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Critical Length for a Beurling Type Theorem.

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Sunto. – *In un lavoro recente* [3] *C. Baiocchi, V. Komornik e P. Loreti hanno ottenuto una generalizzazione dell'identità di Parseval utilizzando delle differenze divise. Noi dimostriamo l'ottimalità del loro teorema.*

Summary. – *In a recent paper* [3] *C. Baiocchi, V. Komornik and P. Loreti obtained a generalisation of Parseval's identity by means of divided differences. We give here a proof of the optimality of that theorem.*

1. – Introduction.

Let us give a sequence $(\lambda_n)_{n \in \mathbb{Z}}$ of real numbers and a non-degenerated interval I of finite length $(0 < |I| < \infty)$. We can define the upper density D^+ by the formula

$$D^+ := \lim_{r \to \infty} \frac{n^+(r)}{r},$$

where $n^+(r)$ denotes the biggest number of occurences of the sequence (λ_n) contained in an interval of length r. This limit is well defined (see [3]).

We say that $(\lambda_n)_{n \in \mathbb{Z}}$ is uniformly discrete if it satisfies for a certain $\delta > 0$ the «gap condition»

$$|\lambda_n - \lambda_m| > \delta \quad \text{for all} \quad n \neq m. \tag{1.1}$$

Then we have a celebrated theorem which gives the critical length for a generalisation of the Parseval identity (see e.g. [6]):

THEOREM 1.1. – *Let $(\lambda_n)_{n \in \mathbb{Z}}$ be a uniformly discrete sequence. For I of length $|I| > 2\pi D^+$, $(e^{i\lambda_n \cdot})$ forms a Riesz sequence, that is there exist two constants $c_1, c_2 > 0$, such that*

$$c_1 \sum_{n=-\infty}^{\infty} |b_n|^2 \leq \int_I |f(t)|^2 \, dt \leq c_2 \sum_{n=-\infty}^{\infty} |b_n|^2 \tag{1.2}$$

for every sum

$$f(t) = \sum_{n=-\infty}^{\infty} b_n e^{i\lambda_n t} \tag{1.3}$$

with square summable coefficients b_n. *On the other hand,* $(e^{i\lambda_n \cdot})$ *doesn't form a Riesz sequence anymore if* $|I| < 2\pi D^+$.

For certain applications in theoretical control, we have to consider sequences which do not satisfy the gap condition (1.1). The question is now the following: what happens if we do not have this condition anymore? In fact, if $(e^{i\lambda_n \cdot})$ forms a Riesz sequence, then (λ_n) is uniformly discrete (see [6]).

One idea was then to use the divided differences (which will be further explained). This approach was introduced by Ullrich in [7] in some particular cases, then a general answer was given by Baiocchi, Komornik and Loreti in [3]. The theorem takes the form:

THEOREM 1.2. – *If* $|I| > 2\pi D^+$, *then the divided differences* (e_n) *form a Riesz sequence.*

The question we will discuss is the following: what happens if $|I| < 2\pi D^+$?

Always in [3], the authors indicated that $2\pi D^+$ is really the optimal length, that is, if $|I| < 2\pi D^+$, the divided differences don't form a Riesz sequence and that one could find a proof by adapting a method of [5]. We propose here to prove this result by adapting a simpler method developed in [4].

2. – Main result.

In order to formulate the result announced in the introduction in a more precise way, we have to define the divided differences. Assume first that $D^+ < \infty$. We can then suppose by a rearranging argument that $(\lambda_n)_{n \in \mathbb{Z}}$ is an increasing sequence. At this stage, we have the following characterization (see [3]):

PROPOSITION 2.1. – *Given an increasing sequence* (λ_n), *we have* $D^+ < \infty$ *if and only if there exist* $\gamma > 0$ *and* $M > 0$ *such that*

$$\lambda_{n+M} - \lambda_n > M\gamma \text{ for every } n \in \mathbb{Z}. \tag{2.1}$$

We now define the divided differences like in [3].

For each $j \geq 1$ and all reals $\mu_1, \dots, \mu_j$, we can define:

(2.2) $$[\mu_1, \dots, \mu_j](t) := (it)^{j-1} \int_0^1 \int_0^{s_1} \dots \int_0^{s_{j-2}} \exp\left(i[s_{j-1}(\mu_j - \mu_{j-1}) + \dots + s_1(\mu_2 - \mu_1) + \mu_1]\, t\right) \mathrm{d}s_{j-1} \dots \mathrm{d}s_1 .$$

If $\mu_1, \dots, \mu_j$ are pairwise distinct, then (2.2) is equivalent to the familiar definition

$$[\mu_1, \dots, \mu_j](t) := \sum_{p=1}^{j} \left[\prod_{q=1}^{j}{}' (\mu_p - \mu_q) \right]^{-1}$$

where the sign $'$ in the products indicates the omission of the zero factor corresponding to $p = q$.

Now let $\gamma > 0$ such that (2.1) holds. We fix a number $0 < \gamma' \leq \gamma$. For $j = 1, \dots, M$ and $m \in \mathbb{Z}$ we say that $\lambda_m, \dots, \lambda_{m+j-1}$ forms a γ' $-$closed exponents chain if

$$\begin{cases} \lambda_m - \lambda_{m-1} \geq \gamma', \\ \lambda_n - \lambda_{n-1} < \gamma' \qquad \text{for } n = m+1, \dots, m+j-1, \\ \lambda_{m+j} - \lambda_{m+j-1} \geq \gamma'. \end{cases}$$

For $j = 1, \dots, M$ and $m \in \mathbb{Z}$ such that $\lambda_m, \dots, \lambda_{m+j-1}$ forms a γ'-closed exponents chain, then we define the divided differences by:

$$e_\ell = [\lambda_m, \dots, \lambda_\ell],$$

for $\ell = m, \dots, m+j-1$.

Then we verify that e_n is well defined for each $n \in \mathbb{Z}$. The sequence (e_n) is called the sequence of divided differences (relative to γ'), associated to the sequence (λ_n).

We can now formulate the theorem that we want to prove:

THEOREM 2.2. – *Let (λ_n) be a sequence such that we have* (2.1) *for a certain γ. Then for any $0 < \gamma' \leq \gamma$, the sequence of divided differences relative to γ' doesn't form a Riesz sequence in $L^2(I)$ for $|I| < 2\pi D^+$, that is there don't exist constants $c_1, c_2 > 0$, such that*

$$c_1 \sum_{n=-\infty}^{\infty} |a_n|^2 \leq \int_I |f(t)|^2 \mathrm{d}t \leq c_2 \sum_{n=-\infty}^{\infty} |a_n|^2$$

for every finite sum

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e_n(t) .$$

By *finite sum* we mean a sum having only a finite number of nonzero elements.

Now what happens if $D^+ = \infty$? The definition of the divided differences then depends on the enumeration of the sequence (λ_n) (see [1]). In that case, for any enumeration, the divided differences do not form a Riesz sequence. Indeed, if $D^+ = \infty$, choose enough and not too much elements of (λ_n) according to a fixed enumeration of (λ_n) such that the upper density D_0 of these elements satisfies: $|I| < 2\pi D_0 < \infty$. Now, the sequence of divided differences coming from the selected elements of (λ_n) is included in the sequence of divided differences of the whole sequence (λ_n) and doesn't form a Riesz sequence thanks to the theorem 2.2, so the whole sequence doesn't form a Riesz sequence, either.

3. – Proof.

In order to prove the theorem, we use a method developed in [4].

Let be given a sequence (λ_n), a real $\gamma > 0$ and an integer M such that we have (2.1). We recall that $D^+ < \infty$. Let $0 < \gamma' \leq \gamma$. We suppose that the sequence of divided differences (e_n) relative to γ' forms a Riesz sequence in $L^2(I)$. We want to prove that $|I| \geq 2\pi D^+$.

We introduce some notations. We write $\Lambda := (\lambda_n)$. We introduce the trigonometric system over $L^2(I)$ given by $f_n := \exp\left(\frac{i2\pi n \cdot}{|I|}\right)$ and we write $\Gamma := (\gamma_n) = \left(\frac{2\pi n}{|I|}\right)$. We also write $n_\Lambda^+(r)$, $D^+(\Lambda)$ to precise that we talk about the upper density associated to Λ. We thus want to show that $|I| \geq 2\pi D^+(\Lambda)$. We can already remark that

$$D^+(\Gamma) = \frac{|I|}{2\pi}.$$

For $y \in \mathbb{R}$ and $r > 0$, we note $\Lambda_r := \Lambda \cap (y - r, y + r)$ and $\Gamma_r := \Gamma \cap (y - r, y + r)$.

The result will be deduced from a comparison theorem:

THEOREM 3.1. – *For every $\varepsilon > 0$, there exists $R > 0$ such that for all $r \geq 0$ and for all $y \in \mathbb{R}$, we have*

$$(1 - \varepsilon)\,\mathrm{Card}(\Lambda_r) \leq \mathrm{Card}(\Gamma_{r+R}). \tag{3.1}$$

Before proving this theorem, let show how it applies to prove Theorem 2.2. From the inequality (3.1), we obtain

$$(1 - \varepsilon)\, n_\Lambda^+(r) \leq n_\Gamma^+(r + R),$$

then

$$(1-\varepsilon)\,D^{+}(\Lambda) \leqslant \lim_{r\to\infty} \frac{n_{\Gamma}^{+}(r+R)}{r+R}\cdot\frac{r+R}{r}\,.$$

Thus

$$(1-\varepsilon)\,D^{+}(\Lambda) \leqslant D^{+}(\Gamma) = \frac{|I|}{2\pi}\,.$$

Since ε can be arbitrarily small, we obtain effectively that

$$|I| \geqslant 2\pi D^{+}(\Lambda)\,.$$

It now remains to prove the comparison theorem. The strategy here is to introduce associated finite dimensional spaces, to define an operator between these spaces and the inequality (3.1) will be derived from the estimation of the trace of this operator obtained in two different manners.

So we consider, for $y \in \mathbb{R}$ and $r > 0$ the linear hull V_r of the vectors e_n with $\lambda_n \in \Lambda_r$ (we recall that $e_n = [\lambda_m, \dots, \lambda_n]$). Similarly, we define the linear hull W_r of the vectors f_n with $\gamma_n \in \Gamma_r$ (we recall that $f_n = e^{i\gamma_n \cdot}$).

Since Λ_r and Γ_r are finite sets, these spaces are effectively finite dimensional. Then we define the orthogonal projections

$$P_r \colon L^2(I) \to V_r$$

and

$$Q_{r+R} \colon L^2(I) \to W_{r+R}\,.$$

Denoting by i the injection

$$i : V_r \hookrightarrow L^2(I)\,,$$

then we can define the endomorphism S_r of V_r by

$$S_r = P_r \circ Q_{r+R} \circ i\,.$$

The aim is to estimate the trace of S_r (which will be denoted by $\operatorname{tr}(S_r)$) in two different manners in order to obtain (3.1).

LEMMA 3.2. – *For every* $R > 0$, $r \geqslant 0$ *and* $y \in \mathbb{R}$, *we have*

$$|\operatorname{tr}(S_r)| \leqslant \operatorname{Card}(\Gamma_{r+R})\,.$$

PROOF. – We have

$$\|S_r\| \leqslant \|P_r\|\|Q_{r+R}\| \leqslant 1\,.$$

Thus the eigenvalues of S_r have their moduli less than 1. So, we have:

$$|\operatorname{tr}(S_r)| \leqslant \operatorname{rang}(S_r) \leqslant \dim W_{r+R}.$$

Since $\dim W_{r+R} = \operatorname{Card}(\Gamma_{r+R})$, the lemma follows. ∎

In order to obtain the inverse inequality, we use a homogeneous approximation lemma.

We recall e.g. from [8] a lemma about Riesz sequences.

LEMMA 3.3. – *If* (g_n) *is a Riesz sequence in a Hilbert space* H, *then it admits a biorthogonal bounded sequence.*

Then we apply this lemma to the divided differences sequence (e_j) and we call by (φ_j) the associated biorthogonal sequence. It is then possible to express the trace of S_r in terms of (φ_j).

LEMMA 3.4. – *We have*

$$\operatorname{tr}(S_r) = \operatorname{Card}(\Lambda_r) + \sum_{\lambda_j \in \Lambda_r} ((Q_{r+R} - \mathrm{Id})\, e_j \,|\, P_r \varphi_j).$$

PROOF. – Using the biorthogonal sequence we have:

$$\begin{aligned}
\operatorname{tr}(S_r) &= \sum_{\lambda_j \in \Lambda_r} (S_r e_j \,|\, v_j) \\
&= \sum_{\lambda_j \in \Lambda_r} (Q_{r+R} e_j \,|\, P_r \varphi_j) \\
&= \sum_{\lambda_j \in \Lambda_r} (e_j \,|\, P_r \varphi_j) + \sum_{\lambda_j \in \Lambda_r} ((Q_{r+R} - \mathrm{Id})\, e_j \,|\, P_r \varphi_j).
\end{aligned}$$

Since $P_r e_j = e_j$, we obtain that $(e_j \,|\, P_r \varphi_j) = (P_r e_j \,|\, \varphi_j) = 1$ and the result follows. ∎

Then we use the following homogeneous approximation lemma:

LEMMA 3.5. – *For every* $\varepsilon > 0$, *there exists* $R > 0$ *such that for all* $r > 0$, $y \in \mathbb{R}$ *and* ℓ *such that* $\lambda_\ell \in \Lambda_r$, *we have*

$$\|(Q_{r+R} - \mathrm{Id})\, e_\ell\| \leqslant \varepsilon .$$

PROOF. – Since the trigonometric system (f_p) of $L^2(I)$ is orthonormal and since Q_{r+R} is an orthogonal projection over W_{r+R}, we obtain:

$$\|(Q_{r+R} - \mathrm{Id})\, e_\ell\|^2 = \sum_{|\gamma_p - y| > r+R} |(e_\ell \,|f_p)|^2.$$

Now we have:

$$(e_\ell \,|f_p) = \int_I g(t)\, e^{i\lambda_\ell t} e^{-i\gamma_p t}\, \mathrm{d}t$$

with

$$g(t) = [\lambda_m - \lambda_\ell, \ldots, \lambda_\ell - \lambda_\ell](t).$$

Integrating by parts over $I = (a, b)$ we obtain

$$(e_\ell \,|f_p) = \left[\frac{1}{i\lambda_\ell - i\gamma_p} g(t)\, e^{i\lambda_\ell t} e^{-i\gamma_p t} \right]_a^b - \int_I \frac{1}{i\lambda_\ell - i\gamma_p} g'(t)\, e^{i\lambda_\ell t} e^{-i\gamma_p t}\, \mathrm{d}t\,.$$

Now, by a direct computation from the formula (2.2), given an integer $r \geqslant 1$ and reals $\mu_1, \ldots, \mu_r$, we have:

$$[\mu_1, \ldots, \mu_r]'(t) \leqslant \frac{(r-1)\, t^{r-2}}{(r-1)!} +$$

$$(|\mu_r - \mu_{r-1}| + \ldots + |\mu_2 - \mu_1| + |\mu_1|)\, \frac{t^{r-1}}{(r-1)!}\,.$$

Thus, in our case, thanks to the γ' −closed exponents property, we have:

$$|g'(t)| \leqslant (\ell - m)\, \frac{t^{\ell - m - 1}}{(\ell - m)!} + (\ell - m)\, \gamma'\, \frac{t^{\ell - m}}{(\ell - m)!}$$

At this stage, we can find a constant C depending only on γ', M, a and b such that

$$|(e_\ell \,|f_p)|^2 \leqslant \frac{C}{|\lambda_\ell - \gamma_p|^2}\,.$$

Recalling that $\lambda_\ell \in \Lambda_r$, we obtain:

$$\begin{aligned} \|(Q_{r+R} - \mathrm{Id})\, e_\ell\|^2 &\leqslant \sum_{|g_p - y| > r+R} \frac{C}{|l_\ell - y + y - \gamma_p|^2} \\ &\leqslant \sum_{|g_p - y| > r+R} \frac{C}{||y - \gamma_p| - r|^2} \\ &\leqslant \sum_{p \in \mathbb{Z}} \frac{C}{\left| \dfrac{2\pi |p|}{|I|} + R \right|^2}\,. \end{aligned}$$

Since this last expression doesn't depend on r, y and tends to 0 as $R \to 0$, the lemma follows. ∎

Now we can finish the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. – Let $\varepsilon > 0$. By combining the two preceding lemmas, since (φ_j) is bounded, we obtain

$$\operatorname{tr}(S_r) \geqslant (1-\varepsilon)\,\mathrm{Card}\,(\Lambda_r)\,.$$

Then the theorem follows from Lemma 3.2. ∎

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