furthermore we find that for the BMO-norm of monotone functions only the «boundary» intervals $[0, t]$ and $[t, 1]$ are important. For functions on $[0, 1]^N$ we can state $\|f^*\|_{BMO} \le \sup_{Q \in \mathcal{Q}} \fint_Q |f - f_Q|$ and $\|f^*\|_{BMO} \le \sup_{P \in \mathcal{P}} \fint_P |f - f_P|$. This are the first sharp, higher-dimensional versions of (12).

REMARK 5.8 (Generalized reverse inequalities). – The results can easily be generalized to reverse inequalities with convex functions more general than $x \mapsto x^q$, $x \mapsto |x - y|^q$ or $x \mapsto x^{-1/(q-1)}$, for example to **reverse Jensen inequalities**. Also weighted reverse inequalities can be treated in the same way (see Remark 1.6).

The special form of the reverse inequalities is not necessary for the reduction to the monotone situation (for the calculation of the sharp bound in the monotone situation it is, of course, important). What matters is a convex arrangement like $(\fint f, \fint f^q)$. For example inequalities like $\fint_I f^q \le ch(\fint_I f) + d$ with constants c, d and a function h imply the same inequalities for f^*.

A function f satisfies reverse Hölder inequalities if the upper boundary $B(f, \mathcal{I})$ and hence also $B(f^*, \mathcal{I})$ is bounded from above by the graph of cx^q. This is mainly a condition for small and for large values of f, since $f^*(t)$ may not increase to fast to infinity for $t \to 0$ and may not decrease to fast to zero for $t \to 1$. We believe that in many applications a condition on the small values is unnatural. Our results make clear that small and large values can be treated separately. For example, it is possible that $RH(c, q, \mathcal{I})$ is a condition which is too strong for the small values of f, but that f satisfies $⨍_I f^q \leq c \max\left((⨍_I f)^q, (⨍_0^1 f)^q\right)$. For large values this is the same condition as $RH(c, q, \mathcal{I})$ and would imply the same integrability.

For Theorem 5.1 we do not need functions with nonnegative values. For $f : [0,1] \to \mathbb{R}$ inequalities like $⨍_I |f|^q \leq c_1 |⨍_I f|^q + c_2$ can be reduced to the monotone situation.

With similar arguments we can treat other oscillation terms. Thus it is not hard to prove that $⨍_I ⨍_J |f - g| \leq c(⨍_I f)(⨍_J g)$ for $I \in \mathcal{I}$ imply the same inequalities for f^* and g^*.

Since our approach gives the best known results for reverse inequalities, we also expect improvements and sharp results for the important class of **weak reverse inequalities** (reverse inequalities with enlarged domain of integration). Unfortunately, mean value sets like $\{(⨍_{x-r}^{x+r} f, ⨍_{x-r/2}^{x+r/2} f^q); [x-r, x+r] \subset [0,1]\}$ do not behave as nicely as the mean value sets treated above and we do not know how to utilize our additional knowledge in this situation, yet. Thus an interesting problem remains open.

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