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Maria De Falco, Carmela Musella

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Some Lattice Properties of Normal-by-Finite Subgroups.

MARIA DE FALCO - CARMELA MUSELLA

Sunto. – *Un sottogruppo H di un gruppo G è detto normale-per-finito se il nocciolo H_G di H in G ha indice finito in H. È stato provato da Buckley, Lennox, Neumann, Smith e Wiegold che se ogni sottogruppo di un gruppo G è normale-per-finito, allora G è abeliano-per-finito, supposto che le sue immagini omomorfe periodiche siano localmente finite. In questo articolo si descrive la struttura dei gruppi G per i quali l'insieme parzialmente ordinato $nf(G)$ dei sottogruppi normali-per-finito verifica alcune rilevanti proprietà.*

Summary. – *A subgroup H of a group G is said to be normal-by-finite if the core H_G of H in G has finite index in H. It has been proved by Buckley, Lennox, Neumann, Smith and Wiegold that if every subgroup of a group G is normal-by-finite, then G is abelian-by-finite, provided that all its periodic homomorphic images are locally finite. The aim of this article is to describe the structure of groups G for which the partially ordered set $nf(G)$ consisting of all normal-by-finite subgroups satisfies certain relevant properties.*

1. – Introduction.

A subgroup H of a group G is said to be *normal-by-finite* if the core H_G of H in G has finite index in H. It has been proved by Buckley, Lennox, Neumann, Smith and Wiegold ([1], [2]) that if every subgroup of a group G is normal-by-finite, then G is abelian-by-finite, provided that all its periodic homomorphic images are locally finite (in particular, this is the case if the group G is locally soluble). Clearly the intersection of finitely many normal-by-finite subgroups is likewise normal-by-finite, while the join of two normal-by-finite subgroups need not be normal-by-finite. In fact, every finite subgroup of an arbitrary group is obviously normal-by-finite, and any periodic simple group G which is not locally finite must contain two finite subgroups X and Y such that $\langle X, Y \rangle$ is infinite. On the other hand, if G is a locally finite group, the set $nf(G)$ of all normal-by-finite subgroups of G is a sublattice of the subgroup lattice $\mathfrak{L}(G)$. A similar property does not hold in general for soluble non-periodic groups, as the following example shows. Let $A = \langle a \rangle \times \langle b \rangle$ be a free abelian group of rank 2, and let x and y be the automorphisms of A defined by the properties $a^x = a^{-1}$, $b^x = b^{-1}$, $a^y = b$, $b^y = a$; in the semidirect product $\langle x, y\rangle \ltimes A$ the infinite core-free subgroup $\langle a, x\rangle$ is generated by two subgroups of order 2.

The aim of this article is to describe the structure of groups G for which the partially ordered set $nf(G)$ consisting of all normal-by-finite subgroups satisfies certain relevant properties. Since all subgroups of a finite group are normal-by-finite, we will restrict the attention to situations that have already been studied in the case of subgroup lattices of finite groups. Our results should also be seen in relation to the corresponding statements for the lattice of normal subgroups or for other relevant lattices of generalized normal subgroups that have recently been considered (see for instance [4], [5], [9]).

Most of our notation is standard and can be found in [10]. We will use the monograph [11] as a general reference for results on subgroup lattices.

2. – Decomposable ordered sets.

A partially ordered set is said to be *decomposable* if it is isomorphic to the direct product of two non-trivial partially ordered sets. It has been proved by Suzuki [12] that a group G has decomposable subgroup lattice if and only if G is periodic and $G = G_1 \times G_2$, where G_1 and G_2 are coprime non-trivial subgroups of G. Groups with decomposable lattice of normal subgroups have been characterized by Curzio [3], while Franciosi and de Giovanni [7] described groups for which the partially ordered set of subnormal subgroups is decomposable.

THEOREM 2.1. – *Let G be a group. The partially ordered set $nf(G)$ of all normal-by-finite subgroups of G is decomposable if and only if $G = G_1 \times G_2$, where G_1 and G_2 are non-trivial subgroups of G such that if N_1 and N_2 are normal subgroups of G_1 and G_2, respectively, the factor groups G_1/N_1 and G_2/N_2 have no elements of the same prime order.*

PROOF. – Suppose first that $nf(G)$ is decomposable, and let

$$\varphi : nf(G) \to \mathfrak{L}_1 \times \mathfrak{L}_2$$

be an order isomorphism, where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are non-trivial partially ordered sets. Clearly $\mathfrak{L}_i$ has a smallest element O_i and a largest element I_i ($i = 1, 2$), and the preimages $G_1 = \varphi^{-1}(I_1, O_2)$ and $G_2 = \varphi^{-1}(O_1, I_2)$ are non-trivial normal-by-finite subgroups of G such that $G_1 \cap G_2 = \{1\}$. Moreover G has no proper normal-by-finite subgroups containing both G_1 and G_2. If a_1 and a_2 are elements of $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively, we have

$$(a_1, a_2) = \sup\{(a_1, O_2), (O_1, a_2)\}$$

and

$$(a_1, O_2) = \inf\{(a_1, a_2), (I_1, O_2)\}, \qquad (O_1, a_2) = \inf\{(a_1, a_2), (O_1, I_2)\}.$$

It follows that, if X is any normal-by-finite subgroup of G, then X coincides with the smallest normal-by-finite subgroup of G containing $\langle X \cap G_1, X \cap G_2 \rangle$. Let g be any element of G_1. Then G_2^g is the smallest normal-by-finite subgroup of G, containing

$$\langle (G_2^g \cap G_1), (G_2^g \cap G_2) \rangle = G_2^g \cap G_2,$$

so that $G_2^g = G_2^g \cap G_2 \leqslant G_2$ and G_2 is a normal subgroup of $\langle G_1, G_2 \rangle$. A similar argument shows that also G_1 is normal in $\langle G_1, G_2 \rangle$, so that $\langle G_1, G_2 \rangle = G_1 \times G_2$ and $\langle G_1, G_2 \rangle$ is a normal-by-finite subgroup of G. Therefore $G = G_1 \times G_2$ is the direct product of its non-trivial subgroups G_1 and G_2. If X is any normal-by-finite subgroup of G, it follows that also $\langle X \cap G_1, X \cap G_2 \rangle$ is normal-by-finite, and hence $X = (X \cap G_1) \times (X \cap G_2)$. Let N_1 and N_2 be normal subgroups of G_1 and G_2, respectively, and assume by contradiction that there exist elements $g_1 \in G_1$ and $g_2 \in G_2$ such that the cosets $g_1 N_1$ and $g_2 N_2$ have the same prime order p. Put $K = \langle g_1, g_2 \rangle$. Then $KN_1 N_2/N_1 N_2$ is an elementary abelian group of order p^2, and the interval $[KN_1 N_2/N_1 N_2]$ is contained in the set $nf(G)$. It follows that $KN_1 N_2/N_1 N_2$ has decomposable subgroup lattice, a contradiction.

Conversely, suppose that the group $G = G_1 \times G_2$ satisfies the condition of the statement. If N_1 and N_2 are normal subgroups of G_1 and G_2, respectively, such that G_1/N_1 and G_2/N_2 have non-trivial centre, it follows from the hypotheses that $Z(G_1/N_1)$ and $Z(G_2/N_2)$ are periodic and coprime; thus the lattice of all normal subgroups of the group G is decomposable and $N = (N \cap G_1) \times (N \cap G_2)$ for each normal subgroup N of G (see [11], Theorem 9.1.5). Let X be any normal-by-finite subgroup of G, and put $X_1 = X \cap G_1$ and $X_2 = X \cap G_2$. Assume that the subgroup $X_1 X_2$ is properly contained in X, so that X_1 is a proper subgroup of $K_1 = XG_2 \cap G_1$ and X_2 is a proper subgroup of $K_2 = XG_1 \cap G_2$. Moreover the factor groups K_1/X_1 and K_2/X_2 are isomorphic (see [11], Theorem 1.6.1). Consider now the core X_G of X in G, and write $N_1 = X_G \cap G_1$ and $N_2 = X_G \cap G_2$. On the other hand, $X_G = N_1 N_2$ and X_G has finite index in X, so that also the indices $|K_1 : N_1|$ and $|K_2 : N_2|$ are finite. As $K_1/X_1 \simeq K_2/X_2$, it follows that there exists a prime number p dividing the orders of both finite groups K_1/N_1 and K_2/N_2, a contradiction. Therefore $X = (X \cap G_1) \times (X \cap G_2)$ for every normal-by-finite subgroup X of G, and hence the map

$$X \mapsto (X \cap G_1, X \cap G_2)$$

is an order isomorphism between the partially ordered sets $nf(G)$ and $nf(G_1) \times nf(G_2)$, so that $nf(G)$ is decomposable. ■

Combining Theorem 2.1 and Suzuki's result quoted in the introduction, we get the following corollary.

COROLLARY 2.2. – *Let G be a periodic group. The partially ordered set $nf(G)$ is decomposable if and only if the lattice $\mathfrak{L}(G)$ is decomposable.*

3. – Complemented ordered sets.

Let $\mathfrak{L}$ be a lattice with smallest element O and largest element I, and let a be an element of $\mathfrak{L}$; an element x of $\mathfrak{L}$ is said to be a *complement* of a if $a \wedge x = O$ and $a \vee x = I$. The lattice $\mathfrak{L}$ is called *complemented* if every element of $\mathfrak{L}$ has a complement.

Recall that a group G is a *K-group* if its subgroup lattice $\mathfrak{L}(G)$ is complemented, and G is a *C-group* if every subgroup X of G has a complement Y in G such that $XY = YX$. Extending these concepts, we shall say that a non-empty subset $\mathfrak{I}$ of $\mathfrak{L}(G)$ is *complemented* (respectively, *permutably complemented*) if for every element X of $\mathfrak{I}$ there exists $Y \in \mathfrak{I}$ such that $\langle X, Y \rangle = G$ and $X \cap Y = \{1\}$ (respectivley, $XY = G$ and $X \cap Y = \{1\}$).

In order to characterize groups G for which the set of all normal-by-finite subgroups is a permutably complemented subset of $\mathfrak{L}(G)$ we need a series of lemmas. The first of them shows in particular that any periodic group with permutably complemented set of normal-by-finite subgroups is metabelian.

LEMMA 3.1. – *Let G be a group such that the set $nf(G)$ is permutably complemented. Then the subgroup G'' is torsion-free.*

PROOF. – Let x be any element of finite order of G'', and let K be a normal-by-finite subgroup of G such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K = \{1\}$. The factor group G/K_G is a finite C-group, so that it is metabelian, and G'' is contained in K. It follows that $x = 1$, and hence the subgroup G'' is torsion-free. ■

LEMMA 3.2. – *Let G be a group such that the set $nf(G)$ is permutably complemented, and let N be a torsion-free normal subgroup of G. Then every G-invariant subgroup of N has a G-invariant complement in N.*

PROOF. – Let H be any G-invariant subgroup of N, and let K be a normal-by-finite subgroup of G such that $G = HK$ and $H \cap K = \{1\}$. Clearly $K \cap N$ is a complement of H in N. There exists also a normal-by-finite subgroup L of G such that $G = (K_G \cap N) L$ and $K_G \cap N \cap L = \{1\}$, so that $N = (K_G \cap N)(L \cap N)$ and $N/K_G \cap N \simeq L \cap N$ is torsion-free. On the other hand, $K \cap N/K_G \cap N$ is finite, and hence $K \cap N = K_G \cap N$ is a normal subgroup of G. The lemma is proved. ■

COROLLARY 3.3. – *Let G be a torsion-free group. Then the set $nf(G)$ is permutably complemented if and only if G is a direct product of simple groups.*

PROOF. – Suppose first that $nf(G)$ is permutably complemented. Then the lattice of all normal subgroups of G is complemented by Lemma 3.2, and it follows that G is a direct product of simple groups (see [11], Theorem 9.1.8).

Conversely, if G is a direct product of (torsion-free) simple groups, it is well-known that all factor groups of G are likewise torsion-free. Therefore every normal-by-finite subgroup of G is normal, and hence the set $nf(G)$ is (permutably) complemented. ■

LEMMA 3.4. – *Let G be a periodic group such that the set $nf(G)$ is permutably complemented. Then G is a C-group, and its socle S has finite index.*

PROOF. – The group G is metabelian by Lemma 3.1, so that in particular the subgroup S is abelian and G' is contained in S (see [11], Lemma 3.1.7). Let K be a normal-by-finite subgroup of G such that $G = SK$ and $S \cap K = \{1\}$; then K is an abelian group with complemented subgroup lattice, and hence it is the direct product of subgroups of prime order. Let N be a minimal normal subgroup of G, and let E be any finite non-trivial subgroup of N. By hypothesis there exists a normal-by-finite subgroup L of G such that $G = EL$ and $E \cap L = \{1\}$, so that $N = E \times (L \cap N)$; moreover, $L \cap N$ is normal in $G = NL$ and it is properly contained in N, so that $L \cap N = \{1\}$ and $N = E$. It follows that every minimal normal subgroup of G has prime order, so that $S = Dr_{i \in I} N_i$, where each N_i is a normal subgroup of G of prime order. Therefore G is a C-group (see [11], Theorem 3.2.5). Since K does not contain minimal normal subgroups of G, the core of K in G is trivial (see [11], Lemma 3.1.7), and hence K is finite. ■

LEMMA 3.5. – *Let G be a group such that the set $nf(G)$ is permutably complemented. Then G'' is perfect and G/G'' is a C-group. Moreover, G'' is the subgroup generated by all torsion-free minimal normal subgroups of G.*

PROOF. – Since $G/G^{(3)}$ is a soluble group in which every normal subgroup has a complement, we have that $G/G^{(3)}$ is a K-group (see [11], Theorem 3.1.14), so that in particular it is periodic (see [6]). Thus $G/G^{(3)}$ is a C-group by Lemma 3.4, and hence it is metabelian, so that $G'' = G^{(3)}$ is perfect and G/G'' is a C-group. As G/G'' is periodic, G'' contains the subgroup K generated by all torsion-free minimal normal subgroups of G, and by Lemma 3.2 there exists a normal subgroup L of G such that $G'' = K \times L$. Assume by contradiction that K

is properly contained in G'', and consider a non-trivial element x of L. Let M be a G-invariant subgroup of L which is maximal with respect to the condition that $x \notin M$. Again by Lemma 3.2 we have that L contains a G-invariant subgroup V such that $L = M \times V$. Write $x = ab$ with $a \in M$ and $b \in V \setminus \{1\}$. Let N be any non-trivial G-invariant subgroup of V. Then x belongs to $MN = M \times N$ and hence $b \in N$. It follows that the normal closure $\langle b \rangle^G$ is a minimal normal subgroup of G, so that it is contained in K. This contradiction proves that $G'' = K$ is generated by the torsion-free minimal normal subgroups of G. ∎

We can now prove the main result of this section.

THEOREM 3.6. – *Let G be a group. The set $nf(G)$ of all normal-by-finite subgroups of G is permutably complemented if and only if $G = E \ltimes (N \times A)$, where E is a finite C-group, N is a direct product of torsion-free non-abelian minimal normal subgroups of G, and A is a direct product of normal subgroups of G of prime order.*

PROOF. – Suppose first that the set $nf(G)$ of all normal-by-finite subgroups of G is permutably complemented, so that it follows from Lemma 3.5 that the subgroup $N = G''$ is a direct product of torsion-free minimal normal subgroups of G. Let K be a normal-by-finite subgroup of G such that $G = NK$ and $N \cap K = \{1\}$. Then K is a C-group by Lemma 3.5, and so its socle S is abelian. The socle A of the core K_G of K contains every abelian normal subgroup of K (see [11], Lemma 3.1.7), so that in particular $S \cap K_G$ is contained in A; thus K_G/A is finite by Lemma 3.4, and hence also K/A is finite. Moreover,

$$A = \operatorname*{Dr}_{i \in I} A_i,$$

where each A_i is a normal subgroup of prime order of K (see [11], Lemma 3.1.7). Since $NA = N \times A$, every A_i is also normal in $G = NK$. Let E be a normal-by-finite subgroup of G such that $G = (NA)E$ and $NA \cap E = \{1\}$. Clearly NA has finite index in G, and so E is a finite C-group. Conversely, suppose that the group $G = E \ltimes (N \times A)$ satisfies the conditions of the statement, and let X be any normal-by-finite subgroup of G. Put $Y = X \cap NA$; as N is torsion-free and Y/Y_G is finite, we have $Y \cap N = Y_G \cap N$ and by Remak's theorem (see [10], p. 86) there exists a normal subgroup M of G such that $N = (Y \cap N) \times M$. Moreover, A contains a G-invariant subgroup B such that $A = (YN \cap A) \times B$ (see [11], Lemma 3.1.8). It follows that the normal subgroup MB of G is a complement of Y in NA (see [11], Lemma 3.1.4). Since E is a C-group, there exists a subgroup L of E such that $E = (XNA \cap E)L$ and $E = XNA \cap E \cap L = \{1\}$. Thus the normal-by-finite subgroup $K = MBL$ is a complement of X in G (see

[11], Lemma 3.1.4). Finally,

$$XK = X(YMB)L = XNAL = XNA((XNA \cap E)L) = XNAE = G,$$

so that $XK = KX$ and the set $nf(G)$ is pemutably complemented. ■

Observe that the elements of finite order of a group with permutably complemented set of normal-by-finite subgroups need not form a subgroup. In fact, if N_1 and N_2 are isomorphic simple torsion-free groups and α is any automorphism of $N = N_1 \times N_2$ such that $N_1^\alpha = N_2$, $N_2^\alpha = N_1$ and $\alpha^2 = 1$, the semidirect product $G = \langle \alpha \rangle \ltimes N$ satisfies the conditions of Theorem 3.6 and hence $nf(G)$ is a permutably complemented subset of $\mathfrak{L}(G)$.

It has been proved by Napolitani [8] that if G is a soluble group in which every normal subgroup has a complement, then the lattice $\mathfrak{L}(G)$ is complemented. Here we note the following slight generalization of this result.

PROPOSITION 3.7. – *Let G be a soluble-by-finite group. If every normal-by-finite subgroup of G has a complement, then G is a K-group.*

PROOF. – Let K be the largest soluble normal subgroup of G. Since every finite homomorphic image of G is a K-group, we may suppose that K is not trivial, so that there is a non-negative integer n such that $A = K^{(n)}$ is an abelian non-trivial normal subgroup of G. Let L be a complement of A in G. Then $L \simeq G/A$ is a K-group by induction on the derived length of K. Moreover, A is a direct product of minimal normal subgroups of G (see [11], Lemma 3.1.7), and hence G itself is a K-group (see [11], Lemma 3.1.9). ■

In the soluble-by-finite case it is also possible to describe groups for which the set of all normal-by-finite subgroups is a complemented subset of the lattice of all subgroups.

THEOREM 3.8. – *Let G be a soluble-by-finite group. Then the following statements are equivalent:*

(a) *every normal-by-finite subgroup of G has a normal-by-finite complement;*

(b) *every subgroup of G has a normal-by-finite complement;*

(c) $G = E \ltimes A$, *where A is a direct product of abelian minimal normal subgroups of G and E is a finite K-group.*

PROOF. – Suppose first that (a) holds, and consider the subgroup A generated by all abelian minimal normal subgroups of G. Let E be a normal-by-finite subgroup of G such that $G = AE$ and $A \cap E = \{1\}$. Assume that E is infinite, so that also its core E_G is infinite, and hence E contains an abelian non-trivial G-invariant subgroup B of G. On the other hand, A contains all abelian normal subgroups of G (see [11], Lemma 3.1.7), a contradiction since $A \cap E = \{1\}$. Therefore $E \simeq G/A$ is a finite K-group, and G satisfies (c).

Suppose now that condition (c) holds, and let X be any subgroup of G. Since E is a K-group, there exists a subgroup Y of E such that $E = \langle X \cap E, Y \rangle$ and $X \cap E \cap Y = \{1\}$. Moreover, $\langle X, E \rangle \cap A$ is a normal subgroup of $G = AE$, and so there exists a G-invariant complement B of $\langle X, E \rangle \cap A$ in A (see [11], Lemma 3.1.8). It follows that YB is a complement of X in G (see [11], Lemma 3.1.4), and YB is a normal-by-finite subgroup of G since E is finite. Thus (b) holds.

Finally it is clear that (a) is a consequence of (b). ■

Let $\mathfrak{L}$ be a lattice with smallest element O and largest element I. A non-empty subset X of $\mathfrak{L}$ is called *boolean* if it is a complemented subset of $\mathfrak{L}$ and

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

for all elements x, y, z of X. Clearly a sublattice of $\mathfrak{L}$ is a boolean subset if and only if it is a boolean lattice and contains O, I. However, the above definition applies also to arbitrary subsets of $\mathfrak{L}$.

COROLLARY 3.9. – *Let G be a soluble-by-finite group such that $nf(G)$ is a boolean subset of $\mathfrak{L}(G)$. Then G is a periodic abelian group whose non-trivial primary components have prime order. In particular, $\mathfrak{L}(G)$ is a boolean lattice.*

PROOF. – By Theorem 3.8, we have $G = E \ltimes A$, where A is a direct product of abelian minimal normal subgroups of G and E is a finite K-group. In particular, G is locally finite and hence $nf(G)$ is a sublattice of $\mathfrak{L}(G)$; thus $nf(G)$ is a boolean lattice, so that it is uniquely complemented. It follows that E is a normal subgroup of G, so that $G = E \times A$ and $\mathfrak{L}(G)$ is a boolean lattice. Then the group G is abelian, and it has the required structure. ■

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Dipartimento di Matematica e Applicazioni, Università degli Studi
di Napoli Federico II, Complesso Universitario di Monte S. Angelo
Via Cintia, I 80126 Napoli (Italy)

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