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Michele Ciarletta, Gerardo Iovane

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Hypersingular Integral Equations and Applications to Porous Elastic Materials with Periodic Cracks.

MICHELE CIARLETTA - GERARDO IOVANE (*)

Sunto. – *In questo lavoro viene presentato lo studio di equazioni integrali con nucleo ipersingolare, che hanno rilevanti applicazioni in molti problemi di dinamica ondulatoria, elasticità e meccanica dei fluidi, con condizioni al contorno miste. Il primo obiettivo del presente lavoro è l'individuazione delle soluzioni quando il nucleo è caratteristico ed il successivo sviluppo di un metodo numerico orientato al trattamento di tali equazioni nel caso di nuclei non caratteristici. In particolare, a partire dalle soluzioni analitiche per equazioni ipersingolari con nucleo caratteristico si effettua l'ottimizzazione dei parametri del metodo numerico che verrà poi utilizzato nel caso di nucleo non caratteristico. Il secondo obiettivo è lo studio di tali problemi nel contesto dei continui porosi quando al loro interno è presente una struttura periodica di cracks. In tale ambito a partire dalla teoria di Cowin e Nunziato ed applicando la trasformata di Fourier alle equazioni rappresentative del problema, tale analisi comporta lo studio di equazioni integrali ipersingolari. Nel caso specifico di deformazione piana operiamo con il metodo numerico diretto per il trattamento di equazioni integrali a nucleo ipersingolare non caratteristico. Infine, studiamo il fattore di concentrazione dello stress ed investighiamo il suo comportamento rispetto alla porosità.*

Summary. – *In this work a treatment of hypersingular integral equations, which have relevant applications in many problems of wave dynamics, elasticity and fluid mechanics with mixed boundary conditions, is presented. The first goal of the present study is the development of an efficient analytical and direct numerical collocation method. The second one is the application of the method to the porous elastic materials when a periodic array of co-planar cracks is present. Starting from Cowin-Nunziato model and by applying Fourier integral transform the problem is reduced to some integral equations. For the plane-strain problem we operate with a direct numerical treatment of a hypersingular integral equation. We also study stress-concentration factor, and investigate its behaviour versus porosity of the material.*

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1. – The method.

Let us consider the hypersingular equation of the following type

$$\int_{-1}^{1} \varphi(t)\left[\frac{1}{(x-t)^2} + K_0(x,t)\right] dt = f(x), \qquad |x| < 1, \tag{1.1}$$

where $K_0(x, t)$ is a regular part of the kernel. Direct numerical treatment of Eq. (1.1) is not easy. A particular case is the hypersingular equation with the characteristic kernel

$$\int_a^b \frac{g(t)\,dt}{(x-t)^2} = f'(x), \qquad x \in (a, b), \;\; f(x) \in C_2(a, b). \tag{1.2}$$

We find a solution of this equation bounded on the both ends $x = a, b$. The latter can be defined in exact explicit form, which is given by the following

THEOREM 1. – *Bounded solution of Eq.* (1.2) *is unique, being given by the following inversion formula*

$$g(x) = \frac{\sqrt{(x-a)(b-x)}}{\pi^2} \int_a^b \frac{f(t)\,dt}{\sqrt{(t-a)(b-t)}(x-t)}. \tag{1.3}$$

The previous theorem is demonstrated in [1]. Moreover, it justifies that any bounded solution of Eq. (1.2) vanishes at $x \to a, b$. The previous solution is possible only for a characteristic equation. In general the exact analytical solution cannot be constructed. Some numerical approach has to be developed. To construct a direct collocation technique to numerically solve equation (1.2) for arbitrary right-hand side, we divide the interval (a, b) to n small equal subintervals by the nodes $a = t_0, t_1, t_2, \dots, t_{n-1}, t_n = b$. Then the length of each interval is $h = (b-a)/n$, and the nodes $t_j = a + jh$, $j = 0, 1, \dots, n$. Let us denote the central points of each sub-interval (t_{i-1}, t_i) by x_i, so that $x_i = a + (i - 1/2)h$, $i = 1, \dots, n$. Therefore, we try to approximate Eq. (1.2) by the linear algebraic system

$$\sum_{j=1}^{n} g(t_j)\left(\frac{1}{x_i - t_j} - \frac{1}{x_i - t_{j-1}}\right) = f'(x_i), \qquad i = 1, \dots, n. \tag{1.4}$$

In [1] we demonstrated that by assuming $x \in (a, b)$, the difference between solution $g(x)$ of the system (1.4) at the point x and solution given by explicit formula (1.3) tends to zero, when $n \to \infty$.

In practice, to use this theorem, one may put the number of nodes to be increasing so that the considered point x be always among the nodes. The advantages of our method are: i) The system is finite; ii) The coefficients are given in an explicit form; iii) The coefficients are not double integrals. It can be proved the eq. (1.4) admits the explicit solution

$$g(x_l) = \frac{\Delta_l}{\Delta} = (x_e - t_0) \sum_{m=1}^{n} \frac{\sum_{k=1}^{m} f'(x_k)}{t_m - t_0} \times \frac{\prod_p (x_l - t_p) \prod_q (x_q - t_m)}{(x_l - t_m) \prod_{p \neq m} (t_m - t_p) \prod_{q \neq l} (x_q - x_l)} . \tag{1.5}$$

If the number of the nodes tends to zero then $g(x_l) \to g(x)$.

When we consider the full equation

$$\int_a^b \left[\frac{1}{(x-t)^2} + K_0(x, t) \right] g(t)\, dt = f'(x), \qquad x \in (a, b), \tag{1.6}$$

where

$$K_0(x, t) = \frac{\partial K_1(x, t)}{\partial x} . \tag{1.7}$$

no analytical solution can be easily obtained; and then the previous numerical methods tested on exact (characteristic) problem are the only chance.

Its bounded solution can be constructed by applying inversion of the characteristic part, that reduces Eq. (1.6) to a second-kind Fredholm integral equation:

$$g(x) + \int_a^b N_1(x, t)\, g(t)\, dt = f_1(x), \qquad x \in (a, b), \tag{1.8}$$

where

$$N_1(x, t) = \frac{\sqrt{(x-a)(b-x)}}{\pi^2} \int_a^b \frac{K_1(\tau, t)\, d\tau}{\sqrt{(\tau - a)(b-\tau)}\,(x-\tau)} , \tag{1.9}$$

$$f_1(x) = \frac{\sqrt{(x-a)(b-x)}}{\pi^2} \int_a^b \frac{f(\tau)\, d\tau}{\sqrt{(\tau - a)(b-\tau)}\,(x-\tau)} . \tag{1.10}$$

It is known from the classical theory of the Cauchy-type integrals that if $f(x) \in C_1(a, b)$, $K_1(x, t) \in C_1[(a, b) \times (a, b)]$ then also $f_1(x) \in C_1(a, b)$; $N_1(x, t) \in C_1[(a, b) \times (a, b)]$. This results in the following

THEOREM 2. – *Let* $f(x) \in C_1\ (a, b)$; $K_1(x, t) \in C_1[(a, b) \times (a, b)]$. *Then for any* $x \in (a, b)$ *the difference between solution* $g(x)$ *of the linear algebraic system*

$$\sum_{j=1}^{n} \left[\frac{1}{x_i - t_j} - \frac{1}{x_i - t_{j-1}} + hK_0(x_i, t_j) \right] g(t_j) = f'(x_i), \qquad i = 1, \dots, n \tag{1.11}$$

and the bounded solution of equation (1.6) *tends to zero when* $h \to 0$ *(i.e.* $n \to \infty$*).*

2. – The application to crack mechanics.

The linear theory of homogeneous and isotropic elastic material with voids is described by the following system of partial differential equations (where $\phi = \nu - \nu_0$ is the change in volume fraction from the reference one) [2], [3]

$$\begin{cases} \mu \varDelta \, \overline{u} + (\lambda + \mu) \text{ grad div } \overline{u} + \beta \text{ grad } \phi = 0 \\ \alpha \varDelta \, \phi - \xi\phi - \beta \text{ div } \overline{u} = 0 \,, \end{cases} \tag{2.1}$$

where μ and λ are classical elastic constants; α, β and ξ – some constants related to porosity of the medium. Besides, $\overline{u}$ denotes the displacement vector. Obviously, if $\beta = 0$, then the elastic and the «porosity» fields are independent. Thus, in the case $\beta = 0$ the stress-strain state is insensitive to the function ϕ. The components of the stress tensor are defined, in terms of the functions $\overline{u}$ and ϕ, by the following relations (δ_{ij} is the Kronecker's delta)

$$\begin{cases} \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu\varepsilon_{ij} + \beta\phi\delta_{ij} \\ \varepsilon_{ij} = \dfrac{1}{2}(u_{i,\,j} + u_{j,\,i}) \,. \end{cases} \tag{2.2}$$

Let us formulate the plane-strain boundary value problem for the porous (granular) medium. In this case $\overline{u} = \{u_x(x,\,y),\, u_y(x,\,y),\, 0\}$, and the basic system (2.1) can be rewritten as follows

$$\begin{cases} \dfrac{\partial^2 u_x}{\partial x^2} + c^2 \dfrac{\partial^2 u_x}{\partial y^2} + (1-c^2)\dfrac{\partial^2 u_y}{\partial x \partial y} + H\dfrac{\partial \phi}{\partial x} = 0 \\ \dfrac{\partial^2 u_y}{\partial y^2} + c^2 \dfrac{\partial^2 u_y}{\partial x^2} + (1-c^2)\dfrac{\partial^2 u_x}{\partial x\,\partial y} + H\dfrac{\partial \phi}{\partial y} = 0 \\ l_1^2\left(\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right) - \dfrac{l_1^2}{l_2^2}\phi - \left(\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y}\right) = 0 \,, \end{cases} \tag{2.3}$$

with

$$\begin{cases} \dfrac{\sigma_{xx}}{\lambda + 2\mu} = \dfrac{\partial u_x}{\partial x} + (1-2c^2)\dfrac{\partial u_y}{\partial y} + H\phi \,, \\ \dfrac{\sigma_{yy}}{\lambda + 2\mu} = (1-2c^2)\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} + H\phi \,, \\ \dfrac{\sigma_{xy}}{\mu} = \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \,, \end{cases} \tag{2.4}$$

where

$$(2.5)\qquad c^2 = \frac{\mu}{\lambda + 2\mu}, \qquad H = \frac{\beta}{\lambda + 2\mu}, \qquad l_1^2 = \frac{\alpha}{\beta}, \qquad l_2^2 = \frac{\alpha}{\xi},$$

where the first two numbers c, H are dimensionless and the quantities l_1, l_2 have the dimension of length. Let us consider a thin crack of the length $2a$ with plane faces, dislocated over the segment $-a < x < a$ along the x-axis. Let the plane-strain deformation of this crack be caused by a constant stress $\sigma_{yy} = \sigma_0$ applied at infinity (i.e. at $y = \pm\infty$). Then, due to linearity of the problem, it is easily seen that both the shape of the crack's faces and the stress concentration at the crack's edges are the same as in the problem with a solution decaying at infinity and the following boundary conditions corresponding to the case when the normal load $-\sigma_0$ is symmetrically applied to the faces of the crack and there is no load at infinity. For the last problem the boundary conditions over the line $y = 0$ are

$$(2.6)\qquad \sigma_{xy} = 0, \quad \frac{\partial \phi}{\partial y} = 0 \ (|x| < \infty), \quad \sigma_{yy} = -\sigma_0 \ (|x| < a), \quad u_y = 0 \ (|x| > a).$$

Let us apply the Fourier transform along the x-axis to relations (2.3), (2.4) and the boundary conditions (2.6). Then the problem can be reduced to the following hypersingular equation:

$$(2.7)\qquad \int_{-b}^{b} g(\xi)\, K(x - \xi)\, d\xi = -(1 - N)^2 \frac{\sigma_0}{2\mu}, \qquad |x| < b.$$

where g is the opening of the crack face, $N = (l_2^2/l_1^2)\, H$, $(0 \leq N < 1)$ and

$$(2.8)\qquad \begin{cases} K(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{|s|}{q(s)} [2Nc^2 s^2 (q - |s|) + (1 - N)(1 - N - c^2)\, q]\, e^{-isx} ds = \\ \qquad\qquad \dfrac{1}{\pi} \displaystyle\int_0^{\infty} L(s) \cos(sx)\, ds, \quad q = q(s) = \sqrt{s^2 + 1 - N}, \\ L(s) = \dfrac{s}{q(s)} [2Nc^2 s^2 (q - s) + (1 - N)(1 - N - c^2)\, q]. \end{cases}$$

The kernel (2.8) admits explicit representation that permits direct estimate of its singular properties.

This method find a good application in the cracks problem; In fact thanks to this approach in [1] it was investigated the effects of the porosity in a material respect to the stress concentration.

By investigating the influence of the porosity to the stress concentration factor shown in figure, we can discover, this factor in the medium with voids is

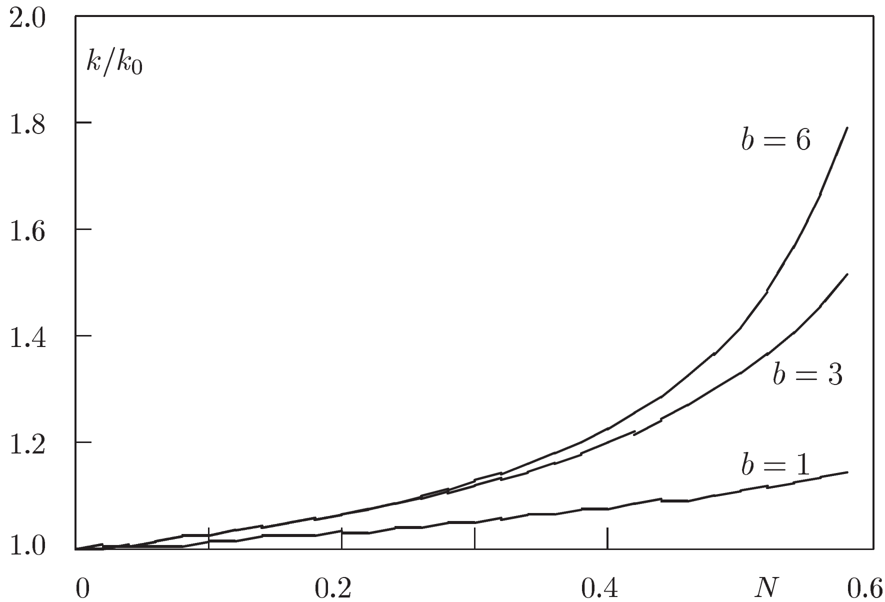

Fig. 1. – Relative value of the stress concentration factor k with respect to k_0 in classical elastic medium versus coupling number, plane linear crack: $c^2 = 0.4$.

always higher, under the same conditions, than in the classical elastic medium made of material of the skeleton. Further, as can be seen, influence of the porosity becomes more significant for larger cracks.

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Michele Ciarletta, Gerardo Iovane: D.I.I.M.A., University of Salerno, Italy

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