Bollettino
Unione Matematica Italiana

Cinzia Casagrande

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **7-B** (2004), n.3, p. 663–671.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2004_8_7B_3_663_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2004.

Bollettino U. M. I.
(8) **7**-B (2004), 663-671

On Some Numerical Properties of Fano Varieties.

CINZIA CASAGRANDE(*)

Sunto. – *Questa nota è il testo di una conferenza tenuta al XVII Convegno dell'Unione Matematica Italiana, tenutosi a Milano,* 8-13 *settembre* 2003. *Parlo di alcune congetture e teoremi sulle relazioni tra l'indice, lo pseudo-indice e il numero di Picard di una varietà di Fano. I risultati in questione fanno parte di un lavoro in collaborazione con Bonavero, Debarre e Druel.*

Summary. – *This is the text of a talk given at the XVII Convegno dell'Unione Matematica Italiana held at Milano, September* 8-13, 2003. *I would like to thank Angelo Lopez and Ciro Ciliberto for the kind invitation to the conference. I survey some numerical conjectures and theorems concerning relations between the index, the pseudo-index and the Picard number of a Fano variety. The results I refer to are contained in the paper* [3], *wrote in collaboration with Bonavero, Debarre and Druel.*

1. – Introduction.

Let X be a smooth, complex projective variety of dimension n. Recall that the Picard group $\operatorname{Pic} X$ is the group of isomorphism classes of line bundles on X, and the anticanonical bundle $-K_X \in \operatorname{Pic} X$ is the determinant of the tangent bundle of X. X is called a *Fano variety* if $-K_X$ is ample, or equivalently if $c_1(X)$ is represented by a positive form. When X is Fano, $\operatorname{Pic} X \simeq H^2(X, \mathbb{Z})$ is a free abelian group of rank ϱ, the *Picard number* of X.

Examples of Fano varieties are:

1) the projective space $\mathbb{P}^n$;

2) the complete intersections $X = Y_1 \cap \ldots \cap Y_r$, Y_i a generic hypersurface of degree d_i in $\mathbb{P}^N$, with $d_1 + \ldots + d_r \leq N$;

3) homogeneous varieties, namely varieties acted on transitively by a connected linear algebraic group (for instance, grassmannians and flag varieties);

(*) Comunicazione presentata a Milano in occasione del XVII Congresso U.M.I.

4) any degree d Galois cyclic cover $X \to \mathbb{P}^n$, ramified over a smooth hypersurface $Y \subset \mathbb{P}^n$ of degree dh, with $h(d-1) \leq n$;

5) the moduli spaces $M(r, L, C)$ of stable vector bundles of rank r on a fixed curve C (smooth, of genus at least 2), with determinant a fixed line bundle $L \in \mathrm{Pic}\, C$ such that $(\deg L, r) = 1$;

6) all (finite) products of Fano varieties.

Fano varieties have a very rich geometry and have been classically intensively studied, see the book [IP99] for a complete survey on the subject.

Up to dimension 3, Fano varieties are classified: in dimension 1 there is only $\mathbb{P}^1$. In dimension 2, there are 10 deformation types: $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow-ups of $\mathbb{P}^2$ in d generic points, $d \in \{0, \dots, 8\}$. For $n = 3$ there are 105 deformation types (the classification is due to Iskovskikh, 1977-78, in the case $\varrho = 1$; to Mori and Mukai, 1981 (1), in the case $\varrho \geq 2$; see [IP99], Ch. 4 and §7.1).

It is well-known that for $n \geq 3$, not all Fano varieties are rational. For instance, the generic cubic hypersurface in $\mathbb{P}^4$ is not rational (Clemens-Griffiths, 1972; see [IP99], Ch. 8 and [Kol96], V.5). Anyway, Fano varieties are close to the projective space in the sense that they contain «lots» of *rational curves* (by a rational curve we mean the image of a non-constant morphism $\mathbb{P}^1 \to X$). This is formalized saying that every Fano variety is *rationally connected* (Campana and Kollár-Miyaoka-Mori, 1992; see [IP99], Corollary 6.2.11 and [Kol96], V.2), namely any two points in X can be joined by a rational curve.

This result implies that in any dimension n there is a finite number of deformation types of Fano varieties, with an explicit bound in n (Nadel, Campana, Kollár-Miyaoka-Mori, 1990-1992; see [IP99], §6.2 and [Kol96], V.2.2.4 for a history of the result).

2. – Toric Fano varieties.

A toric variety is a normal, complex algebraic variety, acted on by the group $(\mathbb{C}^*)^n$, and having a dense orbit. (Toric varieties do not need to be Fano, they don't even need to be projective.)

Toric Fano varieties are very special among Fano varieties; here are some of their properties:

1) there is a finite number of them in each dimension (Batyrev, 1982, see [Bat99] and references therein);

2) they are classified up to dimension 4 (for $n = 3$ the classification is due to Batyrev, 1981, and Watanabe-Watanabe, 1982, see [Oda88], §2.3 p. 90;

(1) Mori and Mukai noticed in 2002 that there is a family missing from their original list.

for $n = 4$ the classification is due to Batyrev [Bat99], see also [Sat00], example 4.7 for a missing case in Batyrev's list);

3) they are rational;

4) they are rigid, namely they do not have non-trivial infinitesimal deformations. This is because for any smooth toric projective variety X, the Bott vanishing holds (see [Oda88], §3.3), namely $H^p(\Omega_X^q \otimes L) = 0$ for any $p > 0$, $q \geq 0$ and $L \in \operatorname{Pic} X$ ample. If X is Fano, this gives $H^1(X, T_X) = 0$ (T_X the tangent bundle of X).

Some examples of toric Fano varieties are: $\mathbb{P}^n$; the blow-up of $\mathbb{P}^2$ in 1, 2 or 3 points; the blow-up of $\mathbb{P}^n$ along a linear subspace; any (finite) product of toric Fano varieties.

To any toric Fano variety one can associate an n-dimensional convex polytope (a so-called *Fano polytope*), in such a way that the variety is determined by its polytope. Hence, when studying toric Fano varieties, one can use – together with the standard geometric tecniques – also their combinatorial features. This makes toric Fano varieties easier and more explicit to study; their are a good testing ground for conjectures about general Fano varieties. For more on toric Fano varieties, see the surveys [Deb03, Wiś02] and references therein.

3. – Index and pseudo-index of a Fano variety.

An important invariant of Fano varieties is the *index*, defined as

$$r := \max\{m \in \mathbb{Z} \mid \text{there exists } H \in \operatorname{Pic} X \text{ such that } -K_X = mH\}.$$

It is well known that (Kobayashi-Ochiai, 1970, see [IP99], Corollary 3.1.15):

1) $r \in \{1, \ldots, n+1\}$;

2) $r = n+1$ if and only if $X = \mathbb{P}^n$;

3) $r = n$ if and only if $X \subset \mathbb{P}^{n+1}$ is a smooth quadric.

There are other classified cases:

4) $r = n-1$: this case has been classified by Iskovskikh in dimension 3 and by Fujita for general n (see [IP99], §3.2); for $n \geq 7$ there are only 4 deformation types in any dimension, all with $\varrho = 1$.

5) $r = n-2$: the classification is due Wiśniewski in the case $\varrho \geq 2$ (see [IP99], Theorems 7.2.1 and 7.2.2) and mainly to Mukai in the case $\varrho = 1$ (see [IP99], §5.2). Again, for $n \geq 11$ there are only 5 deformation types in any dimension, all with $\varrho = 1$.

Observe that in dimension 4, the only non classified case is $r=1$.

The criterion that emerges from these results is that: *Fano varieties with bigger index are simpler.* In 1988 Mukai formulated the following:

CONJECTURE M ([Muk88]). – *Let X be a Fano variety of dimension n, Picard number ϱ and index r. Then*

$$\varrho(r-1) \leqslant n,$$

and equality holds if and only if $X = (\mathbb{P}^{r-1})^{\varrho}$.

In 1990 Wiśniewski [Wiś90], proving a case of Conjecture M (property (c) below), introduced a new invariant of X, closely related to the index. This is the *pseudo-index*, defined as:

$$\iota := \min\{-K_X \cdot C \,|\, C \text{ rational curve in } X\}.$$

Observe that $\iota \geqslant 1$ by Kleiman's criterion of ampleness. Moreover r divides ι, because $-K_X = rH$, so for any curve C in X you have

$$-K_X \cdot C = r(H \cdot C).$$

It can be $r < \iota$: for instance, $\mathbb{P}^1 \times \mathbb{P}^2$ has index 1 and pseudo-index 2.

Basic properties of ι are:

(a) $\iota \leqslant n+1$ (Mori, 1979, see [Kol96], Theorem V.1.1.6);

(b) $\iota = n+1$ if and only if $X = \mathbb{P}^n$ [CMSB02];

(c) if $\iota > \frac{1}{2}n + 1$, then $\varrho = 1$ [Wiś90].

This last property, as Wiśniewski implicitly noticed in [Wiś90], leads to formulate the following stronger conjecture:

CONJECTURE GM ([BCDD03]). – *Let X be a Fano variety of dimension n, Picard number ϱ and pseudo-index ι. Then*

$$\varrho(\iota-1) \leqslant n,$$

and equality holds if and only if $X = (\mathbb{P}^{\iota-1})^{\varrho}$.

Observe that the inequality is meaningful only if $\iota > 1$.

Observe also that, by properties (a) and (b), Conjecture GM holds if $\varrho = 1$.

If $\varrho = 2$, property (c) gives the inequality $\iota \leqslant \frac{1}{2}n + 1$. If moreover $\iota = \frac{1}{2}n + 1$, then $X = (\mathbb{P}^{n/2})^2$ (this is due to Wiśniewski [Wiś90] if $r = \iota$ and to Occhetta [Occ03] in general). Hence Conjecture GM holds for $\varrho = 2$ too.

Conjecture GM remains open in full generality, but it has been proved in a number of cases:

THEOREM 1 ([BCDD03]). – *Let X be a Fano variety of dimension n, Picard number ϱ and pseudo-index ι. Conjecture GM holds in the following cases*:

1) $n \leqslant 4$;
2) *X is toric and* $n \leqslant 7$;
3) *X is toric and* $\iota \geqslant \frac{1}{3}n + 1$;
4) *X is a homogeneous variety.*

Recently, Andreatta, Occhetta and Chierici have proved some more cases:

THEOREM 2 ([ACO03]). – *Let X be a Fano variety of dimension n, Picard number ϱ and pseudo-index ι. Conjecture GM holds in the following cases*:

1) $n = 5$;
2) $\iota \geqslant \frac{1}{3}n + 1$ *and X has a fiber type extremal contraction*;
3) $\iota \geqslant \frac{1}{3}n + 1$ *and X has no small extremal contractions.*

4. – Families of rational curves.

The basic tool in the proof of Theorem 1 is Mori theory, and more generally the study of families of rational curves on X. We describe here a part of our approach to the problem. The reference for this subject is the book [Kol96].

Let X be a smooth, complex projective variety of dimension n. There is a variety $\mathrm{RatCurves}^{\mathrm{n}}(X)$ parametrizing birational morphisms $\mathbb{P}^1 \to X$, modulo automorphisms of $\mathbb{P}^1$. This is contructed as follows: consider the Hilbert scheme $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ of birational morphisms from $\mathbb{P}^1$ to X and consider its normalization. Then $\mathrm{RatCurves}^{\mathrm{n}}(X)$ is the quotient of this normalization under the action of $\mathrm{Aut}(\mathbb{P}^1)$.

An irreducible component V of $\mathrm{RatCurves}^{\mathrm{n}}(X)$ is called a *family of rational curves*; curves parametrized by V are all deformation of a same rational curve in X, so they are algebraically and numerically equivalent. Hence, they all have the same anticanonical degree, which we denote by $\deg_{-K_X} V$.

The family V is called *unsplit* if and only if V is proper (compact) as a variety; this is equivalent to asking that curves parametrized by V do not deform to reducible curves in X.

Being unsplit is a very strong property. If X is Fano, a family V such that

$\deg_{-K_X} V < 2\iota$ is necessarily unsplit: indeed, if a rational curve C deform to a reducible curve $C_1 \cup C_2$, then $-K_X \cdot C \geqslant -K_X \cdot C_1 - K_X \cdot C_2 \geqslant 2\iota$.

Conversely, an unsplit family can not have «too high» anticanonical degree:

THEOREM 3 ([BCDD03]). – *Let X be a smooth, complex projective variety of dimension n. Let $V_1, \dots, V_k$ be unsplit families of rational curves in X such that the classes of $V_1, \dots, V_k$ are algebraically independent. For any $x \in X$ define*

$$L(V_1, \dots, V_k)_x := \{y \in X \,|\, \textit{there exist curves } C_1, \dots, C_k \textit{ in } X \textit{ such that } x \in C_1 \textit{ and } y \in C_k,\ C_j \textit{ is in } V_j \textit{ and } C_j \cap C_{j+1} \neq \emptyset \textit{ for all } j\}.$$

If $L(V_1, \dots, V_k)_x \neq \emptyset$, then $\deg_{-K_X} V_1 + \ldots + \deg_{-K_X} V_k \leqslant \dim L(V_1, \dots, V_k)_x + k$.

Theorem 3 gives the following general approach to Conjecture GM:

COROLLARY 4. – *Let X be a Fano variety of Picard number ϱ. Assume that there exist unsplit families $V_1, \dots, V_\varrho$ of rational curves in X such that*

(i) *the classes of $V_1, \dots, V_\varrho$ are algebraically independent*;

(ii) *there exists curves $C_1, \dots, C_\varrho$ in X such that C_i is in V_i and $C_i \cap C_{i+1} \neq \emptyset$ for all j.*

Then Conjecture GM holds for X.

PROOF. – By (ii), there exists $x_1 \in X$ such that $L(V_1, \dots, V_\varrho)_{x_1} \neq \emptyset$. If ι is the pseudo-index of X, we have $\deg_{-K_X} V_j \geqslant \iota$ for all j, so Theorem 3 yields

$$\varrho\iota \leqslant \deg_{-K_X} V_1 + \ldots + \deg_{-K_X} V_\varrho \leqslant \dim L(V_1, \dots, V_\varrho)_{x_1} + \varrho \leqslant n + \varrho,$$

namely $\varrho(\iota - 1) \leqslant n$. Assume now that $\varrho(\iota - 1) = n$. Then $n + \varrho = \varrho\iota$, hence all inequalities above are equalities. In particular we have $\deg_{-K_X} V_j = \iota$ for all j and $\dim L(V_1, \dots, V_\varrho)_{x_1} = n$, so $L(V_1, \dots, V_\varrho)_{x_1} = X$ ($L(V_1, \dots, V_\varrho)_{x_1}$ is a closed subset, see [BCDD03], §5). This means that for every point $y \in X$ there is a curve belonging to V_ϱ and passing through y, namely that V_ϱ is a *covering* family.

Now choose a curve C'_ϱ in V_ϱ passing through x_1, and $x_\varrho \in C'_\varrho$. By construction $L(V_\varrho, V_1, \dots, V_{\varrho-1})_{x_\varrho} \neq \emptyset$, so applying again Theorem 3, we see that $L(V_\varrho, V_1, \dots, V_{\varrho-1})_{x_\varrho} = X$ and that $V_{\varrho-1}$ is a covering family. Proceeding in this way, for each $j = \varrho, \dots, 2$ we find x_j such that $L(V_j, \dots, V_\varrho, V_1, \dots, V_{j-1})_{x_j} = X$, so V_{j-1} is a covering family.

Thus $V_1, \dots, V_\varrho$ are covering families of degree ι, and Theorem 1 of [Occ03] yields $X \cong (\mathbb{P}^{\iota-1})^\varrho$.

5. – Other properties of the pseudo-index.

The pseudo-index has some remarkable properties also in relation to morphisms.

PROPOSITION 5 ([BCDD03]). – *Let X be a Fano variety of pseudo-index ι_X, Y a smooth variety and $f : X \to Y$ a surjective morphism with connected fibers.*

If $\dim Y < \iota_X$, *then* $Y = \mathbb{P}^r$ *and* $X = F \times \mathbb{P}^r$, *$F$ a smooth variety.*

Again, we observe the principle that the bigger ι_X is, the stronger conditions we find on X.

Recently Bonavero has studied the behaviour of the pseudo-index under a smooth blow-up $X \to Y$. Assume X and Y are Fano and denote by r_X and ι_X (respectively, r_Y and ι_Y) the index and the pseudo-index of X (respectively, of Y). We have $r_X \leqslant r_Y$, and one would expect a similar behaviour for the pseudo-index. Quite surprisingly, it depends on the dimension of the center of the blow-up:

THEOREM 6 ([Bon03]). – *Let X and Y be Fano varieties of dimension n, such that $X \to Y$ is the blow-up along a smooth subvariety $Z \subset Y$.*

If $\dim Z < \frac{1}{2}(n + \iota_Y - 1)$ *or* $\dim Z > n - 2 - \iota_Y$, *then* $\iota_X \leqslant \iota_Y$.

These bounds are optimal: in [Bon03] you can find examples with $\iota_X > \iota_Y$ and $\dim Z = \frac{1}{2}(n + \iota_Y - 1)$ or $\dim Z = n - 2 - \iota_Y$.

6. – Related open questions.

6.1. – There are no known bounds (even conjecturally, to my knowledge) for the Picard number ϱ of an n-dimensional Fano variety X.

1) Conjecture GM would give $\varrho \leqslant n$ if $\iota > 1$.
2) What happens when $\iota = 1$?

In the toric case, it is known that $\varrho \leqslant 2n\sqrt{2n} + o(n^{3/2})$ [VK85, Deb03], but conjecturally the bound should be linear:

$$\varrho \leqslant \begin{cases} 2n & \text{if } n \text{ is even,} \\ 2n-1 & \text{if } n \text{ is odd.} \end{cases}$$

This bound holds for toric Fano varieties of dimension $n \leqslant 5$ [Bat99, Cas03b].

6.2. – Rational curves C in X having minimal anticanonical degree, namely such that $-K_X \cdot C = \iota$, should be the analogue of lines in projective space. It is reasonable to expect that these curves have special properties:

CONJECTURE. – *Let X be a Fano variety of pseudo-index ι and $C \subset X$ a rational curve. If $-K_X \cdot C = \iota$, then C is extremal.*

This conjecture has been proved for toric Fano varieties [Cas03a].

6.3. – We conclude with a conjecture about characterization of Fano varieties.

CONJECTURE ([Kol96], Conjecture III.1.2.5.4). – *Let X be a smooth projective variety. If $-K_X \cdot C > 0$ for any curve in X, then X is Fano.*

The conjecture is trivially true if X is a toric variety (see [Oda88], Theorem 2.18), and has been proved for Fano varieties of dimension $n \leqslant 3$ (Matsuki, 1987, see [Kol96], Remark III.1.2.5.5).

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Dipartimento di Matematica, Università di Roma Tre
Largo San Leonardo Murialdo 1, 00146 Roma - Italy
casagran@mat.uniroma3.it

Pervenuta in Redazione
il 3 dicembre 2003