Alberto Tognoli

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Topological Manifolds and Real Algebraic Geometry (*).

ALBERTO TOGNOLI

Sunto. – *Si studia il problema di approssimazione, a meno di omotopia, delle varietà topologiche compatte di dimensione* 4 *con varietà algebriche. Come conseguenza si prova che ogni forma quadratica intera non degenere è la forma di intersezione di una varietà algebrica reale di dimensione* 4. *Questi risultati sono legati ai ben noti lavori di Freedman sulla topologia delle varietà compatte, semplicemente connesse di dimensione* 4 (**).

Summary. – *We study the problem of approximating, up to homotopy, compact topological manifolds by real algebraic varieties. As a consequence, we realize any integral non-degenerate quadratic form as the intersection form of a real algebraic variety. This is related to a well-known result, due to Freedman* [F], *on the topology of closed simply-connected topological* 4-*manifolds* (**).

1. – Introduction.

In the theory of real algebraic geometry there are several results about the approximation of smooth manifolds by real algebraic varieties. It seems that the smooth structure is absolutely necessary for these results. In this paper we study some relations between topological manifolds and real algebraic varieties.

One of the results, we shall prove, is the following

THEOREM. – *Let M^4 be a compact simply-connected topological* 4-*dimensional manifold. Then there exists a real algebraic variety V such that:*

i) *V has at most one singular point;*

ii) *V has the homotopy type of M.*

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Combined with Freedman's theorem [F] [FQ] this implies that any unimodular integral symmetric bilinear form on a finitely generated free $\mathbb{Z}$-module can be realized by a real algebraic variety as its intersection form.

2. – Some results in real algebraic geometry.

In the following by *real algebraic variety* we shall mean a real affine variety, i.e. the locus of zero of a polynomial in some $\mathbb{R}^n$. Throughout the paper, all the algebraic varieties will be real. By *algebraic map* $\varphi : V \to W$ between real algebraic varieties $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ we mean that φ is locally the restriction of a rational regular map.

Let M be a smooth manifold and $X \subset M$ a closed subset. Then we say that the pair (M, X) has an *algebraic structure* if there exist a regular algebraic variety V and an algebraic subvariety $Y \subset V$ such that (M, X) is diffeomorphic to (V, Y).

If X is a locally finite union of smooth submanifolds, i.e. $X = \cup_i X_i$, then we say that the sets X_i are in *general position* if the X_i's and all the finite intersections Y', Y'' of them are mutually transverse or $Y'' \subset Y'$.

We recall now some results:

THEOREM 1. – *Let (M, X) be a pair, where M is a compact smooth manifold and $X = \bigcup_i X_i$ is a finite family of closed smooth submanifolds in general position. Then (M, X) has an algebraic structure.*

See [T1] for the proof.

THEOREM 2. – *Let V be a real algebraic variety and X a closed algebraic subvariety of V. Let $\varphi : X \to Y$ be a proper surjective algebraic map. Then there exist a real algebraic variety W and an algebraic map $\psi : V \to W$ such that:*

i) $Y \subset W$ *and* $\psi|_X = \varphi$;

ii) $\psi(V) = W$; *and*

iii) $\psi : V \backslash X \to W \backslash Y$ *is an algebraic isomorphism.*

See [B] for the proof.

THEOREM 3. – *Let M be a smooth compact manifold and τ a smooth triangulation of M. Let $K_1, K_2, \ldots, K_n$ be a finite family of subcomplexes of τ and $\mathcal{R}$ the equivalence relation on M defined by:*

$$x \mathcal{R} y \Leftrightarrow x = y \quad \text{or} \quad \{x, y\} \subset K_i \quad \text{for some} \quad i.$$

Then the quotient space $\widetilde{M} = M/\mathcal{R}$ *is homeomorphic to a real algebraic variety that has at most* n *singular points.*

See [AK] or [T1] for the proof.

3. – Some results on topological manifolds.

In the following by *topological n-manifold* we shall mean a metrizable compact connected topological n-dimensional manifold without boundary.

THEOREM 4. – *Let* M^n *be a closed topological n-manifold, for* $n \geqslant 5$. *Then* M *has a smooth structure if and only if the Kirby-Siebenmann invariant* $ks(M) \in H^4(M; \mathbb{Z}_2)$ *vanishes.*

See [KS] for the proof.

We recall that any topological n-manifold has a smooth structure for $n \leqslant 3$, and that Theorem 4 is false in general for $n = 4$. We recall also that in dimension 4 any topological manifold has a smooth structure if and only if it has a PL structure (or equivalently, it has a handle decomposition) as shown for example in [FQ].

Let now M^4 be an orientable compact connected topological 4-manifold without boundary. Then the Poincaré duality induces the symmetric bilinear form:

$$\lambda_M \colon H_2(M; \mathbb{Z})/\mathrm{Tors} \times H_2(M; \mathbb{Z})/\mathrm{Tors} \to \mathbb{Z}$$

defined by:

$$\lambda_M(x, y) = \langle \mathrm{PD}(x) \cup \mathrm{PD}(y), [M] \rangle$$

where $\mathrm{PD}(x) \in H^2(M; \mathbb{Z})$ denotes the Poincaré dual of x, and $[M]$ is the fundamental class of M. The form λ_M is called the *intersection form* of M. We remark that if M is smooth, then we can represent the homology classes x and y by smoothly embedded oriented surfaces X and Y in general position in M. In this case, we can understand the above pairing as the intersection form on $H_2(M; \mathbb{Z})/\mathrm{Tors}$. Let $p_1, \dots, p_r$ be the intersection points of X and Y. To each point p_i we assign a coefficient $+1$ or -1 according to whether the tangent bundle $TX|_{p_i} \oplus TY|_{p_i}$ has the same or opposite orientation as $TM|_{p_i}$. Then we have $\lambda_M(x, y) = \sum_{i=1}^{r} \varepsilon(p_i)$. Given a basis $(e_1, \dots, e_n)$ of $H_2(M; \mathbb{Z})/\mathrm{Tors}$, the intersection form determines the intersection matrix $A_{\lambda_M} = (\lambda_M(e_i, e_j)) \in \mathcal{M}(n; \mathbb{Z})$. As one can easily verify, the intersection form λ_M is *unimodular* (or, equivalently, non-degenerate) that is any intersection matrix of it has determinant ± 1.

We recall now some definitions about bilinear forms on a finitely generated free abelian group. Let Λ be an abelian group which is free and finitely generated. Let $\lambda : \Lambda \times \Lambda \to \mathbb{Z}$ be a symmetric nondegenerate bilinear form. Then λ is called *even* if $\lambda(x, x)$ is even for any $x \in \Lambda$; otherwise, λ is called *odd*. The *rank* $rk(\lambda)$ of λ is by definition the rank of the abelian group Λ. Since the form λ is non-degenerate, its rank equals the rank of any matrix A_λ representing it. We recall that A_λ is equivalent (over $\mathbb{R}$) to a diagonal matrix, D_λ say. Then the *signature* $\sigma(\lambda)$ of λ is the signature of D_λ over $\mathbb{R}$, that is the number of positive terms minus the number of negative terms in the diagonal of D_λ. It is well-known that the signature of a symmetric non-degenerate even form is divisible by 8. The form λ is called *definite* if the rank of λ is equal to the absolute value of its signature; otherwise, the form is said to be *indefinite*. Two forms $\lambda_i : \Lambda_i \times \Lambda_i \to \mathbb{Z}$, for $i = 1, 2$, are *isomorphic* if there is an isomorphism $\eta : \Lambda_1 \to \Lambda_2$ such that $\lambda_1(x, y) = \lambda_2(\eta(x), \eta(y))$ for any $x, y \in \Lambda_1$. In this case, we shall write $\lambda_1 \cong \lambda_2$.

The following theorem, due to Serre, gives the complete classification, up to isomorphism, of non-degenerate indefinite forms.

THEOREM 5. – *Let Λ be a finitely generated free abelian group and λ an indefinite non-degenerate symmetric form of rank $rk(\lambda) = p + q$ and signature $\sigma(\lambda) = p - q$, for $p, q > 0$. Then we have:*

i) *If λ is odd, then $\lambda \cong p(1) \oplus q(-1)$;*

ii) *If λ is even, then*

$$\lambda \cong aE_8 \oplus b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $a = \sigma(\lambda)/8$, $b = (rk(\lambda) - |\sigma(\lambda)|)/2$, and E_8 is the well-known symmetric unimodular positive definite matrix of rank (and signature) 8.

A complete classification of non-degenerate definite forms is not known. However, there is a characterization of such forms if they are realized by closed connected smooth 4-manifolds (see Theorem 9 below).

The intersection form λ_M of a closed connected topological 4-manifold M is a very important invariant to study the structure of M. In fact, the following are well-known results due to Milnor [Mi] and Freedman [F].

THEOREM 6. – *Let M and N be closed simply-connected topological 4-manifolds. Then M is homotopically equivalent to N if and only if λ_M is isomorphic to λ_N. Moreover, M is homeomorphic to N if and only if $\lambda_M \cong \lambda_N$ and $ks(M) = ks(N)$.*

THEOREM 7. – *Given a finitely generated free abelian group Λ and a bilinear symmetric unimodular form $\lambda : \Lambda \times \Lambda \to \mathbb{Z}$, there exists a smooth compact 4-manifold $\overline{M}$ with non-empty boundary which satisfies the following properties:*

i) *$\overline{M}$ is simply-connected;*

ii) *λ is isomorphic to $\lambda_{\overline{M}}$;*

iii) *$\partial \overline{M}$ has the same homology of the standard 3-sphere $\mathbb{S}^3$;*

iv) *$\partial \overline{M}$ is the boundary of a compact contractible topological 4-manifold Δ^4;*

v) *The union $M = \overline{M} \bigcup_{\partial \overline{M}} \Delta^4$ is a closed simply-connected topological 4-manifold such that $\lambda \cong \lambda_M \cong \lambda_{\overline{M}}$.*

We recall now a famous result of Quinn [Q].

THEOREM 8. – *Let M^4 be a topological 4-manifold. Then there exists a subset K in M such that:*

i) *K contains at most one point lying in any compact connected component of M;*

ii) *The complement $M \setminus K$ has a smooth structure.*

The following celebrated result, due to Donaldson [D1] [D3], shows that Theorem 4 does not hold in general for dimension 4.

THEOREM 9. – *Let M^4 be a a smooth compact oriented 4-manifold with positive definite intersection form λ_M. Then λ_M is isomorphic (over $\mathbb{Z}$) to the standard diagonal form $(1) \oplus \ldots \oplus (1)$.*

See [D1] and [D3] for the proof.

We remark that, using Theorem 9, it was possible to construct topological compact 4-manifolds M such that $ks(M) = 0$, but M has no smooth structures (see for example [Q]). In general, Donaldson's theorem says that if the intersection form λ_M of a compact 4-manifold M is even and positive definite, then M is not smoothable. For example, the closed simply-connected topological 4-manifold $\|2E_8\|$ realizing the intersection form $2E_8 = E_8 \oplus E_8$ (whose rank and signature are 16, hence $ks(\|2E_8\|) = 0$) has no smooth structures.

There are also results for certain classes of indefinite forms. The following theorem was proved in [D2].

THEOREM 10. – *Let M^4 be a smooth compact oriented 4-manifold such that the first integral homology group $H_1(M;\mathbb{Z})$ has no 2-torsion. Suppose that the positive part of the intersection form λ_M has rank 2, i.e. $\sigma(\lambda_M) = p - q$,* *where* $p=2$. *Then we have the isomorphism*

$$\lambda_M \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Donaldson [D4] proved that the indefinite form $(1)\oplus 9(-1)$ is realized not only by $\mathbb{C}P^2\#9(-\mathbb{C}P^2)$ but also by the Dolgachev surfaces (for a discussion about such complex surfaces we refer to [HKK]). Moreover, these surfaces are proved to be homotopically equivalent (and therefore topologically homeomorphic and h-cobordant) but not diffeomorphic to $\mathbb{C}P^2\#9(-\mathbb{C}P^2)$. So the smooth h-cobordism theorem fails in dimension 4 since these manifolds are smoothly h-cobordant.

Finally, we recall the following classification theorem given in [F].

THEOREM 11. – *Let Λ be a finitely generated free abelian group. Any even (odd) symmetric unimodular integral bilinear form $\lambda : \Lambda\times\Lambda\to\mathbb{Z}$ is realized as intersection form exactly by one (two) closed simply-connected topological 4-manifold(s). In the odd case, these manifolds are distinguished by their Kirby-Siebenmann invariants. Hence, if M and N are closed, simply-connected and smooth, then M is homeomorphic to N if and only if $\lambda_M\cong\lambda_N$.*

For the proof see [F]. We refer also to [FQ] for more results on the topology of 4-manifolds; survey papers on this topic are for example [C], [Ma], and [S].

4. – Algebraic structures on topological manifolds.

Let M be a compact connected topological n-manifold. We shall say that M is *almost smoothable* if it admits a decomposition $M=S\cup C$ where S is a smooth compact connected n-manifold with boundary $\partial S=S\cap C$, and C is compact and contractible. Of course, C is a topological n-dimensional submanifold of M, and ∂S is a (smooth) homology $(n-1)$-sphere. By Freedman's theorem any closed simply-connected topological 4-manifold is homotopy equivalent to an almost smoothable 4-manifold. Two compact topological n-manifolds M_0 and M_1 are said to be *almost diffeomorphic* if there exist decompositions $M_0=S_0\cup C_0$ and $M_1=S_1\cup C_1$, as above, such that $(S_0, \partial S_0)$ is diffeomorphic to $(S_1, \partial S_1)$. It was proved in [CFHS] that any two smooth h-cobordant (or, equivalently, homotopy equivalent) simply-connected compact 4-manifolds are almost diffeomorphic. A partial extension of this result for the non-simply-connected case can be found in [CHS].

If X is a topological space and $Y\subset X$, then X/Y denotes the quotient space obtained from X by identifying Y to a point.

The following result states that any compact almost smoothable topological manifold con be appoximated, up to homotopy, by a compact real algebraic variety. More precisely, we have

THEOREM 12. – *Let M^n be a compact connected almost smoothable topological n-dimensional manifold. Then there exists a compact real algebraic variety V which satisfies the following properties:*

i) *V has at most one singular point; and*

ii) *V has the same homotopy type of M.*

In fact, *V is homeomorphic to the quotient space $S/\partial S$ obtained from S by identifying ∂S to a point. Moreover, if such a variety V exists, then M is almost smoothable.*

PROOF. – Let $M = S \cup C$ be a decomposition of M as above. Clearly we can always suppose that ∂S is non-empty. Let $N = S \cup \mathrm{id}_{\partial S} S'$ be the double of the smooth compact manifold pair $(S, \partial S)$. It is well-known that N is smooth. Furthermore, N has a smooth triangulation τ such that S' is a subcomplex of τ. Let $\mathcal{R}$ be the equivalence relation in N defined by

$$x \mathcal{R} y \Leftrightarrow x = y \quad \text{or} \quad \{x, y\} \subset S'$$

for any $x, y \in N$. Then we shall denote by $\widetilde{N}$ the quotient space $N/\mathcal{R}$ and by $\pi : N \to \widetilde{N}$ the natural map. By construction, $\widetilde{N}$ is homeomorphic to the quotient space M/C obtained by contracting C to a point in M (in fact, both spaces $\widetilde{N}$ and M/C are homeomorphic to the Alexandroff compactification of $S \backslash \partial S$). Now Theorem 3 implies that $\widetilde{N}$ is homeomorphic to a real algebraic variety V that has at most one singular point. It remains to prove that V has the same homotopy type of M. For this, we must define two continuous maps:

$$p : M \to \widetilde{M} := M/C \quad \text{and} \quad h : \widetilde{M} \to M$$

such that $h \circ p$ and $p \circ h$ are homotopic to the corresponding identity maps. For $p : M \to \widetilde{M}$ we choose the natural projection from M to the quotient space $\widetilde{M} = M/C$. To define h we first recall the following facts:

1) The boundary ∂S has a collar in S, hence there exists a neighbourhood U of ∂S in S that can be identified to a product $\partial S \times I$, where $I = [0, 1]$. From now on, we shall identify ∂S with $\partial S \times \{1\} \subset U$. The neighbourhood U is in fact the union of $U' := \partial S \times [1/2, 1]$ and $U'' := \partial S \times [0, 1/2]$;

2) The subset C is contractible, hence there exists a continuous map

$$\alpha : C \times I \to C$$

such that α_0 is the identity map and α_1 maps C to a point, x_0 say, in C;

3) By construction, we have $U = \bigcup_{t \in I} \partial S \times \{t\}$. Let us denote by $\widetilde{U}_t = p(\partial S \times \{t\}) \subset \widetilde{M}$. If $t \neq 1$, then the restriction map $p| : \partial S \times \{t\} \to \widetilde{U}_t$ is an homeomorphism.

Finally, we can define h as follows:

$$h(x) = \begin{cases} x_0 & \text{if } x = \widetilde{x}_0 \\ \alpha_{2t-1}(p^{-1}(x), 1) & \text{if } 1/2 \leqslant t < 1,\ x \in \widetilde{U}_t \\ (p^{-1}(x), 2t) & \text{if } 0 \leqslant t \leqslant 1/2,\ x \in \widetilde{U}_t \\ p^{-1}(x) & \text{if } x \in p(M \setminus U) \end{cases}$$

where $\widetilde{x}_0$ denotes the equivalence class of C in $\widetilde{M}$. It is easy to verify that p and h satisfy the conditions $h \circ p \simeq \mathrm{id}_M$ and $p \circ h \simeq \mathrm{id}_{\widetilde{M}}$, where $\simeq$ means «is homotopic to». So the first assertion is proved. Let now suppose that the real algebraic variety $V \cong M/C$ exists, and let $v \in V$ be a point such that $V \setminus \{v\}$ is a regular algebraic variety. As proved in [BR], it is possible to find a small sphere Σ centered in v which cuts transversally $V \setminus \{v\}$, and bounds a contractible set in V. This completes the proof. ■

REMARK. – To prove that M and M/C have the same homotopy type one may observe that the map $p : M \to M/C$ induces an isomorphism on the homotopy (or equivalently, on the fundamental group and on homology with local coefficients). So the claim follows from the Whitehead theorem (see for example [Ma]).

Theorem 12 combined with Freedman's theorem permits to realize any symmetric unimodular integral bilinear form by a real algebraic variety. In fact, we have

COROLLARY 1. – *Let Λ be a finitely generated free abelian group, and $\lambda : \Lambda \times \Lambda \to \mathbb{Z}$ be a symmetric unimodular integral bilinear form. Then there exists a compact connected real algebraic variety V which satisfies the following properties:*

i) *V is simply-connected;*

ii) *V has at most one singular point; and*

iii) *λ_V is isomorphic (over $\mathbb{Z}$) to λ.*

PROOF. – By Theorem 6 there exists a closed simply-connected almost smoothable topological 4-manifold M which realizes λ as its intersection form, i.e. $\lambda \cong \lambda_M$. So the corollary follows from Theorem 12. ∎

COROLLARY 2. – *Let M be a closed simply-connected topological 4-manifold. Then there exists a compact real algebraic variety V such that:*

i) *V has the same homotopy type of M; and*

ii) *V has at most one singular point.*

PROOF. – The claim follows from Corollary 1 and Theorem 5. In fact, the form λ_M can be realized by a closed simply-connected almost smoothable manifold M' which has the same homotopy type of a compact simply-connected real algebraic variety V satisfying ii). But M and V have the same intersection form, hence they have the same homotopy type. ∎

REMARK. – If the almost smoothable topological manifold of Theorem 12 has dimension 4 and it does not admit a smooth structure, then M/C can not be homeomorphic to M. In fact, M/C can be triangulated (it is homeomorphic to a real algebraic variety), hence it has a handle decomposition. This would imply that M/C has a smooth structure if it is a topological manifold. In particular, if the intersection form λ_M does not satisfy the condition of Theorem 9 (resp. Theorem 10), then M/C can not be a topological manifold.

A natural question which arises from the above arguments is the following:

Is it possible to generalize Theorem 12 For example, let M be a compact smooth manifold and X a compact tame embedded topological submanifold of M. Is it possible to give an algebraic structure to the pair (M, X)?

We are going to prove that the above question has in general a negative answer.

PROPOSITION 1. – *There exists a smooth compact manifold M containing a compact tame embedded topological copy X of the complex projective line $\mathbb{C}P^1$ such that the pair (M, X) does not admit an algebraic structure. Moreover, the pair (M, X) is not homeomorphic to a pair of compact smooth manifolds.*

PROOF. – Let M' be a compact topological 4-manifold which has no smooth structures, but $ks(M') = 0$ (see [Q] and [FQ]). Let us suppose that M' is connected. Then M' has a «desingularisation». This means that there exist a smooth compact 4-manifold M and a continuous map $p : M \to M'$ which satisfy the following properties:

1) There exists a point $x_0 \in M'$ such that $M' \setminus \{x_0\}$ has a smooth structure;

2) $X = p^{-1}(x_0)$ is homeomorphic to $\mathbb{C}P^1 \cong \mathbb{S}^2$ and it is tamely embedded in M;

3) The restriction map $p|_{M \setminus X}: M \setminus X \to M' \setminus \{x_0\}$ is a diffeomorphism, and a neighbourhood of X in M is diffeomorphic to a neighbourhood of a displacement of the standard $\mathbb{C}P^1$ in $\mathbb{C}P^2$;

4) M' is homeomorphic to M/X, and topologically $M = M' \# \mathbb{C}P^2$.

See [Q] for the proof of these facts. Since M' has no smooth structures, it does not admit a handle decomposition. If the pair (M, X) would be homeomorphic to a pair $(\widetilde{M}, \widetilde{X})$ of compact smooth manifolds, then by Theorem 1 the pair (M, X) has an algebraic structure. Hence $M' \cong M/X$ is triangulable. This gives a contracdiction since M' has no smooth structures by hypothesis. So the statement is proved. ■

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Alberto Tognoli: Dipartimento di Matematica, Università di Trento
Via Sommarive 14, 38050 Povo (Trento), Italy
E-mail: tognoli@science.unitn.it

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