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On the Derivation of the Gross-Pitaevskii Equation.

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Sunto. – *Il presente articolo riproduce nel contenuto la conferenza che l'autore ha tenuto al* XVII Congresso dell'Unione Matematica Italiana, *svoltosi a Milano,* 8-13 *settembre* 2003. *Vengono presentati alcuni recenti risultati sul problema della derivazione dell'equazione di Gross-Pitaevskii in dimensione uno a partire dalla dinamica di un sistema quantistico che contiene un grande numero di bosoni identici. Sono spiegati i motivi per alcune scelte particolari (forma del potenziale di interazione, riscalamento, dato iniziale). Sono evidenziati i problemi aperti e sottolineate le difficoltà e gli ostacoli all'applicazione della stessa strategia in dimensione superiore.*

Summary. – *This article reflects in its content the talk the author gave at the* XVII Congresso dell'Unione Matematica Italiana, *held in Milano,* 8-13 *September* 2003. *We review about some recent results on the problem of deriving the Gross-Pitaevskii equation in dimension one from the dynamics of a quantum system with a large number of identical bosons. We explain the motivations for some peculiar choices (shape of the interaction potential, scaling, initial datum). Open problems are pointed out and difficulties and hindrances in replicating the strategy in higher dimension are put in evidence.*

1. – Introduction.

Since four decades the Gross-Pitaevskii equation (GPE) has been used to represent the ground state of a Bose-Einstein condensate (BEC). Indeed, even though the seminal papers are now eighty years old ([B], [E]), a satisfactory proposal for an effective equation for the state of the condensate has been achieved only fourty years later ([G], [P]). Nevertheless, a rigorous derivation for such an equation from microscopic dynamics is still lacking, at least for the time-dependent model. Here we report on a recent attempt in this direction, whose details can be found in [ABGT].

As a rough idea, one can think of a BEC as a three-dimensional system consisting of a large number of identical bosons at very low temperature: as predicted by Bose and Einstein there exists a critical value of the temperature

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under which a significant fraction of the particles of the system collapses to a well-defined quantum state. The wave function $\psi(t, x)$ associated to this state is supposed to be the stationary, lowest energy solution of the GPE

$$i\hbar \partial_t \psi(t, x) = -\frac{\hbar^2}{2m} \Delta \psi(t, x) + \alpha |\psi(t, x)|^2 \psi(t, x) + V_{\mathrm{ext}}(x)\, \psi(t, x) \tag{1.1}$$

where $\hbar$ is the Planck's constant, m is the mass of the generic particle, $t \in \mathbb{R}$ is the time variable, $x \in \mathbb{R}^3$ is the spatial coordinate, $\alpha > 0$, and V_{ext} represents an external potential term, whose role is to trap the condensate in some bounded spatial region.

Equation (1.1) has been widely investigated from various points of view, including the rigorous derivation of the corresponding stationary problem ([LSY]).

Our perspective is different, since we look for a rigorous derivation of the time-dependent problem (1.1).

In this spirit, it is worth noticing that eq. (1.1) refers to a one-particle system, whereas, according to standard rules of quantum mechanics, the evolution in time for a system consisting of a large number of particles subject to the potential field V_{ext} and interacting via a two-body potential V, is described by the Schrödinger equation

$$\begin{aligned} i\hbar \partial_t \Psi_N(t, X_N) = -\frac{\hbar^2}{2m} \sum_{j=1}^{N} \Delta_{x_j} \Psi_N(t, X_N) + \alpha \sum_{1 \leq i < j \leq N} V(x_i - x_j)\, \Psi_N(t, X_N) \\ + \sum_{j=1}^{N} V_{\mathrm{ext}}(x_j)\, \Psi_N(t, X_N). \end{aligned} \tag{1.2}$$

Here the unknown N-particle wave function $\Psi_N(t, X_N)$ depends on time and on the string $X_N = (x_1, \ldots, x_N)$, where x_j represents the spatial coordinate of the j-th particle.

Therefore, the following question naturally arises: how -and under which hypotheses- is it possible to derive the non-linear, one-particle equation (1.1) from the linear, N-particle equation (1.2)?

Problems like this are usually dealt with in the framework of the scaling limits: the «mesoscopic regime» (i.e. GPE) emerges from the «microscopic regime» (i.e. Schrödinger equation) as the number of particles of the systems increases ($N \to \infty$), the time and space scales are suitably magnified and the strength of the potential correspondingly rescaled.

The nature of the scaling to apply in our analysis is suggested by an old result by Spohn ([S]), and some new ones due to Bardos, Golse, Mauser, Erdos, Yau ([BGM], [EY], [BEGMY]), who, for quite a general choice of φ, proved that the mesoscopic Hartree equation

$$i\hbar\partial_t \psi(t, x) = -\frac{\hbar^2}{2m}\Delta\psi(t, x) + (\varphi * |\psi(t)|^2)(x)\,\psi(t, x) \tag{1.3}$$

can be rigorously derived from the Schrödinger equation (1.2) with $V_{\text{ext}} = 0$, under the three following assumptions:

(1) A suitable choice for the interacting potential: $V = \varphi$.

(2) A mean-field scaling: $V \to \dfrac{V}{N}$.

Notice that by such a scaling the interaction becomes weaker and weaker as the number of particles increases. In fact, one refers to this scaling as to the «weak coupling» one.

(3) The Hartree Ansatz, namely a peculiar choice of the initial datum Ψ_N^I for equation (1.2):

$$\Psi_N(0, X_N) = \Psi_N^I(X_N) = \prod_{j=1}^{N} \psi^I(x_j) \tag{1.4}$$

It is worth remarking that we are modelling a system of identical bosons, therefore the wave function of the system must be symmetric under exchange of pair of particles. Once we choose to start from a factorized state, the Hartree Ansatz comes automatically. Besides, one can think of (1.4) as the quantum version of the hypothesis of «molecular chaos», which is currently adopted in derivation of classical Boltzmann equation from newtonian dynamics. More precisely, in eq. (1.4) we suppose that every particle lies in a pure quantum state, whilst the classical molecular chaos involves more general states.

In order to derive the GPE (1.1), one could be tempted by replacing the potential φ with a Dirac's delta potential and applying the machinery developed in [BGM] to derive equation (1.3).

Unfortunately, in the case of three spatial dimensions a system of quantum bosons interacting one another via a delta potential is affected by serious problems of well-posedness, lying in the fact that the hamiltonian operator associated to the system is not lower bounded if N is large enough (Thomas effect, see e.g. [MF], [AGH-KH]). This effect prevents a priori the possibility of considering such a system as a suitable quantum model.

The Thomas effect is not present in space dimension two ([DFT]), however the definition of the hamiltonian is quite non trivial since a delta potential cannot be introduced via the corresponding quadratic form without a renormalization procedure. Anyway, the model constructed in that way is treatable, but the technicalities required are quite hard.

Therefore, as a starting point we consider the one-dimensional case, where systems of particles interacting via a delta potential are well defined ([LL]) and the domain of the energy is the ordinary space $H^1(\mathbb{R}^N)$.

Furthermore, we neglect the external potential V_{ext}, whose possible presence is not relevant for the strategies and the techniques employed.

In the sequel of the paper we will give the result obtained in [ABGT] (sec. 2) and we will sketch the main steps of the proof (sec. 3). The last section is devoted to perspectives and open problems.

2. – The result.

We start considering the Schrödinger equation for a system of N particles interacting through a delta potential in the mean field scaling

$$i\hbar\partial_t \Psi_N(t, X_N) = - \frac{\hbar^2}{2m} \Delta_{X_N} \Psi_N(t, X_N) + \frac{\alpha}{N} \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)\, \Psi_N(t, X_N) \tag{2.1}$$

where we introduced the shorthand notation $\Delta_{X_N} = \sum_{j=1}^{N} \Delta_{x_j}$.

Like in classical contexts (e.g. Boltzmann equation), the mechanism one invokes in deriving the mesoscopic from the microscopic regime is the so-called «propagation of chaos», which can be illustrated as follows.

As already noticed, the factorization of the initial datum given by the Hartree Ansatz represents the quantum version of the molecular chaos.

When time evolution (2.1) is turned on, factorization is destroyed due to the presence of the mutual interaction between the particles. However, the symmetry under exchange of particle pairs remains preserved.

Propagation of chaos states that in the limit $N \to \infty$ factorization is restored at any time and the elementary factor $\psi(t, x)$ of the resulting factorized state solves the mesoscopic equation (1.1).

Tipically, the proof of such a mechanism is done considering a subsystem of n $(\leq N)$ particles, letting the number of the particle outside the subsystem grow to infinity, and seeing what happens to the subsystem under investigation.

If propagation of chaos is verified, one should find that the state of the subsystem factorizes.

In order to proceed along this line, one is forced to abandon the description in terms of the wave function Ψ_N and to adopt the formalism of the density matrix ϱ_N, which is well fitting for the study of open (i.e. non isolated) systems.

Let us explain such formalism. The whole N-particle system is described by the N-particle wave function $\Psi_N(t; X_N)$ obtained as the solution of equation (2.1) with the initial datum specified by the Hartree Ansatz (1.4). The corresponding density matrix is the orthogonal projection on the linear space spanned by $\Psi_N(t)$ as an element of $L^2(\mathbb{R}^N)$, namely the integral operator $\widehat{\varrho}_N(t)$ whose kernel reads

$$\varrho_N(t; X_N; Y_N) \equiv (\Psi_N(t) \otimes \overline{\Psi}_N(t))(X_N; Y_N) \tag{2.2}$$

Such kernel inherits from $\Psi_N(t)$ the dynamics given by (2.1), i.e.

$$i\hbar \partial_t \varrho_N(t; X_N; Y_N) = -\frac{\hbar^2}{2m}(\Delta_{X_N} - \Delta_{Y_N})\, \varrho_N(t; X_N; Y_N) + \frac{\alpha}{N} \sum_{1 \leqslant i < j \leqslant N} [\delta(x_i - x_j) - \delta(y_i - y_j)]\, \varrho_N(t; X_N; Y_N). \tag{2.3}$$

According to quantum mechanics, the state of the subsystem consisting of the first n particles is described by means of the «reduced density matrix» $\widehat{\varrho}_{N,n}(t)$, namely the trace class operator on $L^2(\mathbb{R}^n)$ whose integral kernel reads

$$\varrho_{N,n}(t; X_n; Y_n) = \int_{\mathbb{R}^{N-n}} dZ_N^{n+1}\, \varrho_N(t; X_n, Z_N^{n+1}; Y_n, Z_N^{n+1}) \tag{2.4}$$

where $Z_N^{n+1} = (z_{n+1}, \ldots, z_N)$.

From (2.3) it follows that for any $1 \leqslant n \leqslant N$ the function $\varrho_{N,n}$ defined in (2.4) solves

$$\begin{aligned} i\hbar \partial_t \varrho_{N,n}(t; X_n; Y_n) = &-\frac{\hbar^2}{2m}(\Delta_{X_n} - \Delta_{Y_n})\, \varrho_{N,n}(t; X_n; Y_n) + \\ &\frac{\alpha}{N} \sum_{1 \leqslant i < j \leqslant n} [\delta(x_i - x_j) - \delta(y_i - y_j)]\, \varrho_{N,n}(t; X_n; Y_n) + \\ &\alpha \frac{N-n}{N} \sum_{1 \leqslant i \leqslant n} [\varrho_{N,n+1}(t; X_n, x_i; Y_n, x_i) - \varrho_{N,n+1}(t; X_n, y_i; Y_n, y_i)] \end{aligned} \tag{2.5}$$

which is called the Finite Schrödinger Hierarchy (FSH).

We stress that in general no closed equation is available for the reduced density matrix; furthermore notice that, since only binary interactions are taken into account, then the equation for $\varrho_{N,n}$ does not involve any $\varrho_{N,k}$ but $\varrho_{N,n+1}$.

FSH is completed defining $\widehat{\varrho}_{N,n}(t) \equiv 0$ if $n > N$ and considering equation (2.5) for any $n \in \mathbb{N}$. We remark that for the hierarchy constructed in this way the existence and uniqueness of the solution is immediatly established, since FSH is equivalent to a system of N linear PDE's. Moreover, such solution is global in time and is obtained by the solution $\Psi_N(t)$ of (2.1) and definitions (2.2) and (2.4).

Now let us introduce the Infinite Schrödinger Hierarchy (ISH)

$$(2.6)\qquad i\hbar\partial_t \varrho_n(t; X_n; Y_n) = -\frac{\hbar^2}{2m}(\Delta_{X_n} - \Delta_{Y_n})\,\varrho_n(t; X_n; Y_n) + \alpha \sum_{1 \le i \le n} [\varrho_{n+1}(t; X_n, x_i; Y_n, x_i) - \varrho_{n+1}(t; X_n, y_i; Y_n, y_i)]$$

with N fixed, $n \in \mathbb{N}$. Notice that the family of factorized density matrices

$$(2.7)\qquad \varrho_n(t; X_n; Y_n) = \prod_{j=1}^{n} \psi(t, x_j)\,\overline{\psi(t, y_j)} \quad n \in \mathbb{N}^*$$

solves the infinite hierarchy (2.6) if and only if ψ solves equation (1.1) with no external potential term. By exhibiting such solution we solve the problem of existence for (2.6), while the problem of the uniqueness remains open.

In [ABGT] the following result has been proven.

THEOREM 2.1 (Convergence of the hierarchies). – *Any limit point for* $N \to \infty$ *of* $\widehat{\varrho}_{N,n}(t)$ *is a trace class operator* $\widehat{\varrho}_n(t)$ *whose integral kernel* $\varrho_n(t; X_n; Y_n)$ *solves the ISH* (2.6) *in the sense of distributions* $\mathcal{D}'(\mathbb{R}^{2n+1})$.

Here, limit points are to be understood in the sense of the weak-$*$ topology for the spaces E_n, $n \in \mathbb{N}$, which will be introduced in the following section.

Let us stress that without a proof of the uniqueness for the solution of the ISH (2.6) we cannot really prove that equation (1.1) corresponds to a limit problem for (2.1), but only that the hierarchy (2.6) is a limit for the hierarchy (2.5). Unfortunately, the uniqueness for (2.6) seems to be a hard problem.

3. – A sketch of the proof.

The two main steps in the proof of theorem (2.1) consist in proving the following statements:

(1) For $N \to \infty$ the sequence of solutions $\{\widehat{\varrho}_{N,n}\}$ of FSH converges, in a sense to be specified, to some $\widehat{\varrho}_n$.

(2) The integral kernel of such $\widehat{\varrho}_n$ solves ISH.

Point (1) is accomplished considering an energy-type estimate. As pointed out by Lieb and Liniger ([LL]) the conserved energy reads

$$(3.1)\qquad \mathcal{E}_N(\Psi_N(t)) = \frac{\hbar^2}{2m}\|\nabla_N \Psi_N(t)\|^2_{L^2(\mathbb{R}^N;\,\mathbb{C}^N)} + \frac{\alpha}{N}\sum_{1 \le i < j \le N} \int_{\mathbb{R}^{N-1}} |\Psi_N(t, X_N^{ij})|^2\, dX_N^j$$

where $\Psi_N(t, X_N^{ij})$ denotes the trace of $\Psi_N(t)$ on the hyperplane $x_i = x_j$ and $dX_N^j = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_N$. This fact, together with identity of particles, implies that the L^2-norm of $\partial_j \Psi_N(t)$ can be bounded uniformly in N and t and so a uniform estimate for the reduced density matrix $\widehat{\varrho}_{N,n}(t)$ is provided:

$$\mathrm{Tr}\,|(\mathbb{I} - \partial_n^2)^{1/2} \widehat{\varrho}_{N,n}(t) (\mathbb{I} - \partial_n^2)^{1/2}| < M \tag{3.2}$$

where M is independent of N, n, t, the symbol ∂_n denotes the derivation with respect to the n^{th} variable and $\mathbb{I}$ is the identity in $L^2(\mathbb{R}^n)$.

Estimate (3.2) suggests to consider the space E_n of all bounded operators $\widehat{T}$ on $L^2(\mathbb{R}^n)$ such that $(\mathbb{I} - \partial_n^2)^{1/2} \widehat{T} (\mathbb{I} - \partial_n^2)^{1/2}$ is trace class and to endowe it with the norm

$$\|\widehat{T}\|_{E_n} \equiv Tr\,|(\mathbb{I} - \partial_n^2)^{1/2} \widehat{T} (\mathbb{I} - \partial_n^2)^{1/2}| \tag{3.3}$$

Notice that the space E_n is the dual of the space E_{n*} of all operators $\widehat{K}$ such that $(\mathbb{I} - \partial_n^2)^{-1/2} \widehat{K} (\mathbb{I} - \partial_n^2)^{-1/2}$ is a compact operator on $L^2(\mathbb{R}^n)$.

The action of $\widehat{T}$ on $\widehat{K}$ as a linear functional is defined as follows

$$\widehat{K} \to \mathrm{Tr}\,(\widehat{T}\widehat{K})\,. \tag{3.4}$$

Estimate (3.2), together with the Banach-Alaoglu theorem, implies the existence of a subsequence $\widehat{\varrho}_{N_k, n}$ which converges in the weak-$*$ topology of $L^\infty(\mathbb{R}, E_n)$ to some $\widehat{\varrho}_n$, for $k \to \infty$. Besides, a standard diagonal procedure shows that the sequence N_k can be chosen independently of n. In what follows by $\widehat{\varrho}_{N,n}$ we will mean an element of a subsequence $\{\widehat{\varrho}_{N_k,n}\}_k$ that converges for any fixed n.

Let us discuss point (2), namely the proof that a limit point $\widehat{\varrho}_n$ of the sequence $\{\widehat{\varrho}_{N,n}\}_N$, solves ISH (2.6). Our strategy is to demonstrate that, given the weak-* convergence of $\widehat{\varrho}_{N,n}$ to $\widehat{\varrho}_n$ in $L^\infty(\mathbb{R}, E_n)$, then every term of FSH (2.5) converges in $\mathcal{D}'(\mathbb{R}^{2n+1})$ to the corresponding term of ISH (2.6).

Such a convergence is trivial for the l.h.s. and for the first term of the r.h.s. of the FSH. To show that the term $\dfrac{\alpha}{N} \sum_{1 \le i < j \le n} [\delta(x_i - x_j) - \delta(y_i - y_j)]\, \varrho_{N,n}(t; X_n; Y_n)$ converges to zero it is sufficient to consider a test function $\varphi \in \mathcal{D}(\mathbb{R}^{2n+1})$ and to observe that the following estimate holds

$$\left| \int_{\mathbb{R}^{2n+1}} \delta(x_i - x_j)\, \varrho_{N,n}(t; X_n; Y_n)\, \varphi(t; X_n; Y_n)\, dt\, dX_n\, dY_n \right| \le \frac{1 + \sqrt{2}}{2^{3/2}} \cdot \tag{3.5}$$

$$\cdot \mu(\mathrm{Supp}_t[\varphi]) (\|\varphi\|_{L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2n}))} + \|\partial_i \varphi\|_{L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2n}))} + \|\partial_{i+1} \varphi\|_{L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2n}))}) \|\widehat{\varrho}_{N,n}\|_{L^\infty(\mathbb{R}; E_n)}$$

where $\mu(\mathrm{Supp}_t[\varphi])$ is the Lebesgue measure of the support in t of φ. Recalling the uniform bound (3.2) we conclude that, due to the the factor $1/N$, the whole sum vanishes in the limit $N \to \infty$.

It is less evident how to perform the same limit for the last sum in the r.h.s.

of (2.5), namely how to show that for any $\varphi \in \mathcal{D}(\mathbb{R}^{2n+1})$ the limit

$$(3.6)\qquad \int_{\mathbb{R}^{2n+1}} \varphi(t; X_n; Y_n)\, \varrho_{N,\, n+1}(t; X_n, x_i; Y_n, x_i)\, dt\, dX_n\, dY_n \xrightarrow{N\to\infty} \int_{\mathbb{R}^{2n+1}} \varphi(t; X_n; Y_n)\, \varrho_{n+1}(t; X_n, x_i; Y_n, x_i)\, dt\, dX_n\, dY_n$$

holds. The key estimate is

$$(3.7)\qquad \int_{\mathbb{R}^{2n+1}} \varphi(t; X_n; Y_n)\, \varrho_{N,\, n+1}(t; X_n, x_i; Y_n, x_i)\, dt\, dX_n\, dY_n \leqslant C\int_{\mathbb{R}} \sup_{x_j\in\mathbb{R}} \left[\int_{\mathbb{R}^{2n-1}} |\varphi(t, X_n, Y_n)|^2\, dX_n^j\, dY_n \right]^{1/2} \|\widehat{\varrho}_{N,\, n+1}(t)\|_{E_{n+1}}\, dt$$

where $C > 0$ and the integration measure dX_n^j equals $dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n$. The derivation of (3.7) is the subject of lemma 4.3 in [ABGT] and this is the main point where the techniques used are specific to the choice of delta potential and differ from the ones employed in [BGM] and [EY]. For details we refer to [ABGT].

Inequality (3.7) shows that at any t the l.h.s. of (3.6) is the action of some linear functional on $\widehat{\varrho}_{N,\, n+1}(t)$, namely

$$(3.8)\qquad \int_{\mathbb{R}^{2n+1}} \varphi(t; X_n; Y_n)\, \varrho_{N,n+1}(t; X_n, x_i; Y_n, x_i)\, dX_n\, dY_n = \mathrm{Tr}\,[\widehat{B}(t)\, \widehat{\varrho}_{N,n+1}(t)]$$

where $\widehat{B}(t)$ belongs to E^*_{n+1}, which is the space of operators $\widehat{B}$ such that $(\mathbb{I} - \partial_n^2)^{-1/2}\widehat{B}(\mathbb{I} - \partial_n^2)^{-1/2}$ is a bounded operator on $L^2(\mathbb{R}^n)$. The integral kernel of $(\mathbb{I} - \partial_n^2)^{-1/2}\widehat{B}(t)(\mathbb{I} - \partial_n^2)^{-1/2}$ equals, in the Fourier space,

$$(3.9)\qquad \widetilde{\kappa}(t; \Xi_{n+1}; \Lambda_{n+1}) \equiv \frac{\widetilde{\varphi}(t; \Lambda_{i-1}, \lambda_i + \lambda_{n+1} + \xi_{n+1}, \Lambda_n^{i+1}; \Xi_n)}{(1 + \lambda_{n+1}^2)^{1/2}(1 + \xi_{n+1}^2)^{1/2}}$$

where the tilde denotes the Fourier transform ([ABGT], formula (4.21)). Notice that $\widetilde{\kappa}(t)$ is a square integrable function, therefore $(\mathbb{I} - \partial_n^2)^{-1/2}\widehat{B}(\mathbb{I} - \partial_n^2)^{-1/2}$ is not only bounded, but also compact, therefore $\widehat{B}(t)$ belongs to E_{n+1^*}.

Thus we can interpret the l.h.s. of (3.8) as the action of $\widehat{\varrho}_{N,\, n+1}(t)$ intended as a linear operator on $\widehat{B}(t)$.

Moreover, since φ has compact support, by direct computation one can prove that $\int_{\mathbb{R}} dt \|\widehat{B}(t)\|_{E_{n+1}} < \infty$, so $B \in L^1(\mathbb{R}, E_{n+1^*})$ which is the pre-dual of $L^\infty(\mathbb{R}, E_{n+1})$. Thus, we have shown that

$$(3.10)\qquad \int_{\mathbb{R}^{2n+1}} \varphi(t; X_n; Y_n)\, \varrho_{N,\, n+1}(t; X_n, x_i; Y_n, x_i)\, dt\, dX_n\, dY_n = \langle \widehat{\varrho}_{N,\, n+1}, \widehat{B} \rangle$$

where $\langle , \rangle$ denotes the duality product in $B \in L^1(\mathbb{R}, E_{n+1^*})$. Applying the definition of weak-* convergence, the limit (3.6) is proven.

4. – Comments.

At first sight one could be disappointed by the weakness of the convergence established. Nevertheless, let us recall the physical meaning of the density matrix: the expected value of a measurement of an observable A on a system whose state is represented by the density matrix $\widehat{\varrho}$ is given by

$$\langle A \rangle_{\widehat{\varrho}} = \mathrm{Tr}\,(\widehat{A}\,\widehat{\varrho}) \tag{4.1}$$

where $\widehat{A}$ is the self-adjoint operator which represents the observable A. Therefore, what is relevant from a physical point of view is the duality product (4.1) between states and observables, then to obtain the convergence of the expected values the convergence of interest for the density matrix is in the weak topology for trace class operators.

We proved a weak-$*$ convergence for density matrices in a subset of the space of the trace class operators; in particular, we proved convergence for the mean values of a class of observables.

The step from the derivation of the hierarchy ISH to the derivation of equation GPE is non trivial, at least in our opinion. The point is that ISH is the hierarchy associated to a system of infinite, identical, independent particles following GPE, but at this stage we cannot exclude the possibility that the same hierarchy is also associated to some other systems. One cannot get rid of this ambiguity by an a priori argument since we are dealing with infinite particle systems.

This consideration leads to the problem of the uniqueness of the solution for ISH, for which the techniques developed in [BGM], [EY], [BEGMY], and [S1] are not sufficient.

Another problem is the generalization of our results to systems in dimension more than one. As already mentioned, while the two-dimensional case is well-defined although complicated, the three-dimensional is irreparably pathologic. A way to approach it could consist in giving up delta interactions and employing a smooth potential and some sort of short-range scaling.

Anyway, the general procedure due to Bardos, Golse and Mauser ([BGM]), together with estimates developed in [ABGT] can provide a powerful machinery that can enable one to derive various effective, nonlinear, one-particle equations from fundamental, N-particle, linear dynamics.

In this spirit we plan to treat the problem of two species of particles interacting each other and the derivation of the Jona-Lasinio, Presilla, Sjöstrand model ([J-LPS]) of concentrated nonlinearity.

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