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Construction of a Natural Norm for the Convection-Diffusion-Reaction Operator (*).

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Sunto. – *In questo lavoro si costruisce, mediante interpolazione, una norma* naturale *per operatori lineari continui coercivi e* non simmetrici*. Più precisamente, si cerca una norma con stesse le proprietà che ha la* norma dell'energia *quando si considerano operatori simmetrici: si dimostrano cioè, rispetto a tale norma, stime di* continuità *e di* inf-sup *indipendenti dall'operatore. In particolare, si prende in considerazione l'operatore di diffusione-trasporto-reazione lineare: si ottengono quindi stime di continuità e inf-sup indipendenti dai coefficienti dell'operatore, pertanto significative anche nel regime di trasporto dominante. I risultati qui presentati possono servire ad una più approfindita comprensione e analisi di tecniche numeriche per problemi non simmetrici.*

Summary. – *In this work, we construct, by means of the function space interpolation theory, a* natural norm *for a generic linear* coercive *and* non-symmetric *operator. We look for a norm which is the counterpart of the* energy norm *for symmetric operators. The natural norm allows for* continuity *and* inf-sup *conditions independent of the operator. Particularly we consider the convection-diffusion-reaction operator, for which we obtain continuity and inf-sup conditions that are uniform with respect to the operator coefficients, and therefore meaningful in the convection-dominant regime. Our results are preliminary to a deeper understanding and analysis of the numerical techniques for non-symmetric problems.*

1. – Introduction.

In order to clarify the aim of this work, we first recall the well known properties of *coercive* and *symmetric* operators. Denote by $\mathcal{L}_{\mathrm{sym}}$ such an operator, defined on a Hilbert space V into its dual V^*:

$$\mathcal{L}_{\mathrm{sym}}: V \to V^*,$$

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and let $a_{\text{sym}}: V \times V \to \mathbb{R}$ be the associated bilinear form

$$a_{\text{sym}}(w, v) := {}_{V^*}\langle \mathcal{L}_{\text{sym}} w, v \rangle_V \equiv {}_{V^*}\langle \mathcal{L}_{\text{sym}} v, w \rangle_V, \quad \forall w, v \in V.$$

Further, assume that the norm of V is the *energy norm* $\|w\|_V := a_{\text{sym}}(w, w)^{1/2}$. Then $\mathcal{L}_{\text{sym}}$ behaves as an isometry from V into V^*, i.e., it has unitary norm and its inverse has a unitary norm, too. This obvious fact can be expressed in terms of continuity and inf-sup conditions:

$$\begin{aligned} &\text{continuity:} \quad \|\mathcal{L}_{\text{sym}}\|_{V \to V^*} := \sup_{w \in V} \sup_{v \in V} \frac{\langle \mathcal{L}_{\text{sym}} w, v \rangle}{\|w\|_V \|v\|_V} = 1 \\ &\text{inf-sup:} \quad \|\mathcal{L}_{\text{sym}}^{-1}\|_{V^* \to V}^{-1} := \inf_{w \in V} \sup_{v \in V} \frac{\langle \mathcal{L}_{\text{sym}} w, v \rangle}{\|w\|_V \|v\|_V} = 1\,. \end{aligned} \tag{1}$$

In this sense, the energy norm is the natural norm. Consider, for example, the problem $\mathcal{L}_{\text{sym}} u = f$, where u denote the solution for the source term f, and a perturbed problem $\mathcal{L}_{\text{sym}}(u + \delta u) = f + \delta f$, where δf represent a perturbation of the source term, then the relative effect on the solution is bounded by the relative magnitude of the source perturbation:

$$\frac{\|\delta u\|_V}{\|u\|_V} \leqslant \frac{\|\delta f\|_{V^*}}{\|f\|_{V^*}}\,. \tag{2}$$

Moreover, the plain Galerkin F.E.M. for $\mathcal{L}_{\text{sym}} u = f$, i.e.,

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a_{\text{sym}}(u_h, v_h) = {}_{V^*}\langle f, v_h \rangle_V, \quad \forall v_h \in V_h \end{cases} \tag{3}$$

gives an *optimal* discrete solution u_h in the discrete space $V_h \subset V$:

$$\|u - u_h\|_V \leqslant \inf_{w_h \in V_h} \|u - w_h\|_V. \tag{4}$$

Finally, optimal *a-posteriori* residual-based estimates for (3) can be proved (see [16]). Given $\Omega \subset \mathbb{R}^n$, $V = H_0^1(\Omega)$, and the coefficients $\kappa > 0$ and $\varrho \geqslant 0$ we may consider, as an example of $\mathcal{L}_{\text{sym}}$, the reaction-diffusion operator:

$$w \mapsto \mathcal{L}_{\text{sym}} w := -\kappa \Delta w + \varrho w. \tag{5}$$

It is worth noting that the previous results (2) and (4) are independent of the coefficients κ and ϱ in the example (5).

Consider now a coercive and *non-symmetric* operator $\mathcal{L}$, still from V into its dual V^*, and let $a(\cdot, \cdot): V \times V \to \mathbb{R}$ be the associated bilinear form (i.e., $a(w, v) := {}_{V^*}\langle \mathcal{L} w, v \rangle_V$, $\forall w, v \in V$). We can split $\mathcal{L}$ into its *symmetric* part $\mathcal{L}_{\text{sym}}$

and its *skew-symmetric* part $\mathcal{L}_{\text{skew}}$

$$\mathcal{L} = \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{skew}}, \qquad a(w, v) = a_{\text{sym}}(w, v) + a_{\text{skew}}(w, v),$$

in the usual way:

$$\begin{aligned} {}_{V^*}\langle \mathcal{L}_{\text{sym}} w, v\rangle_V &:= a_{\text{sym}}(w, v) := \frac{1}{2}(a(w, v) + a(v, w)), \quad \forall w, v \in V, \\ {}_{V^*}\langle \mathcal{L}_{\text{skew}} w, v\rangle_V &:= a_{\text{skew}}(w, v) := \frac{1}{2}(a(w, v) - a(v, w)), \quad \forall w, v \in V. \end{aligned} \tag{6}$$

The example now is the convection-diffusion-reaction operator: given $V = H_0^1(\Omega)$, $\kappa > 0$, $\varrho \geq 0$ and $\beta : \Omega \to \mathbb{R}^n$ we consider

$$w \mapsto \mathcal{L}w := -\kappa \Delta w + \beta \cdot \nabla w + \varrho w\,; \tag{7}$$

with the assumption $\operatorname{div}(\beta) = 0$, the splitting (6) is

$$\mathcal{L}_{\text{sym}} w = -\kappa \Delta w + \varrho w, \qquad \mathcal{L}_{\text{skew}} = \beta \cdot \nabla w,$$

$$a_{\text{sym}}(w, v) = \kappa \int_\Omega \nabla w \cdot \nabla v + \int_\Omega \varrho w v, \quad a_{\text{skew}}(w, v) = \int_\Omega \beta \cdot \nabla w v;$$

We still assume that the norm on V is the energy norm $\|w\|_V := a(w, w)^{1/2} \equiv a_{\text{sym}}(w, w)^{1/2}$. The aim of this paper is to prove conditions similar to (1) for the non-symmetric operator $\mathcal{L}$; more precisely we construct a natural norm $|||\cdot|||$ such that the *continuity*

$$\sup_{w \in V} \sup_{v \in V} \frac{\langle \mathcal{L}w, v\rangle}{|||w|||\,|||v|||} \leq \mathcal{C}_c < +\infty \tag{8}$$

and the *inf-sup* condition

$$\inf_{w \in V} \sup_{v \in V} \frac{\langle \mathcal{L}w, v\rangle}{|||w|||\,|||v|||} \geq \mathcal{C}_{is} > 0 \tag{9}$$

hold true with constants $\mathcal{C}_c$ and $\mathcal{C}_{is}$ independent of $\mathcal{L}$. Therefore, for the example (7), $\mathcal{C}_c$ and $\mathcal{C}_{is}$ will be independent of the coefficients κ, β and ϱ.

It is clear that now, contrary to the symmetric case, the choice $|||\cdot||| := \|\cdot\|_V$ does not give (8)-(9). In the paper, we will use the function space interpolation theory to obtain a suitable $|||\cdot|||$. As for the symmetric case, the norm $|||\cdot|||$, for which (8)-(9) hold true, depends on $\mathcal{L}$ and gives the *natural* topology for $\mathcal{L}$.

This is the proper framework to understand the behavior of (7) for small values of the diffusivity κ, when the higher order term $-\kappa\Delta$ acts as a singular perturbation on the lower order term $\beta \cdot \nabla + \varrho\,\mathrm{Id}$. Conditions (8)-(9) gives also the proper framework for using some recent numerical methodologies devoted to (7). Particularly, we mention the *least-squares* formulations in the context of finite element methods [4] or in the context of wavelet methods [8], and the *adaptive wavelet methods* [7] (see also [1, 3, 6]).

More generally, (8)-(9) are the starting point for the classical analysis of numerical methods for this class of operators. When the continuity and inf-sup conditions are known for an operator $\mathcal{L}$, then *ideal* numerical methods should preserve them at the discrete level. This is straightforward for symmetric and coercive operators, while in other contexts, e.g., for mixed formulations (see [5]), this requires *ad hoc* numerical methods. On the contrary the error analysis of numerical methods for (7) typically do not follow the classical argument mentioned above and it is not completely satisfactory (see [12]). Then we hope this paper could give some insights for a deeper theoretical understanding of numerical methods for (7) (we refer to [12, § 4], [13] for a further discussion on the topic).

This paper presents some of the results of [14]. Different estimates for the operator (7) have been obtained by other authors: see for example the analysis by Bertoluzza, Canuto and Tabacco in [2, § 2.1], or the paper by Dörfler [9]. The peculiarity of our paper is that both conditions (8)-(9) are obtained for (7).

The outline of the paper is as follows: in § 2 we present our methodology for obtaining (8)-(9) in the case of a generic non-symmetric and coercive operator $\mathcal{L}$; then we apply the theory first, in § 3, to the very simple one-dimensional ($n = 1$) convection-diffusion-reaction model problem, and then, in § 4, to the multi dimensional ($n > 1$) case, and discuss the results.

2. – The abstract framework.

In this section, we present our idea for obtaining uniform continuity and inf-sup conditions (8)-(9).

Let V be a Hilbert space, and let V^* be its dual. In the present section we consider a generic coercive isomorphism $\mathcal{L}: V \to V^*$ and the associated bilinear form

$$a(w, v) := {}_{V^*}\langle \mathcal{L}w, v \rangle_V, \qquad \forall w, v \in V; \tag{10}$$

The problem of solving $\mathcal{L}u = f$ for the unknown $u \in V$ admits the variational formulation:

$$\text{find } u \in V \text{ such that } a(u, v) = {}_{V^*}\langle f, v \rangle_V, \quad \forall v \in V. \tag{11}$$

We also assume that $\|\cdot\|_V$, the norm of V, is the *energy norm* for $\mathcal{L}$, i.e.

$$a(w, w) = \|w\|_V^2, \quad \forall w \in V. \tag{12}$$

We split $\mathcal{L} = \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{skew}}$, and introduce the bilinear forms $a_{\text{sym}}(\cdot, \cdot)$ and $a_{\text{skew}}(\cdot, \cdot)$ on $V \times V$ according to (6). $\mathcal{L}_{\text{sym}}$ is the symmetric part of $\mathcal{L}$ (i.e.,

$a_{\mathrm{sym}}(w, v) = a_{\mathrm{sym}}(v, w)$, $\forall w, v \in V$), and we have

$$\text{(13)} \qquad \begin{aligned} a_{\mathrm{sym}}(w, w) &= \|w\|_V^2, & \forall w \in V, \\ a_{\mathrm{sym}}(w, v) &\leq \|w\|_V \|v\|_V, & \forall w, v \in V, \end{aligned}$$

while $\mathcal{L}_{\mathrm{skew}}$ is the skew-symmetric part of $\mathcal{L}$ (i.e., $a_{\mathrm{skew}}(w, v) = -a_{\mathrm{skew}}(v, w)$, $\forall w, v \in V$).

Finally, we define

$$\text{(14)} \qquad \begin{aligned} \|w\|_{A_0}^2 &:= \|w\|_V^2, & \forall w \in V, \\ \|w\|_{A_1}^2 &:= \|w\|_V^2 + \|\mathcal{L}_{\mathrm{skew}} w\|_{V^*}^2, & \forall w \in V, \end{aligned}$$

where

$$\|\mathcal{L}_{\mathrm{skew}} w\|_{V^*} = \sup_{v \in V} \frac{a_{\mathrm{skew}}(w, v)}{\|v\|_V};$$

we also set $A_0 = A_1 = V$ from the algebraic standpoint; nothe that A_0 and A_1 are the same space with the same topology, but the two norms $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$ are different (even though equivalent, up to constants depending on L).

The following lemma states two basic estimates; we explicitly compute the constants appearing into the estimates to put in light their independence of $\mathcal{L}$.

LEMMA 1. – *Under the hypotheses above, we have*

$$\text{(15)} \qquad a(w, v) \leq 2^{1/2} \|w\|_{A_i} \|v\|_{A_{1-i}}, \qquad \forall w, v \in V,$$

$$\text{(16)} \qquad \sup_{v \in V} \frac{a(w, v)}{\|v\|_{A_{1-i}}} \geq 5^{-1/2} \|w\|_{A_i}, \qquad \forall w \in V,$$

for $i = 0$ *or* $i = 1$.

PROOF. – Let v and w be two generic elements of V.

By using the Cauchy-Schwartz inequality we easily get

$$\begin{aligned} a(w, v) &= a_{\mathrm{sym}}(w, v) + a_{\mathrm{skew}}(w, v) \\ &\leq \|w\|_V \|v\|_V + \|\mathcal{L}_{\mathrm{skew}} w\|_{V^*} \|v\|_V \\ &\leq 2^{1/2} \|w\|_{A_1} \|v\|_{A_0}; \end{aligned}$$

similarly, since $a_{\mathrm{skew}}(w, v) = -a_{\mathrm{skew}}(v, w)$, we also get $a(w, v) \leq 2^{1/2} \|w\|_{A_0} \|v\|_{A_1}$, then (15) follows.

Recalling (12) and (13), we have

$$\text{(17)} \qquad \|w\|_V \leq \sup_{v \in V} \frac{a(w, v)}{\|v\|_V},$$

and

$$\sup_{v \in V} \frac{a_{\mathrm{sym}}(w, v)}{\|v\|_V} \leqslant \sup_{v \in V} \frac{a(w, v)}{\|v\|_V} . \tag{18}$$

Then, we get:

$$\begin{aligned} \|\mathcal{L}_{\mathrm{skew}} w\|_{V^*} &= \sup_{v \in V} \frac{a_{\mathrm{skew}}(w, v)}{\|v\|_V} \\ &\leqslant \sup_{v \in V} \frac{a(w, v)}{\|v\|_V} + \sup_{v \in V} \frac{a_{\mathrm{sym}}(w, v)}{\|v\|_V} \\ &\leqslant 2 \sup_{v \in V} \frac{a(w, v)}{\|v\|_V} , \end{aligned} \tag{19}$$

and, collecting (17) and (19), we get

$$\|w\|_{A_1} \leqslant 5^{1/2} \sup_{v \in V} \frac{a(w, v)}{\|v\|_{A_0}} , \tag{20}$$

which is (16) for $i = 1$. We are left to show that

$$\|w\|_{A_0} \leqslant 5^{1/2} \sup_{v \in V} \frac{a(w, v)}{\|v\|_{A_1}} ; \tag{21}$$

for that purpose, we make use of a duality argument. Reasoning as for (20) we obtain

$$\|\widetilde{w}\|_{A_1} \leqslant 5^{1/2} \sup_{v \in V} \frac{a(v, \widetilde{w})}{\|v\|_{A_0}} , \tag{22}$$

for any $\widetilde{w} \in V$. Given a generic $w \in V$, we associate to it $\widetilde{w} \in V$ such that $a(v, \widetilde{w}) = a_{\mathrm{sym}}(v, w)$, $\forall v \in V$; thanks to (22) we have

$$\|\widetilde{w}\|_{A_1} \leqslant 5^{1/2} \sup_{v \in V} \frac{a(v, \widetilde{w})}{\|v\|_{A_0}} = 5^{1/2} \sup_{v \in V} \frac{a_{\mathrm{sym}}(v, w)}{\|v\|_{A_0}} = 5^{1/2} \|w\|_{A_0},$$

whence

$$\begin{aligned} \|w\|_{A_0}^2 &= a_{\mathrm{sym}}(w, w) = a(w, \widetilde{w}) \\ &\leqslant \sup_{v \in V} \frac{a(w, v)}{\|v\|_{A_1}} \cdot \|\widetilde{w}\|_{A_1} \\ &\leqslant 5^{1/2} \sup_{v \in V} \frac{a(w, v)}{\|v\|_{A_1}} \cdot \|w\|_{A_0}, \end{aligned}$$

which completes the proof. ∎

From Lemma 1 we can obtain a family of intermediate estimates by means of the function spaces interpolation. We follow the notation and the definitions of [15]; for the reader's convenience, we recall the fundamental definition of *interpolated norm*, according to the so-called *K-method*: given $0 < \theta < 1$ and $1 \leqslant p \leqslant +\infty$ we define

$$\|w\|_{(A_0,\, A_1)_{\theta,\, p}} := \left[\int_0^{+\infty} \inf_{\substack{w_0 \in A_0,\, w_1 \in A_1,\\ w_0 + w_1 = w}} (t^{-\theta} \|w_0\|_{A_0} + t^{1-\theta} \|w_1\|_{A_1})^p \frac{dt}{t} \right]^{\frac{1}{p}}. \tag{23}$$

Generally $(A_0, A_1)_{\theta, p}$ is the space of functions $w \in A_0 + A_1$ such that $\|w\|_{(A_0, A_1)_{\theta, p}} < +\infty$. In our particular case, A_0 and A_1 are the same space from the algebraic standpoint ($A_0 \equiv A_1 \equiv V$), and $\|\cdot\|_{(A_0, A_1)_{\theta, p}}$ simply is a new norm on V.

LEMMA 2. – *Given* θ, p *and* p' *such that* $0 < \theta < 1$, $1 \leqslant p \leqslant +\infty$, *and* $1/p + 1/p' = 1$, *under the hypotheses above, we have*

$$a(w, v) \leqslant 2^{1/2} \|w\|_{(A_0, A_1)_{\theta, p}} \|v\|_{(A_0, A_1)_{1-\theta, p'}}, \qquad \forall w, v \in V, \tag{24}$$

$$\sup_{v \in V} \frac{a(w, v)}{\|v\|_{(A_0, A_1)_{1-\theta, p'}}} \geqslant 5^{-1/2} \|w\|_{(A_0, A_1)_{\theta, p}}, \qquad \forall w \in V. \tag{25}$$

PROOF. – Typically interpolation theorems are stated in terms of linear operators instead of bilinear forms. Then it is more convenient to rephrase (15) as

$$\|\mathcal{L}w\|_{A_1^*} \leqslant 2^{1/2} \|w\|_{A_0}, \qquad \|\mathcal{L}w\|_{A_0^*} \leqslant 2^{1/2} \|w\|_{A_1}, \tag{26}$$

and (16) as

$$\|w\|_{A_0} \leqslant 5^{1/2} \|\mathcal{L}w\|_{A_1^*}, \qquad \|w\|_{A_1} \leqslant 5^{1/2} \|\mathcal{L}w\|_{A_0^*}, \tag{27}$$

for all $w \in V$.

From (26) and thanks to Theorem [15, §1.3.3] and [15, §1.11.2], we get (24). Proceeding similarly for $\mathcal{L}^{-1}$, from (27) we obtain

$$\|\mathcal{L}^{-1}\phi\|_{(A_0, A_1)^*_{1-\theta, p'}} \leqslant 5^{1/2} \|\phi\|_{(A_0, A_1)_{\theta, p}},$$

for any $\phi \in V^*$, that gives (25). ■

Thanks to (13), $\mathcal{L}_{\rm sym}$ is an isomorphism from V into $V^* \equiv \mathcal{L}_{\rm sym}(V)$; henceforth, we also assume that $\mathcal{L}_{\rm skew}$ is injective. Then we introduce the two Hilbert spaces C_0 and C_1:

$$
(28)\qquad \begin{aligned} C_0 &:= \mathcal{L}_{\text{skew}}(V), &&\text{with } \|\phi\|_{C_0} := \|\mathcal{L}_{\text{skew}}^{-1}\phi\|_V \\ C_1 &:= \mathcal{L}_{\text{sym}}(V), &&\text{with } \|\phi\|_{C_1} := \|\mathcal{L}_{\text{sym}}^{-1}\phi\|_V = \|\phi\|_{V^*}. \end{aligned}
$$

In the next lemma we analyze the structure of $\|\cdot\|_{(A_0, A_1)_{\theta, p}}$.

LEMMA 3. – *Given* θ, p *and* p' *such that* $0 < \theta < 1$, $1 \leqslant p \leqslant +\infty$, *and* $1/p + 1/p' = 1$, *under the hypotheses above, we have*

$$
(29)\quad 1/10\|w\|^2_{(A_0, A_1)_{\theta, p}} \leqslant \|w\|^2_V + \|\mathcal{L}_{\text{skew}} w\|^2_{(C_0, C_1)_{\theta, p}} \leqslant 2\|w\|^2_{(A_0, A_1)_{\theta, p}}, \qquad \forall w \in V.
$$

PROOF. – Since $\|w\|_V \leqslant \|w\|_{A_i}$ with $i = 0, 1$, then $\|w\|_V \leqslant \|w\|_{(A_0, A_1)_{\theta, p}}$ follows by a straightforward application of the interpolation theorem (e.g., [15, §1.3.3]). We also have

$$
\|\mathcal{L}_{\text{skew}} w\|_{C_0} \leqslant \|w\|_{A_0}, \qquad \|\mathcal{L}_{\text{skew}} w\|_{C_1} \leqslant \|w\|_{A_1},
$$

which gives $\|\mathcal{L}_{\text{skew}} w\|_{(C_0, C_1)_{\theta, p}} \leqslant \mathcal{C}\|w\|_{(A_0, A_1)_{\theta, p}}$, whence $\|w\|^2_V + \|\mathcal{L}_{\text{skew}} w\|^2_{(C_0, C_1)_{\theta, p}} \leqslant 2\|w\|^2_{(A_0, A_1)_{\theta, p}}$.

In order to complete the proof, we directly deal with the definition of interpolated norm (23). For any $t > 0$ consider the two splitting

$$
(30)\qquad \begin{aligned} w &= \widetilde{w}_0(t) + \widetilde{w}_1(t), \quad \text{with } \widetilde{w}_i(t) \in V,\ i = 1, 2, \\ w &= \widehat{w}_0(t) + \widehat{w}_1(t), \quad \text{with } \widehat{w}_i(t) \in V,\ i = 1, 2; \end{aligned}
$$

then define $w_0(t) \in V$ and $w_1(t) \in V$ such that $\mathcal{L} w_i(t) = \mathcal{L}_{\text{sym}} \widetilde{w}_i(t) + \mathcal{L}_{\text{skew}} \widehat{w}_i(t)$, i.e.,

$$
(31)\qquad a(w_i(t), v) = a_{\text{sym}}(\widetilde{w}_i(t), v) + a_{\text{skew}}(\widehat{w}_i(t), v), \qquad \forall v \in V,\ i = 0, 1,
$$

whence $w = w_0(t) + w_1(t)$, $\forall t > 0$.

Thanks to (16) and to the properties of $a_{\text{sym}}(\cdot, \cdot)$ and $a_{\text{skew}}(\cdot, \cdot)$ we have

$$
(32)\qquad \begin{aligned} \|w_0(t)\|_{A_0} &\leqslant 5^{1/2} \sup_{v \in V} \frac{a(w_0(t), v)}{\|v\|_{A_1}} \\ &\leqslant 5^{1/2}\left(\sup_{v \in V} \frac{a_{\text{sym}}(\widetilde{w}_0(t), v) - a_{\text{skew}}(v, \widehat{w}_0(t))}{\|v\|_{A_1}}\right) \\ &\leqslant 5^{1/2}\left(\sup_{v \in V} \frac{a_{\text{sym}}(\widetilde{w}_0(t), v)}{\|v\|_V} + \sup_{v \in V} \frac{a_{\text{skew}}(v, \widehat{w}_0(t))}{\|\mathcal{L}_{\text{skew}} v\|_{V^*}}\right) \\ &\leqslant 5^{1/2}\big(\|\widetilde{w}_0(t)\|_V + \|\widehat{w}_0(t)\|_V\big). \end{aligned}
$$

In a similar way, we have

$$\|w_1(t)\|_{A_1} \leqslant 5^{1/2} \sup_{v\in V} \frac{a(w_1(t),\, v)}{\|v\|_{A_0}} \tag{33}$$

$$\leqslant 5^{1/2}\left(\sup_{v\in V} \frac{a_{\mathrm{sym}}(\widetilde{w}_1(t),\, v) + a_{\mathrm{skew}}(\widehat{w}_1(t),\, v)}{\|v\|_{A_0}}\right)$$

$$\leqslant 5^{1/2}\left(\sup_{v\in V} \frac{a_{\mathrm{sym}}(\widetilde{w}_1(t),\, v)}{\|v\|_V} + \sup_{v\in V} \frac{a_{\mathrm{skew}}(\widehat{w}_1(t),\, v)}{\|v\|_V}\right)$$

$$\leqslant 5^{1/2}\big(\|\widetilde{w}_1(t)\|_V + \|\mathcal{L}_{\mathrm{skew}}\widehat{w}_1(t)\|_{V^*}\big).$$

From (23), by the triangle inequality and using (32)-(33), we have

$$\|w\|_{(A_0,\, A_1)_{\theta,\, p}} \leqslant \left[\int_0^{+\infty} (t^{-\theta}\|w_0(t)\|_{A_0} + t^{1-\theta}\|w_1(t)\|_{A_1})^p \frac{dt}{t}\right]^{1/p}$$

$$\leqslant 5^{1/2}\left[\int_0^{+\infty} (t^{-\theta}\|\widetilde{w}_0(t)\|_V + t^{-\theta}\|\widehat{w}_0(t)\|_V + t^{1-\theta}\|\widetilde{w}_1(t)\| + \right.$$

$$\left. t^{1-\theta}\|\mathcal{L}_{\mathrm{skew}}\widehat{w}_1(t)\|_{V^*})^p \frac{dt}{t}\right]^{1/p}$$

$$\leqslant 5^{1/2}\left[\int_0^{+\infty} \big(t^{-\theta}\|\widetilde{w}_0(t)\|_V + t^{1-\theta}\|\widetilde{w}_1(t)\|_V\big)^p \frac{dt}{t}\right]^{1/p}$$

$$\left[\int_0^{+\infty} (t^{-\theta}\|\mathcal{L}_{\mathrm{skew}}\widehat{w}_0(t)\|_{C_0} + t^{1-\theta}\|\mathcal{L}_{\mathrm{skew}}\widehat{w}_1(t)\|_{C_1})^p \frac{dt}{t}\right]^{1/p};$$

finally, taking the infimum over all $\widetilde{w}_0 \in V$, $\widetilde{w}_1 = w - \widetilde{w}_0 \in V$, $\widehat{w}_0 \in V$ and $\widehat{w}_1 = w - \widehat{w}_0 \in V$, and using [15, 1.3.3.(f)], we finally get

$$\|w\|_{(A_0,\, A_1)_{\theta,\, p}} \leqslant 5^{1/2}\big(\|w\|_V + \|\mathcal{L}_{\mathrm{skew}} w\|_{(C_0,\, C_1)_{\theta,\, p}}\big),$$

completing the proof of (29). ■

When $p = p' = 2$ and $\theta = 1 - \theta = 1/2$, Lemma 2 gives the continuity and inf-sup conditions for $\mathcal{L}$, as stated in the introduction, where $|||\cdot||| = \|\cdot\|_{(A_0,\, A_1)_{1/2,\, 2}}$; in particular, under the hypotheses of Lemma 3, we have the following obvious corollary.

COROLLARY 1. – *Under the assumption of Lemma* 3 *and setting*

$$|||\cdot||| := \left(\|\cdot\|_V^2 + \|\mathcal{L}_{\mathrm{skew}}\cdot\|^2_{(C_0,\, C_1)_{1/2,\, 2}}\right)^{1/2}, \tag{34}$$

we have the continuity and inf-sup conditions (8)-(9) *for* $\mathcal{L}$, *with constants* $\mathcal{C}_c$ *and* $\mathcal{C}_{is}$ *independent of* $\mathcal{L}$.

Actually Lemma 2 establishes a family of continuity and inf-sup conditions for $\mathcal{L}$ (for different values of θ and p) with different norms on the trial space (i.e., $\|\cdot\|_{(A_0,\, A_1)_{\theta,\, p}}$) and on the test space (i.e., $\|\cdot\|_{(A_0,\, A_1)_{1-\theta,\, p'}}$); on the other hand from the numerical standpoint (8)-(9) are mainly interesting, as discussed in [12, §4].

3. – The convection-diffusion-reaction operator.

We now apply the results of the previous section to the convection-diffusion-reaction operator. In Lemma 1-3 we have explicitly computed the constants involved into the estimates, in order to emphasize that the estimates do not depend on $\mathcal{L}$; henceforth, for the sake of simplicity, we will use generic constants denoted by $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$, which are independent on the operator coefficients κ, β and ϱ and on the domain Ω.

3.1. – *The one-dimensional case.*

We start with the analysis of the very simple one-dimensional operator, with constant coefficients $\kappa > 0$ and $\varrho \geq 0$, and unitary velocity. Then, for this subsection only, we will consider a special case of (7), which is

$$w \mapsto \mathcal{L}w := -\kappa w'' + w' + \varrho w\,, \tag{35}$$

where the argument w is a function on the interval $\Omega = [0, 1]$.

We consider first, and with particular emphasis, the ordinary differential equation with homogeneous Dirichlet boundary conditions. The variational formulation (11) reads

$$\text{find } u \in V \text{ such that } a(u, v) = \int_0^1 fv, \ \forall v \in V\,,$$

where

$$\begin{gathered} V = H_0^1(0, 1) \text{ with } \|\cdot\|_V^2 = \kappa|\cdot|^2_{H^1} + \varrho\|\cdot\|^2_{L^2}, \\ a(w, v) = \kappa\int_0^1 w'v' + \int_0^1 w'v + \varrho\int_0^1 wv. \end{gathered} \tag{36}$$

Then $\mathcal{L}_{\text{sym}} w = -\kappa w'' + \varrho w$, $\mathcal{L}_{\text{skew}} w = w'$, $a_{\text{sym}}(w, v) = \kappa \int_0^1 w' v' + \varrho \int_0^1 wv$ and $a_{\text{skew}}(w, v) = \int_0^1 w' v$. Finally $C_0 = L_0^2(0, 1)$ and $C_1 = H^{-1}(0, 1)$ from the algebraic standpoint, where L_0^2 is the subspace of L^2 of zero mean value functions, and its natural norm is $\|\cdot\|_{L_0^2} := \|\cdot\|_{L^2}$, while H^{-1} is the dual of H_0^1, endowed with the dual norm $\|\cdot\|_{H^{-1}} = \sup_{v \in H_0^1(0,1)} \langle \cdot, v \rangle / |v|_{H^1}$ (we recall that $|w|_{H^1} := \left(\int_0^1 (w')^2 \right)^{1/2}$ is a norm on H_0^1). It is easy to see that L_0^2 is a dense subspace of H^{-1}. From Corollary 1 we immediately have the following result.

THEOREM 1. – *For the case* (35)-(36), *uniform continuity and inf-sup conditions* (8)-(9) *hold true with respect to the norm*

$$w \mapsto |||w||| = \left(\kappa |w|_{H^1}^2 + \|w'\|_{(C_0, C_1)_{1/2, 2}}^2 + \varrho \|w\|_{L^2}^2 \right)^{1/2}. \tag{37}$$

Now we focus our attention on $|||\cdot|||$ in (37), in order to better understand its structure. Roughly speaking, the term $\|w'\|_{(C_0, C_1)_{1/2, 2}}$ is related to the skew-symmetric part of $\mathcal{L}$, which is the *first* order derivative. Then we expect $w \mapsto \|w'\|_{(C_0, C_1)_{1/2, 2}}$ to act as a 1/2-order norm uniformly on the operator coefficients κ and ϱ. That is in fact stated in the next theorem: we show that $\|w'\|_{(C_0, C_1)_{1/2, 2}}$ stays between the $H^{1/2}$-seminorm and $H_{00}^{1/2}$-norm, where $H^{1/2} := (L^2, H^1)_{1/2, 2}$ and $H_{00}^{1/2} := (L^2, H_0^1)_{1/2, 2}$ are the two usual Hilbert spaces of order 1/2, endowed with the usual norms given by interpolation (see [11]), and $|w|_{H^{1/2}}$ is the seminorm $\|w - \Pi_0 w\|_{H^{1/2}}$, $\Pi_0 \cdot$ denoting the mean value of its argument.

THEOREM 2. – *For the case* (35)-(36), *we have*

$$\mathcal{C}_1 |w|_{H^{1/2}} \leqslant \|w'\|_{(C_0, C_1)_{1/2, 2}} \leqslant \mathcal{C}_2 \|w\|_{H_{00}^{1/2}}, \quad \forall w \in V. \tag{38}$$

PROOF. – When $\varrho = 0$, (38) follows from (52); we assume henceforth $\varrho > 0$. We consider first the left inequality in (38), i.e.

$$\mathcal{C} |w|_{H^{1/2}} \leqslant \|w'\|_{(C_0, C_1)_{1/2, 2}}, \quad \forall w \in V. \tag{39}$$

It is easy to see that $\|z'\|_{L^2} \simeq \|z\|_{H^1}$ and $\|z'\|_{H^{-1}} \simeq \|z\|_{L^2}$, for any $z \in H^1 \cap L_0^2$; then, thanks to Theorem [15, §1.3.3], [15, §1.11.2] and [15, §1.17.1], the first order derivative is a topological isomorphism from $H^{1/2} \cap L_0^2$ into $(H^{-1}, L^2)_{1/2, 2}$, which means

$$|w|_{H^{1/2}} = \|w - \Pi_0 w\|_{H^{1/2}} \simeq \|w'\|_{(H^{-1}, L^2)_{1/2, 2}}. \tag{40}$$

We introduce now the new space $\widetilde{C}_0$: from the algebraic standpoint we set

$\widetilde{C}_0 := L^2$, and we define $\|\cdot\|_{\widetilde{C}_0} := (\kappa\|\cdot\|^2_{L^2} + \varrho\|\cdot\|^2_{H^{-1}})^{1/2}$. Our next step is to show that

$$\|\phi\|_{(H^{-1}, L^2)_{1/2,2}} \leqslant \mathcal{C}\|\phi\|_{(\widetilde{C}_0, C_1)_{1/2,2}}, \quad \forall \phi \in L^2. \tag{41}$$

For that purpose we split a generic $\phi \in L^2$ into

$$\phi = \phi_{\text{high}} + \phi_{\text{low}}, \tag{42}$$

where $\phi_{\text{high}}, \phi_{\text{low}} \in L^2$ are, roughly speaking, the high frequency part and the low frequency part of ϕ, respectively, in such a way that

$$\kappa^{1/2}\|\phi_{\text{high}}\|_{L^2} + \varrho^{1/2}\|\phi_{\text{low}}\|_{H^{-1}} \leqslant \mathcal{C}\|\phi\|_{\widetilde{C}_0} \tag{43}$$

$$\kappa^{-1/2}\|\phi_{\text{high}}\|_{H^{-1}} + \varrho^{-1/2}\|\phi_{\text{low}}\|_{L^2} \leqslant \mathcal{C}\|\phi\|_{C_1}. \tag{44}$$

For that purpose, we introduce an auxiliary problem: let $\psi \in H^1_0$ the solution of

$$\mathcal{L}_{\text{sym}}\psi = \phi \quad \text{in } (0, 1) \tag{45}$$

and let $\phi_{\text{high}} := -\kappa\psi''$ and $\phi_{\text{low}} := \varrho\psi$.

Multiplying both members of the differential equation (45) by $-\psi''$, integrating over $(0, 1)$ and integrating by parts we get

$$\kappa\|\psi''\|^2_{L^2} + \varrho\|\psi'\|^2_{L^2} = -\int_0^1 \phi\psi'';$$

then, thanks to the Cauchy-Schwartz inequality, we have

$$\|\phi_{\text{high}}\|_{L^2} = \|\kappa\psi''\|_{L^2} \leqslant \|\phi\|_{L^2}. \tag{46}$$

Integrating (45) we have

$$-\kappa\psi' + \kappa\psi'(0) + \varrho\Psi = \Phi,$$

where $\Psi(x) = \int_0^x \psi(t)\,dt$ and analogously $\Phi(x) = \int_0^x \phi(t)\,dt$; after multiplying by $\Psi - \Pi_0\Psi$ both members, integrating over $(0, 1)$ and integrating by parts we obtain

$$\kappa\|\psi\|^2_{L^2} + \varrho\|\Psi - \Pi_0\Psi\|^2_{L^2} = \int_0^1 \Phi(\Psi - \Pi_0\Psi),$$

whence now

$$\|\phi_{\text{low}}\|_{H^{-1}} = \varrho\|\Psi - \Pi_0\Psi\|_{L^2} \leqslant \|\Phi - \Pi_0\Phi\|_{L^2} = \|\phi\|_{H^{-1}}. \tag{47}$$

Collecting (46)-(47) we obtain (43). From (45) it is also easy to obtain the estimate $(\kappa\|\psi'\|_{L^2}^2+\varrho\|\psi\|_{L^2}^2)^{1/2}\leqslant\|\phi\|_{V^*}=\|\phi\|_{C_1}$, which gives (44) straightforwardly.

Consider now the linear operator $\phi\mapsto(\phi_{\text{high}},\phi_{\text{low}})$ from L^2 into $L^2\times L^2$, with $\phi_{\text{high}},\phi_{\text{low}}$ as defined above: by interpolation from the two continuity estimates (43)-(44) we get

$$\|\phi_{\text{high}}\|_{(L^2,H^{-1})_{1/2,2}}+\|\phi_{\text{low}}\|_{(H^{-1},L^2)_{1/2,2}}\leqslant\mathcal{C}\|\phi\|_{(\tilde{C}_0,C_1)_{1/2,2}},\tag{48}$$

whence, by using the triangle inequality and since $\|\cdot\|_{(L^2,H^{-1})_{1/2,2}}=\|\cdot\|_{(H^{-1},L^2)_{1/2,2}}$, we obtain (41). Finally (40) and (41) gives (39).

Now we consider the right equivalence in (38), which is

$$\|w'\|_{(C_0,C_1)_{1/2,2}}\leqslant\mathcal{C}\|w\|_{H_{00}^{1/2}},\qquad\forall w\in V.\tag{49}$$

Given $w\in H_0^1$ it is easy to see that

$$\|w'\|_{C_0}=\|w\|_V=\|w\|_{C_1^*},$$

and

$$\|w'\|_{C_1}=\|w'\|_{V^*}\leqslant\|w\|_{\tilde{C}_0^*},$$

whence (thanks to Theorem [15, §1.11.2])

$$\|w'\|_{(C_0,C_1)_{1/2,2}}\leqslant\|w\|_{(C_1^*,\tilde{C}_0^*)_{1/2,2}}=\|w\|_{(\tilde{C}_0,C_1)^*_{1/2,2}}.\tag{50}$$

Moreover, passing to the duals in (41), still using Theorem [15, §1.11.2], we also have

$$\|w\|_{(\tilde{C}_0,C_1)^*_{1/2,2}}\leqslant\|w\|_{(H^{-1},L^2)^*_{1/2,2}}=\|w\|_{(H^1,L^2)_{1/2,2}}=\|w\|_{(H_{00}^{1/2})}.\tag{51}$$

Inequalities (50)-(51) give (49). ■

REMARK 1. – *It is worth noting that Theorem* 1-2 *allow for* $\varrho=0$ *as well; in that case we have* $\|w'\|_{(C_0,C_1)_{1/2,2}}=\|w'\|_{(H^{-1},L_0^2)_{1/2,2}}$, *since the coefficient* κ *easily cancel when interpolating. Let* $H_\#^1$ *be the subspace of* H^1 *of functions* w *such that* $w(0)=w(1)$, *endowed with the* $\|\cdot\|_{H_\#^1}:=\|\cdot\|_{H^1}$, *and* $H_\#^{1/2}:=(L^2,H_\#^1)_{1/2,2}$ *endowed with the norm given by interpolation. Given* $z\in H_\#^1\cap L_0^2$, *one has* $\|z'\|_{L_0^2}\simeq\|z\|_{H_\#^1}$ *and* $\|z'\|_{H^{-1}}\simeq\|z\|_{L^2}$, *whence (by using Theorem [15, §1.3.3], [15, §1.11.2] and [15, §1.17.1])* $\|z'\|_{(H^{-1},L_0^2)_{1/2,2}}\simeq\|z\|_{(L^2,H_\#^1)_{1/2,2}}$ *and therefore* $\|w'\|_{(H^{-1},L_0^2)_{1/2,2}}\simeq\|w-\Pi_0w\|_{(L^2,H_\#^1)_{1/2,2}}$, *for any* $w\in H_0^1$; *this means that we have the following characterization*:

$$\varrho=0\Rightarrow|w|_{H_\#^{1/2}}:=\|w-\Pi_0w\|_{(L^2,H_\#^1)_{1/2,2}}=\|w'\|_{(C_0,C_1)_{1/2,2}},\qquad\forall w\in V.\tag{52}$$

We may also deal with different kind of boundary conditions; consider the example

$$\begin{cases} \mathcal{L}u = f & \text{in } (0,1) \\ u(0) = u'(1) = 0, \end{cases} \tag{53}$$

where $\mathcal{L}$ is still formally given by (35). The variational formulation (11) now requires

$$V = \{v \in H^1(0,1) \text{ such that } v(0) = 0\}$$

$$a(w, v) = \kappa \int_0^1 w'v' + \int_0^1 w'v + \varrho \int_0^1 wv ;$$

the key point is that the bilinear form $a(\cdot, \cdot)$ is coercive on V; accordingly, we define $\|\cdot\|_V$ as

$$\|w\|_V^2 := a(w, w) = \kappa |w|_{H^1}^2 + \varrho \|w\|_{L^2}^2 + \frac{1}{2} w(1)^2,$$

and we have now

$$a_{\text{sym}}(w, v) = \kappa \int_0^1 w'v' + \varrho \int_0^1 wv + \frac{1}{2} w(1)\, v(1),$$

$$a_{\text{skew}}(w, v) = \int_0^1 w'v - \frac{1}{2} w(1)\, v(1).$$

Then we can still make use of the theory of §2 and obtain uniform inf-sup and continuity conditions from Corollary 1.

When the bilinear form $a(\cdot, \cdot)$ is not coercive, then we can not use the results of §2. This is the case of

$$\begin{cases} -\kappa u'' + u' = f & \text{in } (0,1) \\ u'(0) = u(1) = 0\,, \end{cases} \tag{54}$$

i.e., when $\varrho = 0$ and we prescribe Neumann boundary condition at the *inflow* $x = 0$; then $V = \{v \in H^1(0,1) \text{ such that } v(1) = 0\}$ and

$$a(w, w) = \kappa |w|_{H^1}^2 + \varrho \|w\|_{L^2}^2 - \frac{1}{2} w(1)^2,$$

which is not positive in general, when κ and ϱ are small enough. However, when $f = 1$ the solution of (54) is $u(x) = \kappa(\exp(1/\kappa) - \exp(x/\kappa)) + x - 1$; for

$\kappa \to 0$ we have $\|u\|_{L^2} \approx \kappa \exp(1/\kappa)$, whence we see that (54) is in fact not uniformly well posed with respect to κ.

3.2. – *The multi-dimensional case.*

In this section, we analyze the multi-dimensional convection-diffusion-reaction operator (7) with Dirichlet homogeneous boundary conditions, and the associated bilinear form

$$a(w, v) = \kappa \int_\Omega \nabla w \cdot \nabla v + \int_\Omega \beta \cdot \nabla w \, v + \int_\Omega \varrho w v ,$$

which is defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ (see, e.g., [11]). Under the assumption

$$\varrho - \frac{1}{2} \operatorname{div}(\beta) \geqslant 0 \tag{55}$$

the bilinear form $a(\cdot, \cdot)$ is coercive, whence we set

$$\begin{gathered} V = H_0^1(\Omega) \\ \|w\|_V^2 = a(w, w) = \kappa |w|_{H^1}^2 + \left(\varrho - \frac{1}{2} \operatorname{div}(\beta)\right) \|w\|_{L^2}^2 . \end{gathered} \tag{56}$$

The decomposition (6) gives

$$\begin{cases} a_{\text{sym}}(w, v) = \kappa \displaystyle\int_\Omega \nabla w \cdot \nabla v + \int_\Omega \left(\varrho - \frac{1}{2} \operatorname{div}(\beta)\right) w v, \\ a_{\text{skew}}(w, v) = \displaystyle\int_\Omega \beta \cdot \nabla w \, v + \frac{1}{2} \int_\Omega \operatorname{div}(\beta) \, w v; \end{cases} \tag{57}$$

For the sake of simplicity, we shall consider henceforth the case

$$\operatorname{div}(\beta) = 0 . \tag{58}$$

In order to apply Corollary 1 to this case, we need $\mathcal{L}_{\text{skew}} = \beta \cdot \nabla$ to be injective on V: this is assured, for example, by the assumption

(59) there exists a smooth $\phi : \Omega \to \mathbb{R}$ such that $\nabla \phi \cdot \beta \geqslant \mathcal{C} > 0$;

we refer to [10] for further details. Definition (28) says that, from the algebraic standpoint, C_0 is the space of the streamline derivatives $\beta \cdot \nabla w$ of functions $w \in H_0^1$, while C_1 is H^{-1}. Corollary 1 gives then the following result

THEOREM 3. – *For the case* (56), (58)-(59), *the uniform continuity and inf-sup conditions* (8)-(9) *hold true with respect to the norm*

$$w \mapsto |||w||| = \left(\kappa |w|^2_{H^1} + \|\beta \cdot \nabla w\|^2_{(C_0,\, C_1)_{1/2,\,2}} + \varrho \|w\|^2_{L^2}\right)^{1/2}. \tag{60}$$

Roughly speaking, we expect $\|\beta \cdot \nabla w\|_{(C_0,\, C_1)_{1/2,\,2}}$ to be of order 1/2 in the direction of β, and of order 0 in the directions orthogonal to β (this can be more easily seen for the case $\varrho = 0$), but a rigorous analysis of the structure of $\|\beta \cdot \nabla w\|_{(C_0,\, C_1)_{1/2,\,2}}$ is more difficult now than for the simpler one-dimensional case considered in §3.1. The next result shows that $\|\beta \cdot \nabla w\|_{(C_0,\, C_1)_{1/2,\,1}}$ has some uniform bounds independent of κ and ϱ (though the anisotropy is not investigated). Then we end by a comparison between $\|\beta \cdot \nabla w\|_{(C_0,\, C_1)_{1/2,\,1}}$ and $\|\beta \cdot \nabla w\|_{(C_0,\, C_1)_{1/2,\,2}}$.

PROPOSITION 1. – *For the case* (56), (58)-(59), *we have:*

$$\mathcal{C}_p \|\beta\|^{1/2}_{L^\infty} \|w\|_{L^2} \leq \|\beta \cdot \nabla w\|_{(C_0,\, C_1)_{1/2,\,1}} \leq \mathcal{C} \|\beta\|^{1/2}_{L^\infty} \|w\|_{(L^2,\, H^1_0)_{1/2,\,1}}, \qquad \forall w \in V, \tag{61}$$

where the constant $\mathcal{C}_p$ of the Poincaré-like inequality depends on $\beta/\|\beta\|_{L^\infty}$ and Ω.

PROOF. – Let η be the solution of $\beta \cdot \nabla \eta = \|\beta\|_{L^\infty}$ with $\eta = 0$ on the inflow boundary $\partial\Omega^- := \{\boldsymbol{x} \in \partial\Omega \,|\, \beta(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) < 0\}$, $\boldsymbol{n}$ denoting the outward normal unit vector defined on $\partial\Omega$; the existence of η is guaranteed by (59). Given $w \in H^1_0$, integrating by parts, using the Cauchy-Schwartz inequality and (58) we have

$$\begin{aligned} \|\beta\|_{L^\infty} \|w\|^2_{L^2} &= \int_\Omega \beta \cdot \nabla \eta\, w^2 \\ &= -2 \int_\Omega \eta w\, \beta \cdot \nabla w \\ &\leq 2\, |\eta w|_V \|\beta \cdot \nabla w\|_{V^*}. \end{aligned} \tag{62}$$

We have

$$\|\eta w\|_{L^2} \leq \|\eta\|_{L^\infty} \|w\|_{L^2}, \tag{63}$$

and, using the classical Poincaré inequality, it is easy to get

$$|\eta w|_{H^1} \leq \mathcal{C}(\|\eta\|_{L^\infty} + \|\nabla \eta\|_{(L^\infty)^2})\, |w|_{H^1}. \tag{64}$$

Moreover, thanks to (59), we have $\widetilde{\mathcal{C}}_p := \|\eta\|_{L^\infty} + \|\nabla \eta\|_{(L^\infty)^2} < +\infty$ (e.g., see [10, Theorem 3.2]), where $\widetilde{\mathcal{C}}_p$ depends on η, i.e. on $\beta/\|\beta\|_{L^\infty}$ and Ω. Then

$$\|\eta w\|_V \leq \mathcal{C}\, \widetilde{\mathcal{C}}_p \|w\|_V; \tag{65}$$

substituting back in (62),

$$\|\beta\|_{L^\infty}\|w\|_{L^2}^2 \leqslant \mathcal{C}\widetilde{\mathcal{C}}_p\|w\|_V\|\beta\cdot\nabla w\|_{V^*} = \mathcal{C}\widetilde{\mathcal{C}}_p\|\beta\cdot\nabla w\|_{C_0}\|\beta\cdot\nabla w\|_{C_1},$$

and thanks to Theorem [15, §1.10.1] we obtain

$$\mathcal{C}_p\|\beta\|_{L^\infty}^{1/2}\|w\|_{L^2} \leqslant \|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,1}}, \qquad \forall w \in V,$$

which is the left inequality of (61).

We have, thanks to Theorem [15, §1.3.3]

$$\|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,1}}^2 \leqslant \|\beta\cdot\nabla w\|_{C_0}\|\beta\cdot\nabla w\|_{C_1} \leqslant \kappa^{1/2}\,|w|_{H^1}\|\beta\cdot\nabla w\|_{V^*} + \varrho^{1/2}\|w\|_{L^2}\|\beta\cdot\nabla w\|_{V^*}, \tag{66}$$

and

$$\begin{aligned}\|\beta\cdot\nabla w\|_{V^*} &\leqslant \kappa^{-1/2}\|\beta\cdot\nabla w\|_{H^{-1}} \leqslant \kappa^{-1/2}\|\beta\|_{L^\infty}\|w\|_{L^2},\\ \|\beta\cdot\nabla w\|_{V^*} &\leqslant \varrho^{-1/2}\|\beta\cdot\nabla w\|_{L^2} \leqslant \varrho^{-1/2}\|\beta\|_{L^\infty}\,|w|_{H^1};\end{aligned} \tag{67}$$

from (66)-(67), we get

$$\|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,1}}^2 \leqslant 2\|\beta\|_{L^\infty}\,|w|_{H^1}\|w\|_{L^2},$$

and Theorem [15, §1.10.1] yields

$$\|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,1}} \leqslant \mathcal{C}\|\beta\|_{L^\infty}^{1/2}\|w\|_{(L^2,\, H_0^1)_{1/2,\,1}}, \qquad \forall w \in V,$$

and concludes the proof of (61). ■

In the previous proposition, we have shown uniform bounds (with respect to the operator coefficients) for $\|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,1}}$; as a general result of the interpolation theory (see, e.g., [15, 1.3.3.d]), we have

$$\|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,2}} \leqslant \mathcal{C}\|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,1}}, \qquad \forall w \in V, \tag{68}$$

and similarly

$$\|w\|_{(A_0,\, A_1)_{1/2,\,2}} \leqslant \mathcal{C}\|w\|_{(A_0,\, A_1)_{1/2,\,1}}, \qquad \forall w \in V; \tag{69}$$

the converse inequality of (68), that is $\|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,1}} \leqslant \mathcal{C}\|\beta\cdot\nabla w\|_{(C_0,\, C_1)_{1/2,\,2}}$, does not hold true; on the other hand the converse of (69) holds true, and it is, roughly speaking, *almost* uniform, in the sense that the constant in it only depends on a logarithm of the coefficients, as stated in the next proposition.

PROPOSITION 2. – *Consider the case* (56), (58) *and* (59); *let*

(70) $$\alpha := \max\{\kappa^{1/2}\varrho^{1/2},\ \kappa \operatorname{diam}(\Omega)\}/\|\beta\|_{L^\infty}.$$

When $\alpha \leqslant 1$ *we have*

(71) $$\|w\|_{(A_0, A_1)_{1/2,1}} \leqslant \mathcal{C}\big(1-\log^{1/2}(\alpha)\big)\,\|w\|_{(A_0, A_1)_{1/2,2}}, \qquad \forall w \in V,$$

while for $\alpha > 1$ *we have*

(72) $$\|w\|_{(A_0, A_1)_{1/2,1}} \leqslant \mathcal{C}\|w\|_{(A_0, A_1)_{1/2,2}}, \qquad \forall w \in V,$$

PROOF. – We only consider here the case $\alpha \leqslant 1$, since when $\alpha > 1$ we can set $\alpha := 1$ instead of (70) and follow the proof. First, recall that from the definition (14) we have

(73) $$\|w\|_{A_0} \leqslant \|w\|_{A_1}, \qquad \forall w \in V,$$

and, since (67) and the Poincaré inequality, we also have

(74) $$\alpha\|w\|_{A_1} \leqslant \mathcal{C}\|w\|_{A_0}, \qquad \forall w \in V,$$

By the definition (23) and by the triangle inequality we get

$$\begin{aligned}
\|w\|_{(A_0, A_1)_{1/2,1}} &\leqslant \int_0^{+\infty} (t^{-1/2}\|w_0(t)\|_{A_0} + t^{1/2}\|w_1(t)\|_{A_1})\,\frac{dt}{t}\\
&\leqslant \int_0^{\alpha^2} (t^{-1/2}\|w_0(t)\|_{A_0} + t^{1/2}\|w_1(t)\|_{A_1})\,\frac{dt}{t}\\
&\quad + \int_{\alpha^2}^{1} (t^{-1/2}\|w_0(t)\|_{A_0} + t^{1/2}\|w_1(t)\|_{A_1})\,\frac{dt}{t}\\
&\quad + \int_1^{+\infty} (t^{-1/2}\|w_0(t)\|_{A_0} + t^{1/2}\|w_1(t)\|_{A_1})\,\frac{dt}{t}\\
&= I + II + III,
\end{aligned}$$

for any $w_0(t)$ and $w_1(t)$ with $w = w_0(t) + w_1(t)$, $w_i(t) \in V$, $i = 1, 2$ and $0 < t < +\infty$. Taking $w_0(t) = w$ and $w_1(t) = 0$ for $t \geqslant 1$ we have

$$\begin{aligned}
III &\leqslant \|w\|_{A_0}\int_1^{\infty} t^{-3/2}\,dt\\
&\leqslant 2\|w\|_{A_0}.
\end{aligned}$$

In a very similar way, we deal with the first term, taking $w_1(t) = w$ and $w_0(t) = 0$ for $0 < t < \alpha^2$; thanks to (74) we obtain:

$$\begin{aligned} I &\leqslant \|w\|_{A_1} \int_0^{\alpha^2} t^{-1/2}\, dt \\ &\leqslant 2\alpha \|w\|_{A_1} \\ &\leqslant \mathcal{C} \|w\|_{A_0}. \end{aligned}$$

Thanks to the Cauchy-Schwartz inequality we have

$$\int_{\alpha^2}^{1} (t^{-1/2} \|w_0(t)\|_{A_0} + t^{1/2} \|w_1(t)\|_{A_1}) \frac{dt}{t} \leqslant \left[\int_{\alpha^2}^{1} \frac{dt}{t} \right]^{1/2} \cdot \tag{75}$$

$$\left[\int_{\alpha^2}^{1} (t^{-1/2} \|w_0(t)\|_{A_0} + t^{1/2} \|w_1(t)\|_{A_1})^2 \frac{dt}{t} \right]^{1/2} \leqslant [-2 \log(\alpha)]^{1/2} \cdot$$

$$\left[\int_{\alpha^2}^{1} (t^{-1/2} \|w_0(t)\|_{A_0} + t^{1/2} \|w_1(t)\|_{A_1})^2 \frac{dt}{t} \right]^{1/2}$$

that holds true for any choice of $w_0(t)$ and $w_1(t)$ on $\alpha^2 < t < 1$; taking the infimum on w_0, w_1 we obtain

$$II \leqslant [-2 \log(\alpha)]^{1/2} \|w\|_{(A_0, A_1)_{1/2, 2}}.$$

Finally, we have from (73)

$$\|w\|_{A_0} \leqslant \|w\|_{(A_0, A_1)_{1/2, 2}}$$

and (71) follows from the previous estimates on I, II and III. ■

From Proposition 1-2 we easily derive the next *almost* uniform bounds (still, up to a $\log(\alpha)^{1/2}$ factor, which is, roughly speaking, a *weak* loss of uniformity).

COROLLARY 2. – *For the case* (56), (58)-(59), *given* α *from* (70), *we have*:

$$\mathcal{C}_p \min\{1, |\log(\alpha)|^{-1/2}\} \|\beta\|_{L^\infty}^{1/2} \|w\|_{L^2} \leqslant |||w||| , \quad \forall w \in V, \tag{76}$$

$$\|\beta \cdot \nabla w\|_{(C_0, C_1)_{1/2, 2}} \leqslant \mathcal{C} \|\beta\|_{L^\infty}^{1/2} \|w\|_{(L^2, H_0^1)_{1/2, 1}}, \quad \forall w \in V. \tag{77}$$

where $\mathcal{C}_p$ *depends on* $\beta / \|\beta\|_{L^\infty}$ *and* Ω.

Though (76)-(77) are not sharp estimates as we got in §3.1 for the one dimensional case, they put in evidence the relationship between the norm $|||\cdot|||$ defined in (60), and the skew-symmetric part $\mathcal{L}_{\text{skew}} = \beta\cdot\nabla$ of (7). Recall that $\max\{\kappa^{1/2}, \varrho^{1/2}\}\|w\|_{L^2} \leqslant \mathcal{C}\|w\|_V \leqslant \mathcal{C}\,|||w|||$, while (76) states the bound on the L^2-norm which is mainly due to $\|\beta\cdot\nabla w\|_{(C_0,\,C_1)_{1/2,\,2}}$. Then (76) becomes relevant when κ and ϱ are small.

REFERENCES

[1] A. Barinka - T. Barsch - P. Charton - A. Cohen - S. Dahlke - W. Dahmen - K. Urban, *Adaptive wavelet schemes for elliptic problems-implementation and numerical experiments*, SIAM J. Sci. Comput., **23** (2001), 910-939 (electronic).

[2] S. Bertoluzza - C. Canuto - A. Tabacco, *Stable discretizations of convection-diffusion problems via computable negative-order inner products*, SIAM J. Numer. Anal., **38** (2000), 1034-1055 (electronic).

[3] S. Bertoluzza - M. Verani, *Convergence of a nonlinear wavelet algorithm for the solution of PDE*s, Appl. Math. Lett., **16** (2003), 113-118.

[4] J. H. Bramble -R. D. Lazarov - J. E. Pasciak, *Least-squares for second-order elliptic problems*, Comput. Methods Appl. Mech. Engrg., **152** (1998), 195-210. Symposium on Advances in Computational Mechanics, Vol. 5 (Austin, TX, 1997).

[5] F. Brezzi - M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, New York, 1991.

[6] A. Cohen - W. Dahmen - R. DeVore, *Adaptive wavelet methods for elliptic operator equations: convergence rates*, Math. Comp., **70** (2001), 27-75 (electronic).

[7] A. Cohen -W. Dahmen - R. DeVore, *Adaptive wavelet methods. II. Beyond the elliptic case*, Found. Comput. Math., **2** (2002), 203-245.

[8] W. Dahmen -A. Kunoth - R. Schneider, *Wavelet least squares methods for boundary value problems*, SIAM J. Numer. Anal., **39** (2002), 1985-2013 (electronic).

[9] W. Dörfler, *Uniform a priori estimates for singularly perturbed elliptic equations in multidimensions*, SIAM J. Numer. Anal., **36** (1999), 1878-1900 (electronic).

[10] H. Goering - A. Felgenhauer -G. Lube -H.-G. Roos - L. Tobiska, *Singularly perturbed differential equations*, vol. 13 of Reihe Math. Research, Akademie-Verlag, Berlin, 1983.

[11] J.-L. Lions - E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[12] G. Sangalli, *Analysis of the advection-diffusion operator using fractional order norms*, Tech. Rep. 1221, I.A.N.-C.N.R., 2001. Accepted on Numer. Math.

[13] G. Sangalli, *Quasi-optimality of the SUPG* method for the one-dimensional advection-diffusion problem, Tech. Rep. 1222, I.A.N.-C.N.R., 2001. Accepted on SIAM J. Numer. Anal.

[14] G. Sangalli, *A uniform analysis of non-symmetric and coercive linear operators*, Tech. Rep. 23-PV, I.M.A.T.I.-C.N.R., 2003. Submitted to SIAM J. Math. Anal.
[15] H. Triebel, *Interpolation theory, function spaces, differential operators*, Johann Ambrosius Barth, Heidelberg, second ed., 1995.
[16] R. Verfürth, *Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation*, Numer. Math., 78 (1998), 479-493.

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