Bollettino Unione Matematica Italiana

Mohamed Zohry

On multivalued martingales, multimeasures and multivalued Radon-Nikodym property

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **7-B** (2004), n.2, p. 453–468.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2004_8_7B_2_453_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2004.

Bollettino U. M. I.
(8) **7**-B (2004), 453-468

On Multivalued Martingales, Multimeasures and Multivalued Radon-Nikodym Property.

MOHAMED ZOHRY

Sunto. – *Sia X uno spazio di Banach reale, separabile e $\mathcal{K}_c(X)$ la classe dei sottoinsiemi non vuoti, chiusi, limitati e convessi di X. Si dimostra un risultato di rappresentazione per martingale essenzialmente limitate a valori in $\mathcal{K}_c(X)$. Quindi rivolgiamo la nostra attenzione al legame tra misure multivoche e rappresentazioni di Riesz a valori multivoci. Infine, diamo la versione multivoca del teorema di Radon-Nikodym.*

Summary. – *In this paper we prove a representation result for essentially bounded multivalued martingales with nonempty closed convex and bounded values in a real separable Banach space. Then we turn our attention to the interplay between multimeasures and multivalued Riesz representations. Finally, we give the multivalued Radon-Nikodym property.*

1. – Introduction.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space and X a real separable Banach space with norm $\|\cdot\|$ and the dual space X^*. For each $Y \subseteq X$, $\mathrm{cl}(Y)$ denotes the norm-closure of Y. Let $\mathcal{K}(X)$ (resp. $\mathcal{K}_c(X)$) denote the family of all nonempty closed and bounded (resp. nonempty closed bounded and convex) subsets of X. For Y and Z en $\mathcal{K}(X)$, the distance $d(x, Y)$ of $x \in X$ and Y, the Hausdorff distance $h(Y, Z)$ of Y and Z, the norm $\|Y\|$ of Y, and the support function $\delta^*(.\,|Y)$ of Y are defined by

$$d(x, Y) = \inf\{\|x-y\| : y \in Y\},$$

$$h(Y, Z) = \max\{\sup_{y \in Y} d(y, Z), \sup_{z \in Z} d(z, Y)\},$$

$$\|Y\| = h(Y, \{0\}) = \sup\{\|y\| : y \in Y\},$$

$$\delta^*(x^*|Y) = \sup\{\langle x^*|y\rangle : y \in Y\}, \quad x^* \in X^*.$$

It is easy to check that for Y and Z in $\mathcal{K}_c(X)$,

$$h(Y, Z) = \sup\{\,|\delta^*(x^*\,|Y) - \delta^*(x^*\,|Z)\,| : x^* \in X^*,\ \|x^*\| \leq 1\}.$$

In the sequel we will frequently use the well known result; see for instance [4].

THEOREM 1. – *There is a one-to-one correspondence between nonempty closed convex sets and sublinear $\sigma(X^*, X)$ lower semi-continuous functions on X^* (with values in $]-\infty, \infty]$) which maps A into $\delta^*(\cdot|A)$.* ■

With Y and Z in $\mathcal{K}(X)$, the *closure sum* of Y and Z is the element of $\mathcal{K}(X)$ defined by $Y \dot{+} Z := \mathrm{cl}(Y+Z)$. It is well known that for Y and Z in $\mathcal{K}_c(X)$, $\delta^*(\cdot|Y \dot{+} Z) = \delta^*(\cdot|Y) + \delta^*(\cdot|Z)$. It is not difficult to verify that the operation $\dot{+}$ is associative and commutative on $\mathcal{K}(X)$. Given a sequence $\{Y_n\}_{n\geq 1}$ of members of $\mathcal{K}(X)$, we say that the serie $\dot{\sum}_{n=1}^{\infty} Y_n$ converges to Y if $\lim_n h(Y, S_n) = 0$, where $S_n = Y_1 \dot{+} Y_2 \dot{+} \ldots \dot{+} Y_n = \dot{\sum}_{k=1}^{n} Y_k$.

Let $\mathcal{B}_X$ be the Borel σ-field on X and $\mathcal{B}_{\mathcal{K}(X)}$ the σ-field on $\mathcal{K}(X)$ generated by the sets $\{Y \in \mathcal{K}(X) : Y \cap \mathcal{O} \neq \emptyset\}$ taken for all open subsets $\mathcal{O}$ of X. A multivalued (set-valued) function $\Gamma : \Omega \to \mathcal{K}(X)$ is said to be *measurable* if Γ is $\mathcal{A} - \mathcal{B}_{\mathcal{K}(X)}$ measurable, i.e., $\Gamma^-(\mathcal{O}) = \{\omega \in \Omega : \Gamma(\omega) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{A}$ for every open $\mathcal{O} \subseteq X$. Such a function Γ is called a *multivalued random variable*. It is known that a multivalued function Γ from Ω to $\mathcal{K}(X)$ is measurable if and only if there exists a sequence $\{f_n\}_{n\geq 1}$ of measurable functions f_n from Ω to X such that $\Gamma(\omega) = \mathrm{cl}(\{f_n : n \in \mathbb{N}\})$ for almost surely all $\omega \in \Omega$. Such a sequence $\{f_n\}_{n\geq 1}$ is called a *representation by selections* of Γ. The proofs of these results could be found in [4] and [15]. By $L^1(\Omega, \mathcal{A}; X)$ we mean the Banach space of all (equivalence classes of) $\mathcal{A}$-measurable functions f from Ω to X such that the norm $\|f\|_1 = \int_\Omega \|f(\omega)\| d\mu$ is finite, and $L^1(\Omega, \mathcal{A}; \mathbb{R})$ is denoted by $L^1(\mathcal{A})$. We shall also consider $L^\infty(\Omega, \mathcal{A}; X)$, the Banach space of (equivalence classes of) essentially bounded $\mathcal{A}$-measurable functions f from Ω to X with the norm $\|f\|_\infty = \mathrm{ess\,sup}\{\|f(\omega)\| : \omega \in \Omega\}$, and $L^\infty(\Omega, \mathcal{A}; \mathbb{R})$ will be denoted by $L^\infty(\mathcal{A})$.

For a multivalued random variable Γ from Ω to $\mathcal{K}(X)$, let

$$S_\Gamma^1(\mathcal{A}) = \{f \in L^1(\Omega, \mathcal{A}; X) : f(\omega) \in \Gamma(\omega) \text{ a.s.}\}$$

which is closed subset of $L^1(\Omega, \mathcal{A}; X)$ and is nonempty if and only if $d(0, \Gamma(.))$ is in $L^1(\mathcal{A})$. If $S_\Gamma^1(\mathcal{A}) \neq \emptyset$, its elements are called *selections* of Γ, then there exists a representation by selections of Γ contained in $S_\Gamma^1(\mathcal{A})$. If $S_\Gamma^1(\mathcal{A})$ is nonempty and bounded in $L^1(\Omega, \mathcal{A}; X)$, we say that Γ is *integrably bounded*. Let $\mathcal{L}^1(\Omega, \mathcal{A}; X)$ denote the space of all integrably bounded multivalued random variables from Ω to $\mathcal{K}(X)$. Moreover, we denote by $\mathcal{K}_{cc}(X)$ the family of all compact convex subsets of X. We consider the following subspaces of $\mathcal{L}^1(\Omega, \mathcal{A}; X)$ as follows:

$$\mathcal{L}^1_c(\Omega, \mathcal{A}; X) = \{\Gamma \in \mathcal{L}^1(\Omega, \mathcal{A}; X) : \Gamma(\omega) \in \mathcal{K}_c(X) \text{ a.e.}\},$$

$$\mathcal{L}^1_{cc}(\Omega, \mathcal{A}; X) = \{\Gamma \in \mathcal{L}^1(\Omega, \mathcal{A}; X) : \Gamma(\omega) \in \mathcal{K}_{cc}(X) \text{ a.e.}\}.$$

The Radstrom theorem as cited by Hiai-Umegaki [13, Theorem 3.6] states the following.

THEOREM 2. – *There exists a real (separable) Banach space $\mathcal{Y}$ such that $\mathcal{L}^1_{cc}(\Omega, \mathcal{A}; X)$ can be embedded as a convex cone in $L^1(\Omega, \mathcal{A}; \mathcal{Y})$ in such a way that*

(i) *the embedding is isometric,*

(ii) *addition in $L^1(\Omega, \mathcal{A}; \mathcal{Y})$ induces addition $\dot{+}$ in $\mathcal{L}^1_{cc}(\Omega, \mathcal{A}; X)$,*

(iii) *multiplication by nonnegative real L^∞ functions in $L^1(\Omega, \mathcal{A}; \mathcal{Y})$ induces the corresponding operation in $\mathcal{L}^1_{cc}(\Omega, \mathcal{A}; X)$.* ■

The integral of Γ is defined by $\int_\Omega \Gamma d\mu = \{\int_\Omega f d\mu : f \in S^1_\Gamma(\mathcal{A})\}$, where $\int_\Omega f d\mu$ is the usual Bochner integral. This multivalued integral was introduced by Aumann [1]. For $A \in \mathcal{A}$, let $\int_A \Gamma d\mu$ be the integral of Γ restricted on A. Given a sub-σ-field $\mathcal{B}$ of $\mathcal{A}$ and a $\mathcal{B}$-measurable multivalued function Γ from Ω to $\mathcal{K}(X)$, besides $S^1_\Gamma(\mathcal{A})$ and $\int_\Omega \Gamma d\mu$ taken on $(\Omega, \mathcal{A}, \mu)$, we define on the measure space $(\Omega, \mathcal{B}, \mu)$ the sets

$$S^1_\Gamma(\mathcal{B}) = \{f \in L^1(\Omega, \mathcal{B}; X) : f(\omega) \in \Gamma(\omega) \text{ a.s.}\}, \text{ and}$$

$$\int_\Omega^{(\mathcal{B})} \Gamma d\mu = \left\{ \int_\Omega f d\mu : f \in S^1_\Gamma(\mathcal{B}) \right\}.$$

For $f \in L^1(\Omega, \mathcal{A}; X)$, the conditional expectation of f relative to $\mathcal{B}$ is given (see Chatterji [5], [6] and [7]) as a function $E^{\mathcal{B}}(f) \in L^1(\Omega, \mathcal{B}; X)$ such that

$$\int_B E^{\mathcal{B}}(f)\, d\mu = \int_B f d\mu \quad \text{for all } B \in \mathcal{B}.$$

If Γ is a multivalued random variable from Ω to $\mathcal{K}(X)$ with $S^1_\Gamma(\mathcal{A}) \neq \emptyset$, then it is seen (cf. [13, Theorem 5.1.]) that there exists a unique (in the a.s. sense) $\mathcal{B}$-measurable multivalued function, noted $E^{\mathcal{B}}(\Gamma)$, from Ω to $\mathcal{K}(X)$ satisfying

$$S^1_{E^{\mathcal{B}}(\Gamma)}(\mathcal{B}) = \mathrm{cl}(\{E^{\mathcal{B}}(f) : f \in S^1_\Gamma(\mathcal{A})\}), \quad \text{the closure in } L^1(\Omega, \mathcal{A}; X).$$

We call $E^{\mathcal{B}}(\Gamma)$ the (*multivalued*) *$\mathcal{B}$-conditional expectation* of Γ relative to $\mathcal{B}$

or simply the *conditional expectation* of Γ. This conditional expectation $E^{\mathcal{B}}(\Gamma)$ has the properties analogous to those of the usual conditional expectation (see [13, § 5.]). For example, we have

$$[*] \qquad \mathrm{cl}\left(\int_B^{(\mathcal{B})} E^{\mathcal{B}}(\Gamma)\, d\mu\right) = \mathrm{cl}\left(\int_B \Gamma d\mu\right), \qquad \forall B \in \mathcal{B},$$

and if $\Gamma(\omega) \in \mathcal{K}_c(X)$ a.s., then

$$[**] \qquad \mathrm{cl}\left(\int_B E^{\mathcal{B}}(\Gamma)\, d\mu\right) = \mathrm{cl}\left(\int_B \Gamma d\mu\right), \qquad \forall B \in \mathcal{B}.$$

Note that $E^{\mathcal{B}}(\Gamma)(\omega) = \mathrm{cl}\left(\int_\Omega \Gamma d\mu\right)$ for all $\omega \in \Omega$ when $\mathcal{B} = \{\emptyset, \Omega\}$.

It is known (cf. [13, Theorem 5.4.]) that when X^* is separable and Γ is a multivalued random variable from Ω to $\mathcal{K}_c(X)$, then $E^{\mathcal{B}}(\Gamma)$ is uniquely determined as the $\mathcal{B}$-measurable multifunction taking values in $\mathcal{K}_c(X)$ satisfying the condition [*] or [**]. In [19] the author shows that the assumption X^* is separable may be removed for multivalued random variable Γ *essentially bounded*, that is $\|\Gamma(.)\| \in L^\infty(\mathcal{A})$ by using convex analysis arguments, namely the concept of convex normal integrand and a duality theorem of integrand functionals for separable Banach spaces, to characterize the multivalued integral. This result establishes the following.

THEOREM 3 [19]. – *Let Γ be an essentially bounded $\mathcal{A}$-measurable multivalued random variable from Ω to $\mathcal{K}_c(X)$ and $\mathcal{B}$ a sub-σ-field of $\mathcal{A}$. Then $E^{\mathcal{B}}(\Gamma)$ is the unique (in the a.s. sense) $\mathcal{B}$-measurable multivalued random variable from Ω to $\mathcal{K}_c(X)$ such that*

$$\mathrm{cl}\left(\int_B E^{\mathcal{B}}(\Gamma) d\mu\right) = \mathrm{cl}\left(\int_B \Gamma d\mu\right) \qquad \forall B \in \mathcal{B}. \qquad \blacksquare$$

Throughout this work, we will be dealing with an increasing sequence $\{\mathcal{A}_n\}_{n \geq 1}$ of sub-σ-fields of $\mathcal{A}$ such that $\sigma\left(\bigcup_{n \geq 1} \mathcal{A}_n\right) = \mathcal{A}$. For E in $\mathcal{A}$, we will denote by χ_E the characteristic function of E.

2. – Multivalued martingales.

Continuously studied since its introduction more than sixty years ago, martingale theory is one of the central components of Probability theory. Today martingale theory has become recognized as an important tool in a diversity of topics in mathematical analysis namely optimal control, statistics and mathematical economy. At this stage, martingale theory is having an increasingly important impact particularly in statistics and Banach space theory. Multivalued random variables and multivalued martingales have been studied by many authors. We refer to the interesting work of Caponetti [3], Chatterji [8], Coste [9], Ezzaki [12], Hiai-Umegaki [13, 14], Luu [17] and Neveu [18]. Furthermore, the theory of multivalued martingales is the natural tool in the study of certain problems in the theory of information systems (see [11]) and in mathematical economics.

Let $\{\Gamma_n\}_{n\geq 1}$ be a sequence of $\mathcal{K}_c(X)$-valued random variables adapted to $\{\mathcal{A}_n\}_{n\geq 1}$ such that

$$S^1_{\Gamma_n}(\mathcal{A}_n) = \mathrm{cl}(\{f \in L^1(\Omega, \mathcal{A}_n; X) : f(\omega) \in \Gamma_n(\omega) \text{ a.s.}\}) \neq \emptyset .$$

We say that $\{\Gamma_n, \mathcal{A}_n\}_{n\geq 1}$ is a *multivalued martingale*, if for every $n \geq 1$, it verifies

$$E^{\mathcal{A}_n}(\Gamma_m) = E^n(\Gamma_m) = \Gamma_n \quad \text{for all } n \text{ and } m \in \mathbb{N} \text{ such that } m \geq n.$$

If, in addition, $S^1_{\Gamma_n}(\mathcal{A}_n)$ is bounded in $L^1(\Omega, \mathcal{A}_n; X)$ (for sufficiently large n), the multivalued martingale is said to be *integrably bounded*.

The next result is crucial in our study of multivalued martingales. The first proof was given in [16] for multivalued random variables which selections are Pettis-integrables. We state it here with a new proof for integrably bounded multivalued functions.

PROPOSITION 1. – *Let Γ be a multivalued random variable from Ω to $\mathcal{K}_c(X)$. If $S^1_\Gamma(\mathcal{A})$ is nonempty, then*

$$\forall A \in \mathcal{A},\ \forall x^* \in X^* : \delta^*\Big(x^* \,\Big|\, \int_A \Gamma d\mu\Big) = \int_A \delta^*(x^* \,|\, \Gamma(\omega))\, d\mu .$$

PROOF. – Let $A \in \mathcal{A}$ and $x^* \in X^*$.

First case. – If $\delta^*(x^* \,|\, \int_A \Gamma d\mu) = +\infty$, then for each $M > 0$, there exists f in $S^1_\Gamma(\mathcal{A})$ such that $\langle x^* \,|\, \int_A f d\mu\rangle > M$. Thus

$$\int_A \delta^*(x^* \,|\, \Gamma(\omega))\, d\mu \geq \int_A \langle x^* \,|\, f(\omega)\rangle\, d\mu = \Big\langle x^* \,\Big|\, \int_A f d\mu \Big\rangle > M ,$$

hence $\int_A \delta^*(x^* \,|\, \Gamma(\omega))\, d\mu = +\infty$.

Second case. – If $\delta^*(x^* \,|\, \int_A \Gamma d\mu)$ is finite, then for each $\varepsilon > 0$, fixed in the rest of the proof, there exists f in $S^1_\Gamma(\mathcal{A})$ such that

$$\delta^*\Big(x^* \,\Big|\, \int_A \Gamma d\mu\Big) < \Big\langle x^* \,\Big|\, \int_A f d\mu \Big\rangle + \varepsilon = \int_A \langle x^* \,|\, f(\omega)\rangle\, d\mu + \varepsilon .$$

Let's define $\Delta(\omega) = \{x \in \Gamma(\omega) : \delta^*(x^* \mid \Gamma(\omega)) \leq \langle x^* \mid x \rangle + \varepsilon\}$, which is nonempty, since $\Gamma(\omega)$ is bounded and let $\varphi : \Omega \times X \to \mathbb{R}$ be defined by $\varphi(\omega, x) = \delta^*(x^* \mid \Gamma(\omega)) - \langle x^* \mid x \rangle$. It is easy to see that $\varphi(\omega, x)$ is $\mathcal{A}$-measurable in ω for each x and continuous in x for each ω. Then φ is $\mathcal{A} \otimes \mathcal{B}_X$-measurable (cf. Himmelberg [15, Theorem 6.1.]), and hence $S^1_\Delta(\mathcal{A})$ is nonempty. Then there exists $f \in S^1_\Gamma(\mathcal{A})$ such that

$$\delta^*(x^* \mid \Gamma(\omega)) \leq \langle x^* \mid f(\omega) \rangle + \varepsilon \quad \text{a.s.},$$

then taking integral over A, we obtain

$$\int_A \delta^*(x^* \mid \Gamma(\omega))\, d\mu \leq \int_A \langle x^* \mid f(\omega) \rangle\, d\mu + \varepsilon = \left\langle x^* \,\middle|\, \int_A f d\mu \right\rangle + \varepsilon ,$$

it follows that

$$\int_A \delta^*(x^* \mid \Gamma(\omega))\, d\mu \leq \delta^*\left(x^* \,\middle|\, \int_A \Gamma d\mu\right) + \varepsilon ,$$

thus, combining with a previous inequality and letting ε go to zero, we obtain the result. ∎

As a consequence, we immediately obtain the following result.

Corollary 1. – *If $\{\Gamma_n, \mathcal{A}_n\}_{n \geq 1}$ is an integrably bounded multivalued martingale valued in $\mathcal{K}_c(X)$, then $(\delta^*(x^* \mid \Gamma_n(\cdot)), \mathcal{A}_n)_{n \geq 1}$ is an $L^1(\mathcal{A})$-martingale for $x^* \in X^*$.* ∎

If $\{\Gamma_n, \mathcal{A}_n\}_{n \geq 1}$ is a martingale taking its values in $\mathcal{K}(X)$, a sequence $\{f_n\}_{n \geq 1}$ in $L^1(\Omega, \mathcal{A}; X)$ such that $\{f_n, \mathcal{A}_n\}_{n \geq 1}$ is a martingale and for each $n \geq 1$, f_n is a selection of Γ_n, is said to be a *martingale selection* of $\{\Gamma_n, \mathcal{A}_n\}_{n \geq 1}$. When the sequence $\{\Gamma_n\}_{n \geq 1}$ is $\mathcal{K}_c(X)$-valued and integrably bounded, a result of [17] shows that the set of martingale selections of $\{\Gamma_n, \mathcal{A}_n\}_{n \geq 1}$ is nonempty. In the next result, we obtain [13, Theorem 6.5.] imposing additional hypotheses on the random variables Γ_n but without any separability of the dual space X^*. First, let us recall that a set M of measurable functions $f : \Omega \to X$ is *$\mathcal{A}$-decomposable* if for any $f_1, f_2 \in M$ and $A \in \mathcal{A}$, the function $\chi_A f_1 + \chi_{\Omega \setminus A} f_2$ lies in M.

The next result, see [13, Theorem 3.1], is fundamental for much of what will follow.

Lemma 1. – *Let M be a nonempty closed subset of $L^1(\Omega; X)$. Then there exists a multivalued random variable $\Gamma : \Omega \to X$ such that $M = S^1_\Gamma(\mathcal{A})$ if and only if M is $\mathcal{A}$-decomposable.* ∎

THEOREM 4. – *Let* $\{\Gamma_n, \mathcal{A}_n\}_{n \geq 1}$ *be a martingale of essentially bounded multifunctions taking its values in* $\mathcal{K}_c(X)$ *such that*:

1) $\exists M > 0 : \|\Gamma_n\|_\infty \leq M$ *for all* $n \geq 1$.

2) $\forall A \in \bigcup_{n \geq 1} \mathcal{A}_n$: $\lim_{\mu(A) \to 0} \int_A \|\Gamma_n\| d\mu = 0$ *uniformly in* n.

If X *has the Radon-Nikodym Property, then there exists an* $\mathcal{A}$*-measurable essentially bounded multifunction* Γ *valued in* $\mathcal{K}_c(X)$, *which is the unique* $\mathcal{A}$*-measurable* (*in the a.s. sense*) $\mathcal{K}_c(X)$*-valued multifunction verifying*

$$E^n(\Gamma) = \Gamma_n \quad \text{for all } n \geq 1 .$$

PROOF. – Let $\mathcal{MS}(\Gamma_n)$ denote the family of martingale selections of $\{\Gamma_n\}_{n \geq 1}$ and consider the subset of $L^1(\Omega, \mathcal{A}; X)$

$$\mathcal{H} = \{f \in L^1(\Omega, \mathcal{A}; X) : E^n(f) \in S^1_{\Gamma_n}(\mathcal{A}_n) \text{ for all } n \geq 1\} .$$

Arguing as in [13, Theorem 6.5.], we show that $\mathcal{H}$ is a closed, convex, bounded and $\mathcal{A}$-decomposable subset of $L^1(\Omega, \mathcal{A}; X)$. Then combining Lemma 1 and Corollary 1.6 of [13], we get an $\mathcal{A}$-measurable integrably bounded multifunction Γ from Ω to $\mathcal{K}_c(X)$ such that

$$\mathcal{H} = S^1_\Gamma(\mathcal{A}) = \{f \in L^1(\Omega, \mathcal{A}; X) : f(\omega) \in \Gamma(\omega) \text{ a.s.}\} .$$

From Luu [17], we know that

$$S^1_{\Gamma_k}(\mathcal{A}_k) = \mathrm{cl}(\{f_k : \{f_n, \mathcal{A}_n\}_{n \geq 1} \in \mathcal{MS}(\Gamma_n)\}), \quad k \geq 1 .$$

If $f \in S^1_\Gamma(\mathcal{A})$, then $\{E^n(f), \mathcal{A}_n\}_{n \geq 1}$ is a martingale selection of $\{\Gamma_n, \mathcal{A}_n\}_{n \geq 1}$, which implies that

$$S^1_{E^n(\Gamma)}(\mathcal{A}_n) = \mathrm{cl}(\{E^n(f) : f \in S^1_\Gamma(\mathcal{A})\}) \subseteq S^1_{\Gamma_n}(\mathcal{A}_n).$$

On the other hand, given a martingale selection $\{f_n, \mathcal{A}_n\}_{n \geq 1}$ of $\{\Gamma_n, \mathcal{A}_n\}_{n \geq 1}$, since X has the Radon-Nikodym Property, there exists $f \in L^1(\Omega, \mathcal{A}; X)$ such that $E^n(f) = f_n$ for all $n \geq 1$. Thus $f \in \mathcal{H} = S^1_\Gamma(\mathcal{A})$ and $S^1_{\Gamma_n}(\mathcal{A}_n) \subseteq S^1_{E^n(\Gamma)}(\mathcal{A}_n)$. Therefore we conclude that $E^n(\Gamma) = \Gamma_n$ for all $n \geq 1$ a.s.

For each $f \in \mathcal{H}$, there exists a sequence $\{f_n\}_{n \geq 1}$ in $\mathcal{MS}(\Gamma_n)$ such that $\lim_n \|f - f_n\|_1 = 0$ and since $\{f_n, \mathcal{A}_n\}_{n \geq 1}$ is a martingale,

$$\lim_n \|f(\omega) - f_n(\omega)\| = 0 \text{ a.s.}$$

Then, for any $\omega \in \Omega$, let n_0 (depending on ω) be an integer such that

$$\|f(\omega)\| \leq \|f_{n_0}(\omega)\| + 1 \text{ a.s.},$$

thus,

$$\|f(\omega)\| \leq \|\Gamma_{n_0}(\omega)\| + 1 \leq \|\Gamma_{n_0}\|_\infty + 1 \text{ a.s.}$$

This shows that $f \in L^\infty(\Omega, \mathcal{A}; X)$ with $\|f\|_\infty \leq M + 1$. Therefore we conclude that Γ is essentially bounded. Consider a $\mathcal{K}_c(X)$-valued and $\mathcal{A}$-measurable multifunction Δ such that

$$E^n(\Delta) = \Gamma_n \qquad \text{for all } n \geq 1 .$$

For $f \in S^1_\Delta(\mathcal{A})$, $E^n(f) \in S^1_{\Gamma_n}(\mathcal{A}_n)$ for all $n \geq 1$. Since we have

$$\lim_n \|f(\omega) - E^n(f)(\omega)\| = 0 \qquad \text{a.s.},$$

let n_0 be an integer such that $\|f(\omega)\| \leq 1 + \|E^{n_0}(f)(\omega)\|$. Hence

$$\|f(\omega)\| \leq 1 + \|\Gamma_{n_0}(\omega)\| \leq 1 + \|\Gamma_{n_0}\|_\infty \qquad \text{a.s.},$$

then $\|f\|_\infty \leq 1 + M$, which implies that Δ is essentially bounded.

Now for all $A \in \bigcup_{n \geq 1} \mathcal{A}_n$, choose an integer $n_1 \geq 1$ such that $A \in \mathcal{A}_{n_1}$, then

$$\operatorname{cl}\left(\int_A \Delta d\mu\right) = \operatorname{cl}\left(\int_A E^{n_1}(\Delta)\, d\mu\right) = \operatorname{cl}\left(\int_A \Gamma_{n_1} d\mu\right) = \operatorname{cl}\left(\int_A \Gamma d\mu\right).$$

In light of Proposition 1 and the fact that for any $A \in \mathcal{A}$, there exists a sequence $\{A_k\}_{k \geq 1}$ in $\bigcup_{n \geq 1} \mathcal{A}_n$ such that $\lim_k \|\chi_A - \chi_{A_k}\|_1 = 0$, one obtain the identity

$$\operatorname{cl}\left(\int_A \Delta d\mu\right) = \operatorname{cl}\left(\int_A \Gamma d\mu\right) \qquad \text{for all } A \in \mathcal{A}.$$

We complete the proof using Theorem 3. ■

3. – Multimeasures and Riesz Representations.

A multifunction M from a field $\mathcal{F}$ of subsets of Ω to $\mathcal{K}(X)$ is called additive if $M(E \cup F) = M(E) \dot{+} M(F)$ whenever E and F are disjoint members of $\mathcal{F}$. If, in addition,

$$M\left(\bigcup_{n=1}^{\infty} E_n\right) = \dot{\sum_{n=1}^{\infty}} M(E_n)$$

in the Hausdorff topology of $\mathcal{K}(X)$ for all sequences $\{E_n\}_{n \geq 1}$ of pairwise disjoint members of $\mathcal{F}$ such that $\bigcup_{n \geq 1} E_n \in \mathcal{F}$, then M is termed a *multimeasure*. If this occurs, then the serie $\dot{\sum_{n \geq 1}} M(E_n)$ is unconditionally convergent. We recall that a *selection* of an additive multifunction M from $\mathcal{F}$ to $\mathcal{K}(X)$ is an additive function m from $\mathcal{F}$ to X such that $m(E) \in M(E)$ for all $E \in \mathcal{F}$. An additive multifunction M is said to be *rich* if

$$M(E) = \mathrm{cl}(\{m(E): m \in \mathcal{S}(M)\}) \qquad \text{for all } E \in \mathcal{F},$$

where $\mathcal{S}(M)$ is the family of all selections of M. We mention that a multimeasure with values in $\mathcal{K}(X)$ is rich when X is a separable Banach space or when X has the Radon-Nikodym Property; see [10].

This section is devoted to the study of the multivalued extensions of Radon-Nikodym Theorem, the Riesz Representation Theorem and the interplay between them. Before making this precise, we start with a look at the following result, which is a consequence of [13, Theorem 4.1].

LEMMA 2. – *Let Γ be an $\mathcal{A}$-measurable multifunction taking its values in $\mathcal{K}_c(X)$. Assume that Γ is integrably bounded. Then*

$$\mathrm{cl}\left(\int [\varphi_1 + \varphi_2]\, \Gamma d\mu\right) = \mathrm{cl}\left(\int \varphi_1 \Gamma d\mu\right) \dot{+} \mathrm{cl}\left(\int \varphi_2 \Gamma d\mu\right)$$

for all $\varphi_1, \varphi_2 \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$. ■

REMARK 1. – The assumption of φ_i $(i = 1, 2)$ being nonnegative in the last result cannot be removed as the following simple example shows. Let $A = [1, 2]$, $a = -1$ and $b = 2$, then $0 \notin (a + b)A = A = [1, 2] \subsetneq aA + bA$ but $0 = (-1) \times 2 + 2 \times 1) \in aA + bA$.

At first glance, one can define a concept of a multivalued operator from $L^1(\Omega, \mathcal{A}; \mathbb{R})$ to $\mathcal{K}(X)$ as a map T satisfying $T(\alpha f + g) = \alpha T(f) \dot{+} T(g)$ for all $\alpha \in \mathbb{R}$ and $f, g \in L^1(\Omega, \mathcal{A}; \mathbb{R})$. But, Lemma 2 and Remark 1 illustrate that the map $f \mapsto \mathrm{cl}(\int f \Gamma d\mu)$ is additive only when f is taken in the space $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$. Thus, since we are mostly interested in a generalization of representable operators, by a multivalued operator, we will mean a mapping T from $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ (however, see the observation in the remark below) to $\mathcal{K}(X)$ satisfying the following conditions:

1) $T(f + g) = T(f) \dot{+} T(g)$ for all f, g in $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$.
2) $T(\alpha f) = \alpha T(f)$ for all $f \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ and $\alpha \in \mathbb{R}^+$.

A continuous multivalued operator taking values in $\mathcal{K}(X)$ endowed with the Hausdorff topology is *Riesz representable* (or simply *representable*) if there exists an integrably bounded multifunction Γ with $\|\Gamma(.)\| \in L^\infty(\mathcal{A})$ such that $T(f) = \mathrm{cl}(\int f \Gamma d\mu)$ for all $f \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$.

REMARK 2. – 1) As one might worry about, our definition, restricted to operators acting on $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$, covers the case of representable vector valued operators. In fact for such an operator T defined on the space

$L^1(\Omega, \mathcal{A}; \mathbb{R})$, for any f in $L^1(\Omega, \mathcal{A}; \mathbb{R})$, $T(f)$ can be written as $T(f) = T(f^+) - T(f^-)$. Hence, if T, restricted to $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ is representable, then $T(f^+) = \mathrm{cl}(\int f^+ \Gamma d\mu)$ and $T(f^-) = \mathrm{cl}(\int f^- \Gamma d\mu)$ where Γ is an essentially bounded multifunction. Since in this situation Γ is vector valued we obtain

$$T(f) = \int f^+ \Gamma d\mu - \int f^- \Gamma d\mu = \int f \Gamma d\mu .$$

2) The condition $\|\Gamma(\cdot)\| \in L^\infty(\mathcal{A})$ in the definition of a representable operator is essential since it asserts the non-vacuity of $S^1_{f\Gamma}(\mathcal{A})$ for all f in $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ which allows us to define $\int f\Gamma d\mu$.

3) It is a basic fact that a continuous multivalued operator T gives rise to a multimeasure M, by letting $M(E) = T(\chi_E)$.

PROPOSITION 2. – *Let T be a continuous multivalued operator from $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ to $\mathcal{K}(X)$. For $E \in \mathcal{A}$, define $M(E)$ by*

$$M(E) = T(\chi_E).$$

Then T is representable if and only if there exists an integrably bounded multifunction Γ such that

$$M(E) = \mathrm{cl}\left(\int_E \Gamma d\mu \right)$$

for all $E \in \mathcal{A}$. In this case, $\|\Gamma(\cdot)\| \in L^\infty(\mathcal{A})$ and $T(f) = \mathrm{cl}(\int f\Gamma d\mu)$ for all $f \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$.

PROOF. – If T is representable, then there exists an integrably bounded multifunction Γ such that $T(f) = \mathrm{cl}(\int f\Gamma d\mu)$ for all $f \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$. Thus, if $E \in \mathcal{A}$, then $M(E) = T(\chi_E) = \mathrm{cl}(\int_E \Gamma d\mu)$. This proves the necessity.

For the converse, let $M(E) = T(\chi_E) = \mathrm{cl}(\int_E \Gamma d\mu)$ for some integrably bounded multifunction Γ and all $E \in \mathcal{A}$. Since for $E \in \mathcal{A}$ one has

$$\|M(E)\| = \|T(\chi_E)\| = h(T(\chi_E), \{0\}) \leq k\|\chi_E\|_1 = k\mu(E), \qquad (k \in \mathbb{R})$$

it follows that the variation $|M|$ (which is defined in an obvious way) of M satisfies $|M|(E) \leq k\mu(E)$ for all $E \in \mathcal{A}$. Since for $E \in \mathcal{A}$ and each selection f of Γ, the measure m defined on $\mathcal{A}$ by

$$m(E) = \int_E f d\mu$$

is a selection of M, one has

$$\int_E \|f\|\, d\mu = |m|(E) \leqslant |M|(E) \leqslant k\mu(E),$$

it follows immediately that $\|f(\cdot)\| \leqslant k$ a.s. Hence $\|\Gamma(\cdot)\| \leqslant k$ a.s., and Γ is essentially bounded.

To finish the proof, note that the identity $M(E) = \mathrm{cl}\,(\int_E \Gamma d\mu)$ for every $E \in \mathcal{A}$ says that $T(f) = \mathrm{cl}\,(\int f\Gamma d\mu)$ whenever f is a characteristic function. Then an algebraic manipulation shows that this is also true when f is a simple function. Finally approximate an arbitrary element of $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ by a simple function and use routine properties of Hausdorff distance to complete the proof. ■

In order to apply in the last section the multivalued version of the Radon-Nikodym Property we establish the following result.

THEOREM 5. – *Let* $\{\Gamma_n, \mathcal{A}_n\}_{n \geqslant 1}$ *be a multivalued martingale with values in* $\mathcal{K}_c(X)$ *such that*

$$\exists M > 0 : \|\Gamma_n\|_\infty \leqslant M \quad \text{for all } n \geqslant 1 .$$

Then, for each $\varphi \in L^1(\mathcal{A})$, *the sequence* $\{\mathrm{cl}\,(\int \varphi\Gamma_n d\mu)\}_{n \geqslant 1}$ *converges in the Hausdorff distance topology.*

PROOF. – We divide the proof in two steps. Let $\mathcal{B} = \bigcup_{n \geqslant 1} \mathcal{A}_n$ and $\mathcal{S}(\Omega, \mathcal{B}, \mathbb{R})$ be the subspace of $\mathcal{B}$-measurable simple functions from Ω to $\mathbb{R}$.

First step. – For $f \in \mathcal{S}(\Omega, \mathcal{B}, \mathbb{R})$, the result is a direct consequence of the definition of a multivalued martingale. Moreover, we obtain much more: For $f \in \mathcal{S}(\Omega, \mathcal{B}, \mathbb{R})$ there exists an integer n_0 depending on f such that

$$\mathrm{cl}\left(\int f\Gamma_{n_0} d\mu\right) = \mathrm{cl}\left(\int f\Gamma_n d\mu\right) \text{ for all } n \geqslant n_0,$$

which shows that $\lim_n \mathrm{cl}\,(\int f\Gamma_n d\mu) = \mathrm{cl}\,(\int f\Gamma_{n_0} d\mu)$.

Let T be the map from $\mathcal{S}(\Omega, \mathcal{B}, \mathbb{R})$ to the complete metric space $\mathcal{K}_c(X)$ defined by

$$T(\varphi) = \lim_n \mathrm{cl}\left(\int \varphi\Gamma_n d\mu\right).$$

Then if f_1 and f_2 are elements of $\mathcal{S}(\Omega, \mathcal{B}, \mathbb{R})$, let n_1 and n_2 be integers and $p =$

$\max(n_1, n_2)$ such that

$$T(f_1) = \mathrm{cl}\left(\int f_1 \Gamma_{n_1} d\mu\right) \quad \text{and} \quad T(f_2) = \mathrm{cl}\left(\int f_2 \Gamma_{n_2} d\mu\right).$$

Then for any $x^* \in X^*$, applying Proposition 1 we have,

$$|\delta^*(x^* \,|\, T(f_1)) - \delta^*(x^* \,|\, T(f_2))| =$$

$$|\delta^*\left(x^* \,|\, \mathrm{cl}\left(\int f_1 \Gamma_p d\mu\right)\right) - \delta^*\left(x^* \,|\, \mathrm{cl}\left(\int f_2 \Gamma_p d\mu\right)\right)| =$$

$$|\int \delta^*(x^* \,| f_1(\omega)\, \Gamma_p(\omega))\, d\mu - \int \delta^*(x^* \,| f_2(\omega)\, \Gamma_p(\omega))\, d\mu| \leqslant$$

$$\int |\delta^*(x^* \,| f_1(\omega)\, \Gamma_p(\omega)) - \delta^*(x^* \,| f_2(\omega)\, \Gamma_p(\omega))\,| d\mu.$$

On the other hand,

$$|\delta^*(x^* \,| f_1(\omega)\, \Gamma_p(\omega)) - \delta^*(x^* \,| f_2(\omega)\, \Gamma_p(\omega))| =$$

$$|\delta^*(x^* f_1(\omega) \,|\, \Gamma_p(\omega)) - \delta^*(x^* f_2(\omega) \,|\, \Gamma_p(\omega))| \leqslant$$

$$\|\Gamma_p(\omega)\| \, \|x^* f_1(\omega) - x^* f_2(\omega)\|_{X^*},$$

then integrating, we obtain

$$h(T(f_1), T(f_2) \leqslant \|\Gamma_p\|_\infty \|f_1 - f_2\|_1 \leqslant M\|f_1 - f_2\|_1.$$

Second step. – Thus, since $(\mathcal{K}_c(X), h)$ is a complete metric space, T can be uniquely extended to a continuous map (with the same bound), $\overline{T}$, from the $\|\cdot\|_1$-completion of $\mathcal{S}(\Omega, \mathcal{B}, \mathbb{R})$, which is $L^1(\Omega, \mathcal{A}, \mathbb{R})$, to $\mathcal{K}_c(X)$. Hence, for each f in $L^1(\mathcal{A})$, there is a sequence of elements $\{f_n\}_{n \geqslant 1}$ in $\mathcal{S}(\Omega, \mathcal{B}, \mathbb{R})$ with $f_n \to f$ as $n \to \infty$ and $\overline{T}(f) = \lim_n T(f_n)$. So, for $n \geqslant 1$, we can find $k_n \geqslant 1$ such that for all $k \geqslant k_n$

$$T(f_n) = \mathrm{cl}\left(\int f_n \Gamma_{k_n} d\mu\right) = \mathrm{cl}\left(\int f_n \Gamma_k d\mu\right).$$

Now the result follows by a routine argument. ∎

4. – Multivalued Radon-Nikodym Property.

A nonempty closed convex subset C of X is said to be a *Radon-Nikodym subset*, if given a finite mesure space (E, Σ, ν) and a ν-continuous vector measure $m : \Sigma \to X$ of bounded variation such that $m(A) \subseteq \nu(A) C$ for $A \in \Sigma$ with $\nu(A) > 0$, there exists $f \in L^1(E, \Sigma; X)$ such that $m(A) = \int_A f d\nu$ for all $A \in \Sigma$. With the help of martingales, we see the Radon-Nikodym property of a subset transform itself into an internal geometric property of the subset. According to a result of Chatterji [7] from 1968, a subset C has the Radon-Nikodym property if and only if for every finite measure space (E, Σ, ν) every bounded uniformly integrable martingale in $L^1(\Sigma, \nu; X)$ taking values in C converges in $L^1(E, \nu; X)$-norm. For more details, we refer to the monograph of Bourgin [2].

THEOREM 6. – *Let C be a nonempty convex closed and bounded subset of X and M be a multimeasure from $\mathcal{A}$ to $\mathcal{K}_c(X)$ such that $M(E) \subseteq \mu(E)C$ for all $E \in \mathcal{A}$. If C is a Radon-Nikodym subset, then there exists an $\mathcal{A}$-measurable multifunction Γ with values in $\mathcal{K}_c(X)$ such that*:

(i) $M(E) = \mathrm{cl}\,(\int_E \Gamma d\mu)\ \forall E \in \mathcal{A}$.

(ii) *Γ is essentially bounded.*

PROOF. – Since X is a separable Banach space,

$$M(E) = \mathrm{cl}\,(\{m(E) : m \in \mathcal{S}(M)\}) \qquad \text{for all } E \in \mathcal{A}.$$

If m is a selection of M, then for all $E \in \mathcal{A}$, $m(E) \in \mu(E)\, C$ and hence m is μ-continuous. Thus, since C is a Radon-Nikodym subset, there exists f_m in $L^1(\mathcal{A}, \mu; X)$ such that $m(E) = \int_E f_m\, d\mu$ for all $E \in \mathcal{A}$.

If $\mathcal{H}$ denote the subset of $L^1(\mathcal{A}, \mu; X)$ defined by

$$\mathcal{H} = \{f_m : m \in \mathcal{S}(M)\},$$

then it is easy to check that $\mathcal{H}$ is a nonempty, convexe, $\mathcal{A}$-decomposable, closed and bounded subset of $L^1(\Omega, \mathcal{A}; X)$. Therefore, there exists an integrably bounded multifunction $\Gamma : \mathcal{A} \to \mathcal{K}(X)$ such that

$$\mathcal{H} = S^1_\Gamma(\mathcal{A}) = \{f \in L^1(\Omega, \mathcal{A}; X) : f(\omega) \in \Gamma(\omega) \text{ a.s.}\}.$$

Consequently, since for $E \in \mathcal{A}$,

$$M(E) = \mathrm{cl}\,(\{m(E) : m \in \mathcal{S}(M)\}) = \mathrm{cl}\left(\left\{\int_E f d\mu : f \in S^1_\Gamma(\mathcal{A})\right\}\right),$$

it follows that $M(E) = \mathrm{cl}\,(\int_E \Gamma d\mu)$ for all $E \in \mathcal{A}$. On the other hand for all $\omega \in \Omega$, we have

$$\|\Gamma(\omega)\| = \sup_{f \in \mathcal{H}} \|f(\omega)\| \leq \sup_{f \in \mathcal{H}} \|f\|_\infty \leq \|C\| \qquad \text{a.s.},$$

which shows that Γ is essentially bounded. ■

We can now state the multivalued Radon-Nikodym Property.

THEOREM 7. – *Let C be a nonempty convex closed and bounded subset of X. Then the following conditions are equivalent*:

(i) *C is a Radon-Nikodym subset.*

(ii) *For any multivalued continuous operator T from $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ to $\mathcal{K}_c(X)$ such that*

$$T(\varphi) \subseteq C \int \varphi \, d\mu \quad \text{for all } \varphi \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+),$$

there exists an $\mathcal{A}$-measurable multifunction Γ taking its values in $\mathcal{K}_c(X)$ in such way that Γ is essentially bounded and $T(\varphi) = \mathrm{cl}(\int \varphi \Gamma d\mu)$ for all $\varphi \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$.

(iii) *For any $\mathcal{K}_c(X)$-valued martingale $\{\Gamma_n, \mathcal{A}_n\}_{n \geqslant 1}$ such that*

$$\Gamma_n(\omega) \subseteq C \quad \text{for all } n \geqslant 1 \quad \text{a.s.}$$

there exists a unique (a.s.) $\mathcal{K}_c(X)$-valued and $\mathcal{A}$-measurable multifunction Γ such that Γ is essentially bounded and $E^n(\Gamma) = \Gamma_n$ for all $n \geqslant 1$.

PROOF. – (i) $\Rightarrow$ (ii). Consider an operator T from $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ to $\mathcal{K}_c(X)$ such that $T(\varphi) \subseteq C \int \varphi d\mu$ for all $\varphi \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$. If M is the multimeasure associated to T, then $M(A) = T(\chi_A) \subseteq \mu(A) C$ for all $A \in \mathcal{A}$. By Theorem 6, there exists an $\mathcal{A}$-measurable multifunction Γ with values in $\mathcal{K}_c(X)$ satisfying $\|\Gamma(\cdot)\| \in L^\infty(\mathcal{A})$ and $M(E) = \mathrm{cl}(\int_E \Gamma d\mu)$ for all $E \in \mathcal{A}$. Finally Proposition 2 shows that

$$T(\varphi) = \mathrm{cl}\left(\int \varphi \Gamma d\mu\right) \quad \text{for all } \varphi \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+).$$

(ii) $\Rightarrow$ (iii). Let $\{\Gamma_n, \mathcal{A}_n\}_{n \geqslant 1}$ be a multivalued martingale taking values in $\mathcal{K}_c(X)$ such that $\Gamma_n(\omega) \subseteq C$ for all $n \geqslant 1$ a.s. Then $\|\Gamma_n\|_\infty \leqslant \|C\|$ for all $n \geqslant 1$. By Theorem 5, the mapping T from $L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$ to $\mathcal{K}_c(X)$ defined by

$$T(\varphi) = \lim_n \mathrm{cl}\left(\int \varphi \Gamma_n d\mu\right)$$

is a continuous multivalued operator such that

$$T(\varphi) \subseteq C \int \varphi \, d\mu \quad \text{for all } \varphi \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+).$$

Hence, there exists an essentially bounded multifunction Γ from Ω to $\mathcal{K}_c(X)$ such that $T(\varphi) = \mathrm{cl}(\int \varphi \Gamma d\mu)$ for all $\varphi \in L^1(\Omega, \mathcal{A}; \mathbb{R}^+)$.

Now for every fixed integer k and any $A \in \mathcal{A}_k$, since for $n \geq k$ we have

$$\mathrm{cl}\left(\int_A \Gamma_n \, d\mu \right) = \mathrm{cl}\left(\int_A \Gamma_k \, d\mu \right),$$

it follows that

$$\mathrm{cl}\left(\int_A \Gamma d\mu \right) = T(\chi_A) = \lim_n \mathrm{cl}\left(\int_A \Gamma_n \, d\mu \right) = \mathrm{cl}\left(\int_A \Gamma_k \, d\mu \right),$$

taking into account that $\|\Gamma(\cdot)\|$ and $\|\Gamma_k(\cdot)\|$ are elements of $L^\infty(\mathcal{A})$, Theorem 3 says that $E^k(\Gamma) = \Gamma_k$ for all $k \geq 1$. Finally arguing as in the proof of Theorem 4, we obtain that Γ is the unique in the a.s. sense verifying $E^n(\Gamma) = \Gamma_n$ for all $n \geq 1$.

(iii) $\Rightarrow$ (i). Let $\{f_n, \mathcal{A}_n\}_{n \geq 1}$ be a martingale with values in C, then consider the multivalued martingale $\{\Gamma_n, \mathcal{A}_n\}_{n \geq 1}$ defined by $\Gamma_n(\cdot) = \{f_n(\cdot)\}$ for all $n \geq 1$. Then there exists an $\mathcal{A}$-measurable multifunction Γ with values in $\mathcal{K}_c(X)$ such that $E^n(\Gamma) = \Gamma_n$ for all $n \geq 1$. Since for all $n \geq 1$

$$\mathrm{cl}\left(\{E^n(f) : f \in S^1_\Gamma(\mathcal{A})\}\right) = S^1_{E^n(\Gamma)}(\mathcal{A}_n) = S^1_{\Gamma_n}(\mathcal{A}_n) = \{f_n\},$$

it follows that $E^n(f) = f_n$ for all $n \geq 1$ and $\lim_n \|f - f_n\|_1 = 0$. ∎

Acknowledgement. The author is very grateful to the referee for his invaluable comments and suggestions.

REFERENCES

[1] R. J. AUMANN, *Integrals of set-valued functions*, J. Math. Anal. Appl., **12** (1965), 1-12.

[2] R. D. BOURGIN, *Geometric aspects of convex sets with the Radon-Nikodym property*, Lecture notes in Math., **993**, Springer-Verlag, Berlin, 1983.

[3] D. CAPONETTI, *On Convergence of Multivalued Martingales and Uniform Amarts indexed by Directed Sets*, Boll. Un. Mat. Ital. B (7), **4** (1990), 553-565.

[4] C. CASTAING - M. VALADIER, *Convex Analysis and measurable multifunctions*, Lecture notes in Math., **580**, Springer-Verlag, Berlin, 1977.

[5] S. D. CHATTERJI, *Martingales of Banach-valued random variables*, Bull. Amer. Math. Soc., **66** (1960), 395-398.

[6] S. D. CHATTERJI, *A note on the convergence of Banach space-valued martingales*, Math. Ann., **153** (1964), 142-149.

[7] S. D. CHATTERJI, *Martingale convergence and the Radon-Nikodym theorem in Banach spaces*, Math. Scand., **22** (1968), 21-41.

[8] S. D. Chatterji, *Vector values martingales and their applications*, Proc. of the conf. in Probability Theory in Banach spaces, Oberwolfah (1975), Lecture Notes in Math, **256**, Springer, Berlin (1976), 33-51.

[9] A. Coste, *La Propriété de Radon-Nikodym en intégration multivoque*, C. R. Acad. Sci. Paris, **280** (1975), 1515-1518.

[10] A. Coste, *Contribution à la Théorie de l'intégration multivoque*, Thèse de Doctorat d'Etat, Université Paris VI (1977).

[11] A. Dekorvin - R. Kleyle, *A convergence theorem for convex set valued supermartingales*, Stoch. Anal. Appl., **3** (1985), 433-445.

[12] E. Ezzaki, *A general dominated convergence theorem for unbounded randon sets*, Bull. Ac. Pol. Sc. Math., **44**, **3** (1996).

[13] F. Hiai - H. Umegaki, *Integrals, Conditional expectation, and martingales of multivalued functions*, J. Multivariate Anal., **7** (1977), 149-182.

[14] F. Hiai - H. Umegaki, *Radon-Nikodym theorems for set-valued measures*, J. Multivariate Anal., 8 (1978), 96-118.

[15] C. J. Himmelberg, *Mesurable relations*, Fund. Math., **87** (1975), 53-72.

[16] A. D. Ioffe - V. M. Tihomiron, Uspehi Math., **23**, 6 (1968), 51-116.

[17] D. Q. Luu, *Quelques résultats de représentation des amars uniformes multivoques*, C. R. Acad. Sci. Paris, **300** (1985), 63-65.

[18] J. Neveu, *Convergence presque sûre de martingales multivoques*, Ann. Inst. H. Poincaré, 8, 1 (1972), 1-7.

[19] M. Zohry, *Characterization of Set-Valued Conditional Expectation for a class of Multivalued Random Variables*, Preprint, (2001).

Département de Mathématiques, Faculté des Sciences
Université Abdelmalek Essaâdi - BP 2121 Tetouan, Morocco
e-mail address: zohry@fst.ac.ma

Pervenuta in Redazione
il 23 maggio 2002 e in forma rivista il 25 ottobre 2002