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Quasi-Homeomorphisms, Goldspectral Spaces and Jacspectral Spaces.

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Sunto. – *In questo lavoro vengono studiati i quasi-omeomorfismi tra spazi spettrali, lo spettro primo di Goldman e lo spettro primo di Jacobson di un anello commutativo. Proviamo che, se $g : Y \to X$ è un quasi-omeomorfismo, Z uno spazio sobrio ed $f : Y \to Z$ una mappa continua, allora esiste un'unica mappa continua $F : X \to Z$ tale che $F \circ g = f$. Sia X uno T_0-spazio, $q : X \to {}^sX$ l'iniezione di X nella sua sobrificazione ${}^sX$, allora mostriamo che $q(\mathrm{Gold}(X)) = \mathrm{Gold}({}^sX)$, dove $\mathrm{Gold}(X)$ è l'insieme di tutti punti localmente chiusi di X. Di tali risultati vengono date alcune applicazioni. Lo spettro primo di Jacobson di un anello commutativo R è l'insieme di tutti gli ideali primi di R che si ottengono come intersezione di ideali massimali di R. Uno dei risultati principali di questo lavoro fornisce una risposta, per vari aspetti sorprendente, al problema delle unioni disgiunte di insiemi jacspettrali (insiemi ordinati che sono isomorfi come insiemi ordinati allo spettro primo di Jacobson di un qualche anello commutativo). Sia $\{(X_\lambda, \leqslant_\lambda) : \lambda \in \Lambda\}$ una famiglia di insiemi ordinati disgiunti e sia $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Introduciamo su X una relazione d'ordine dichiarando $x \leqslant y$ se esiste $\lambda \in \Lambda$ tale che $x, y \in X_\lambda$ e $x \leqslant_\lambda y$. Allora, le affermazioni seguenti sono tra loro equivalenti:*
(i) $(X, \leqslant)$ *è jacspettrale.*
(ii) $(X_\lambda, \leqslant_\lambda)$ *è jacspettrale, per ogni* $\lambda \in \Lambda$.

Summary. – *In this paper, we deal with the study of quasi-homeomorphisms, the Goldman prime spectrum and the Jacobson prime spectrum of a commutative ring. We prove that, if $g : Y \to X$ is a quasi-homeomorphism, Z a sober space and $f : Y \to Z$ a continuous map, then there exists a unique continuous map $F : X \to Z$ such that $F \circ g = f$. Let X be a T_0-space, $q : X \to {}^sX$ the injection of X onto its sobrification ${}^sX$. It is shown, here, that $q(\mathrm{Gold}(X)) = \mathrm{Gold}({}^sX)$, where $\mathrm{Gold}(X)$ is the set of all locally closed points of X. Some applications are also indicated. The Jacobson prime spectrum of a commutative ring R is the set of all prime ideals of R which are intersections of some maximal ideals of R. One of our main results is a surprising answer to the problem of ordered disjoint union of jacspectral sets (ordered sets which are isomorphic to the Jacobson prime spectrum of some ring): Let $\{(X_\lambda, \leqslant_\lambda) : \lambda \in \Lambda\}$ be a collection of ordered disjoint sets and $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Partially order X by declaring $x \leqslant y$ to mean that there exists $\lambda \in \Lambda$ such that $x, y \in X_\lambda$ and $x \leqslant_\lambda y$. Then the following statements are equivalent:*
(i) $(X, \leqslant)$ *is jacspectral.*
(ii) $(X_\lambda, \leqslant_\lambda)$ *is jacspectral, for each* $\lambda \in \Lambda$.

Introduction.

We start this introduction by pointing out that it is essential to know what we are talking about, that is, to understand definitions of the terms we are using.

A continuous map $q: X \to Y$ is said to be a *quasi-homeomorphism* [13] if for each open subset U of X there exists a unique open subset V of Y such that $U = q^{-1}(V)$ (equivalently, for each closed subset F of X, there exists a unique closed subset G of Y such that $F = q^{-1}(G)$).

A subset S of a topological space X is said to be *strongly dense* in X, if S meets every nonempty locally closed subset of X. Thus a subset S of X is strongly dense if and only if the canonical injection $S \hookrightarrow X$ is a quasi-homeomorphism. It is well known that a continuous map $q: X \to Y$ is a quasi-homeomorphism if and only if the topology of X is the inverse image by q of that of Y and the subset $q(X)$ is strongly dense in Y [13].

The notion of quasi-homeomorphism is used in algebraic geometry. It is recently shown that this notion arise naturally in the theory of some foliations associated to closed connected manifolds (see our papers written jointly with E. Bouacida and E. Salhi [4], [5], [6]). It is worth noting that quasi-homeomorphisms are also linked with sober spaces. Recall that a topological space X is said to be *sober* if any nonempty irreducible closed subset of X has a unique generic point. Let X be a topological space and ${}^sX$ the set of all irreducible closed subsets of X [13]. Let U be an open subset of X, set $\widetilde{U} = \{F \in {}^sX : U \cap F \neq \emptyset\}$, then the collection ($\widetilde{U}$, U is an open subset of X) gives a topology on ${}^sX$ and the following properties hold:

(*i*) *The map* $q: X \to {}^sX$ *defined by:* $x \mapsto \overline{\{x\}}$ *is a quasi-homeomorphism.*

(*ii*) ${}^sX$ *is a sober space.*

(*iii*) *Moreover,* q *is injective if and only if* X *is* T_0 *and is a homeomorphism when* X *is sober. The topological space* ${}^sX$ *is called the sobrification of* X, *the assignment* $X \to {}^sX$ *defines a functor* [13, 0.2.9].

Jacobson topological spaces are also used in algebraic geometry and are linked with quasi-homeomorphisms. A topological space X is called a *Jacobson space* if the set X_0 of its closed points is strongly dense in X. If X is a topological space, we denote by $\mathrm{Jac}(X)$ the set $\{x \in X : \overline{\{x\}} = \overline{\overline{\{x\}} \cap X_0}\}$. It is obviously seen that $\mathrm{Jac}(X)$ is a Jacobson space; we call it the *Jacobson subspace* of X.

Let R be a ring and $\mathrm{Spec}(R)$ its prime spectrum equipped with the Zariski topology. We denote $\mathrm{Jac}(R)$ the Jacobson subspace of $\mathrm{Spec}(R)$. Following Picavet [23], [24], a prime ideal p of R lies in $\mathrm{Jac}(R)$, if and only if p is the intersection of some maximal ideals of R.

A *jacspectral* space is defined to be a topological space homeomorphic to the Jacobson subspace of Spec(R), for some ring R. A nice topological characterization has been given by Bouacida et al [6]: «*Jacspectral spaces are exactly the quasi-compact Jacobson sober spaces*».

A topology $\mathcal{T}$ on a set X is defined to be *spectral* (and $(X, \mathcal{T})$ is called a *spectral space*) if the following conditions hold:

(*i*) *$\mathcal{T}$ is sober.*

(*ii*) *the quasi-compact open subsets of X form a basis of $\mathcal{T}$.*

(*iii*) *the family of quasi-compact open subsets of X is closed under finite intersections.*

In a remarkable paper, M. Hochster has proved that a topological space is homeomorphic to the prime spectrum of some ring if and only if it is a spectral space [15]. Two years later, M. Hochster gave a topological characterization of the minimal prime spectrum of a ring [16].

Goldman ideals are important objects of investigation in algebra mostly because their role in the study of graded rings and some applications to algebraic geometry. Thus it is important to pay attention to the Goldman prime spectrum of a ring. Recall that a prime ideal of a commutative ring R is said to be a *Goldman ideal* (*G-ideal*) if there exists a maximal ideal M of $R[X]$ such that $p = M \cap R$. If R is an integral domain and (0) is a G-ideal, R is called a *G-domain*.

Over the years, mathematicians have focused attention on G-domains; for instance, O. Goldman and W. Krull used G-ideals for a short inductive proof of the Nullstellensatz [14], [20]. The set of all G-ideals of a commutative ring R is denoted by Gold(R) and called the *Goldman prime spectrum* of that ring.

In [8], A. Conte has proved that Spec(R) is a T_D-space if and only if every prime ideal of R is a G-ideal. He has also proved that if Spec(R) is Noetherian, then Spec(R) is a T_D-space if and only if it is finite.

M. Fontana and P. Maroscia [12] have also established, by topological methods, several properties of the set of the G-ideals of a commutative ring. They also discussed in detail a topological approach to a classification of the class of the commutative rings in which every prime ideal is a G-ideal.

Note also that rings in which every prime ideal is a G-ideal has been studied by G. Picavet in [24].

By a *goldspectral space*, we mean a topological space which is homeomorphic to some Gold(R). Using the notion of sobrification, we have given an intrinsic topological characterization of the Goldman prime spectrum of a commutative ring [10].

In this paper, we are aiming to give some results on quasi-homeomorphisms, goldspectral spaces and jacspectral spaces.

In Section 1, we are interested in finding results on quasi-homeomorphisms in a more general setting. We prove that, if $g: Y \to X$ is a quasi-homeomorphism, Z a sober space and $f: Y \to Z$ a continuous map, then there exists a unique continuous map $F: X \to Z$ such that $F \circ g = f$.

We prove, also, that quasi-homeomorphisms and homeomorphisms are linked by the following fact: «*Let $q: X \to Y$ be a continuous map. Then q is a quasi-homeomorphism if and only if ${}^s q: {}^s X \to {}^s Y$ is a homeomorphism*», where ${}^s q$ is the map defined by ${}^s q(C) = \overline{q(C)}$, for each irreducible closed subset C of X. Some applications are also indicated.

Section 2 is intended to motivate our investigation of the Goldman prime spectrum of a commutative ring. We prove that if X is a T_0-space and $q: X \to {}^s X$ is the injection of X onto its sobrification ${}^s X$, then $\mathrm{Gold}({}^s X) = q(\mathrm{Gold}(X))$, where $\mathrm{Gold}(X)$ is the set of all locally closed points of X. We then derive that, if X is a T_D-space [2] (that is, $\mathrm{Gold}(X) = X$), then X is homeomorphic to $\mathrm{Gold}({}^s X)$ and thus X is goldspectral if and only if X is a T_D-space and ${}^s X$ is spectral.

If an ordered set $(X, \leqslant)$ is the disjoint union of ordered sets $\{(X_\lambda, \leqslant_\lambda), \lambda \in \Lambda\}$, we shall say that X is the *ordered disjoint union* of the X_λ's if $x \leqslant y$ if and only if there is an λ such that $x, y \in X_\lambda$ and $x \leqslant_\lambda y$. Theorem 4.1 of [22] says that if $(X_\lambda, \leqslant_\lambda)$ is spectral for each $\lambda \in \Lambda$, then $(X, \leqslant)$ is spectral. However, Lewis and Ohm have not been able to establish the converse of the previous theorem and they raise the following question «*If $(X, \leqslant)$ is the ordered disjoint union of posets $(X_\lambda, \leqslant_\lambda)$, $\lambda \in \Lambda$, and if $(X, \leqslant)$ is spectral, then are the $(X_\lambda, \leqslant_\lambda)$ also spectral?*».

Ordered disjoint union of dimension $\leqslant 1$ is also discussed by A. Bouvier and M. Fontana in [7].

Section 1 is closed by giving an analogous result to that of Lewis and Ohm for an ordered disjoint union [22, Theorem 4.1].

Section 3 deals with the Jacobson prime spectrum of a commutative ring. Let X be a T_0-space. We prove that X is a quasi-compact Jacobson space if and only if ${}^s X$ is jacspectral. Many constructions are given showing the limits of the results established.

Theorem 3.20 gives a surprising result about the problem of ordered disjoint union of jacspectral sets (ordered sets isomorphic to the Jacobson prime spectrum of a commutative ring): Let $\{X_\lambda : \lambda \in \Lambda\}$ be a collection of ordered disjoint sets and $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Partially order X by declaring $x \leqslant y$ to mean that there exists $\lambda \in \Lambda$ such that $x, y \in X_\lambda$ and $x \leqslant_\lambda y$. Then the following statements are equivalent:

(*i*) *$(X, \leqslant)$ is jacspectral.*

(ii) *$(X_\lambda, \leqslant_\lambda)$ is jacspectral, for each $\lambda \in \Lambda$.*

1. – Quasi-homeomorphisms.

In this section we will look more closely at quasi-homeomorphisms. Some results may be considered as consequence of that proved in [13] by Grothendieck and Dieudonné (for instance Lemma 1.1, Lemma 1.2, and Lemma 1.3). However, our main results Theorem 1.5, Theorem 1.6 and Theorem 1.8 are new. The proofs have been divided into a sequence of lemmata.

First, observe that, combining [4, Proposition 2.3] and [4, Proposition 2.9], we get the following.

LEMMA 1.1. – *Let $q : X \to Y$ be a continuous surjective map. Then the following statements are equivalent:*

(*i*) *q is a quasi-homeomorphism.*
(*ii*) *q is open and for each open subset U of X, $q^{-1}(q(U)) = U$.*
(*iii*) *q is closed and for each closed subset C of X, $q^{-1}(q(C)) = C$.*

The proof of the following is straightforward.

LEMMA 1.2. – *Let X be a topological space and Y a subset of X. Then the following statements are equivalent:*

(*i*) *Y is strongly dense in X.*
(ii) *For each $x \in X$, $x \in \overline{\{x\} \cap Y}$.*

LEMMA 1.3. – *Let $g : Y \to X$ be a quasi-homeomorphism, $f : Y \to Z$ a continuous map. Suppose that there exists a continuous map $F : X \to Z$ such that $F \circ g = f$. Then the following properties hold:*

(1) f is a quasi-homeomorphism if and only if so is F.
(2) *If f is open (resp. closed), then so is F.*

PROOF. – (1) Straightforward.

(2) • We observe that a continuous map $\varphi : X \to Z$ is open if and only if $\varphi^{-1}(\overline{V}) = \overline{\varphi^{-1}(V)}$, for every subset V of Z [13].

• Let us prove that F is open when f is open. Let V be an open subset of Z; we have $f^{-1}(\overline{V}) = \overline{f^{-1}(V)}$. It follows that $g^{-1}(F^{-1}(\overline{V})) = \overline{g^{-1}(F^{-1}(V))}$. Let $g_1 : Y \to g(Y)$ be the map induced by g, then following Lemma 1.1, g_1 is an open quasi-homeomorphism. For this reason

$$g^{-1}(\overline{F^{-1}(V)}) = g_1^{-1}(\overline{F^{-1}(V)} \cap g(Y)) = g_1^{-1}(\overline{F^{-1}(V)}^{g(Y)}) =$$
$$\overline{g_1^{-1}(F^{-1}(V))} = \overline{g^{-1}(F^{-1}(V))}.$$

Hence $g^{-1}(F^{-1}(\overline{V})) = g^{-1}(\overline{F^{-1}(V)})$. From the fact that g is a quasi-homeomorphism, we conclude that $F^{-1}(\overline{V}) = \overline{F^{-1}(V)}$. Thus F is an open map.

Now, let us prove that F is closed when f is closed. Let C be a closed subset of X, we need only show that $F(C) = f(g^{-1}(C))$. We have $f(g^{-1}(C)) = \overline{F(C \cap g(Y))}$. Let $x \in C$. Since $g(Y)$ is strongly dense in X, $\overline{\{x\}} = \overline{\overline{\{x\}} \cap g(Y)}$ (see Lemma 1.2). On the other hand, the continuity of F yields $\overline{\{F(S)\}} = \overline{F(\overline{S})}$, for every subset S of X. Thus $\overline{\{F(x)\}} = \overline{F(\overline{\overline{\{x\}} \cap g(Y)})} = \overline{F(\overline{\{x\}} \cap g(Y))}$. Hence $F(x) \in \overline{F(C \cap g(Y))} = \overline{f(g^{-1}(C))} = f(g^{-1}(C)) = F(C \cap g(Y))$. Therefore $F(C) = F(C \cap g(Y)) = f(g^{-1}(C))$, and F is a closed map. ■

LEMMA 1.4. – *Let $g : Y \to X$ be a quasi-homeomorphism, $f : Y \to Z$ a continuous map. Suppose that Z is a T_0-space, then there exists at most one continuous map $F : X \to Z$, such that $F \circ g = f$.*

PROOF. – Assume that there exist two distinct continuous maps F, $G : X \to Z$ such that $F \circ g = G \circ g = f$. Let $x \in X$ with $F(x) \neq G(x)$. There exists an open subset W of Z such that $F(x) \in W$ and $G(x) \notin W$. Consider the two open subsets $U = F^{-1}(W)$ and $V = G^{-1}(W)$ of X. We thus get $g^{-1}(U) = g^{-1}(V)$; consequently $U = V$, a contradiction. ■

THEOREM 1.5 *(The first continuous extension theorem). – Under the same assumptions of Lemma* 1.3, *if moreover, g is surjective, then there exists one and only one continuous map $F : X \to Z$, such that $F \circ g = f$.*

PROOF. – Let $x \in X$, then there exists $y \in Y$ such that $x = g(y)$. Suppose that there exists $y_1 \in Y$ such that $x = g(y) = g(y_1)$, then $\overline{\{g(y)\}} = \overline{\{g(y_1)\}}$; and since g is closed (see Lemma 1.1), we get $g(\overline{\{y\}}) = g(\overline{\{y_1\}})$. Using again Lemma 1.1, we obtain $\overline{\{y\}} = \overline{\{y_1\}}$.

Now, since f is continuous, $\overline{f(\overline{\{y\}})} = \overline{\{f(y)\}} = \overline{f(\overline{\{y_1\}})} = \overline{\{f(y_1)\}}$. This implies that $f(y) = f(y_1)$ (since Z is a T_0-space). Thus providing a map $F : X \to Z$; $x = g(y) \mapsto f(y)$, with $F \circ g = f$.

It remains to prove that F is continuous. To do so, let V be an open subset of Z. Then $f^{-1}(V) = g^{-1}(F^{-1}(V))$. Hence $F^{-1}(V) = g(f^{-1}(V))$ so that $F^{-1}(V)$ is an open subset of X (since g is open (see Lemma 1.1)). We have thus proved that F is continuous. The uniqueness of that map is obvious. ■

THEOREM 1.6 *(The second continuous extension theorem). – Let $g : Y \to X$ be a quasi-homeomorphism, Z a sober space and $f : Y \to Z$ a continuous map. Then there exists one and only one continuous map $F : X \to Z$ such that $F \circ g = f$.*

PROOF. – Let $g_1 : Y \to g(Y)$ be the map induced by g, then g_1 is a surjective quasi-homeomorphism. From the first continuous extension theorem, there exists a continuous map $\tilde{f} : g(Y) \to Z$ such that $\tilde{f} \circ g_1 = f$. Now, since $g(Y)$ is strongly dense in X, using [6, Theorem 2.21], there exists a continuous extension map $F: X \to Z$ of $\tilde{f}$, and thus $F \circ g = f$. ■

LEMMA 1.7. – *Let $q: X \to Y$ be a quasi-homeomorphism. Then the following properties hold:*

(1) *If X is a T_0-space, then q is injective.*

(2) *If X is sober and Y is a T_0-space, then q is a homeomorphism.*

PROOF. – (1) Let x_1, x_2 be two points of X with $q(x_1) = q(x_2)$. Suppose that $x_1 \neq x_2$, then there exists an open subset U of X such that $x_1 \in U$ and $x_2 \notin U$. Since there exists an open subset V of Y satisfying $q^{-1}(V) = U$, we get $q(x_1) \in U$ and $q(x_2) \notin U$, which is impossible. It follows that q is injective.

(2) • We start with the obvious observation that if S is a closed subset of Y, then S is irreducible if and only if so is $q^{-1}(S)$.

• Let us prove that q is surjective. For this end, let $y \in Y$, according to the above observation, $q^{-1}(\overline{\{y\}})$ is a nonempty irreducible closed subset of X. Hence $q^{-1}(\overline{\{y\}})$ has a generic point x. Thus we have the containments

$$\overline{\{x\}} \subseteq q^{-1}(\overline{\{q(x)\}}) \subseteq q^{-1}(\overline{\{y\}}) = \overline{\{x\}}.$$

Then $q^{-1}(\overline{\{q(x)\}}) = q^{-1}(\overline{\{y\}})$. It follows, from the fact that q is a quasi-homeomorphism, that $\overline{\{q(x)\}} = \overline{\{y\}}$. Since Y is a T_0-space, we get $q(x) = y$. This proves that q is a surjective map, and thus q is bijective. One may see that bijective quasi-homeomorphisms are homeomorphisms. ■

The following indicates the links between quasi-homeomorphisms and homeomorphisms.

THEOREM 1.8. – *Let $q: X \to Y$ be a continuous map. Then the following statements are equivalent:*

(*i*) *q is a quasi-homeomorphism.*

(*ii*) *${}^s q$ is a homeomorphism.*

PROOF. – Remark that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ q_X \downarrow & \circlearrowleft & \downarrow q_Y \\ {}^s X & \xrightarrow{{}^s q} & {}^s Y \end{array}$$

is commutative.

$(i) \Rightarrow (ii)$ Since $q_Y \circ q = {}^s q \circ q_X$ is a quasi-homeomorphism, the map ${}^s q$ is a quasi-homeomorphism (see Lemma 1.3). Thus, following Lemma 1.7, ${}^s q$ is a homeomorphism.

$(ii) \Rightarrow (i)$ Since $q_X = (({}^s q)^{-1} \circ q_Y) \circ q$ and $({}^s q)^{-1} \circ q_Y$ are quasi-homeomorphisms, it follows from Lemma 1.3, that q is a quasi-homeomorphism. ■

Our next concern will be an application of the «second continuous extension theorems», it illustrates the fact that the sobrification is a solution of a universal problem.

PROPOSITION 1.9. – *Let X, T be two topological spaces. Then the following statements are equivalent:*

(*i*) *T is homeomorphic to the sobrification ${}^s X$.*

(*ii*) *T is sober and there exists a quasi-homeomorphism $q_X: X \to T$ such that for each sober space Z and each continuous map $f: X \to Z$, there exists a unique continuous map $F: T \to Z$ with $F \circ q_X = f$ (that is, the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{q_X} & T \\ \searrow & \circlearrowright & \swarrow \\ & Z & \end{array}$$

is commutative).

(*iii*) *T is sober and there exists a quasi-homeomorphism $q_X: X \to T$.*

PROOF. – $(i) \Rightarrow (ii)$ Let $q_X: X \to {}^s X$ be the quasi-homeomorphism defined by; $x \mapsto q_X(x) = \overline{\{x\}}$. Since Z is sober, following the « second continuous extension theorem», there exists a unique continuous map $F: {}^s X \to Z$ such that $F \circ q_X = f$.

$(iii) \Rightarrow (i)$ According to Theorem 1.8, ${}^s q_X$: ${}^s X \to {}^s T \sim T$ is a homeomorphism. ■

2. – The Goldman prime spectrum of a commutative ring.

As mentioned in the introduction, the Goldman prime spectrum of a commutative ring has been studied by several authors (see for instance, [8], [12], [23] and [24]). This section is intended to motivate our investigation of the Goldman prime spectrum. Let us, first, introduce the following terminology.

DEFINITIONS 2.1. – (1) Let X be a topological space, by the Goldman subspace of X, we mean the set $\mathrm{Gold}(X)$ of all locally closed points of X.

(2) An ordered set $(X, \leqslant)$ is said to be *goldspectral* if there exists a ring R such that $(X, \leqslant)$ is order isomorphic to $(\mathrm{Gold}(R, \subseteq))$, where $\mathrm{Gold}(R)$ is the set of G-ideals of R.

(3) We call a topological space X *goldspectral* if there exists a ring R such that X is homeomorphic to $\mathrm{Gold}(R)$ (equipped with the topology inherited by that of Zariski on $\mathrm{Spec}(R)$).

Goldspectral spaces has been introduced and studied in [10]. Our main theorem, in [10], provides an intrinsic topological characterization of goldspectral spaces.

THEOREM 2.2 [10]. – *Let X be a topological space. Then the following conditions are equivalent:*

(1) *X is goldspectral.*

(2) *X satisfies the following properties:*

(*i*) *X is quasi-compact and has a basis of quasi-compact open subsets which is stable under finite intersections.*

(*ii*) *X is a T_D-space.*

A slight change in the proof of the previous result, yields the following.

THEOREM 2.3. – (1) *Let $q : X \to Y$ be an injective quasi-homeomorphism, then* $\mathrm{Gold}(Y) \subseteq q(\mathrm{Gold}(X))$.

(2) *Let X be a T_0-space and $q : X \to {}^sX$ be the injection of X onto its sobrification ${}^sX$. Then* $q(\mathrm{Gold}(X)) = \mathrm{Gold}({}^sX)$.

PROOF. – (1) Straightforward.

(2) According to (1), it remains to prove that $q(\mathrm{Gold}(X)) \subseteq \mathrm{Gold}({}^sX)$. Let $x \in \mathrm{Gold}(X)$, there exists an open subset U of X such that $\{x\} = \overline{\{x\}} \cap U$. We claim that $\widetilde{U} \cap \overline{\{q(x)\}} = \{q(x)\}$. It is easily seen that $\{q(x)\} \subseteq \overline{\{q(x)\}} \cap \widetilde{U}$. Let $C \in \overline{\{q(x)\}} \cap \widetilde{U}$, then $C \cap U \neq \emptyset$. We must prove that $C = q(x)$. Let $y \in C \cap U$ and V be an open subset of X containing y, then $C \in \widetilde{V}$. On the other hand, $C \in \overline{\{q(x)\}}$, thus $q(x) \in \widetilde{V}$ and then $x \in V$. It follows that $y \in \overline{\{x\}}$. Thus we have $y \in \overline{\{x\}} \cap U = \{x\}$, so that $y = x$. Hence $x \in C$ and $\overline{\{x\}} = q(x) \subseteq C$.

For the inverse containment, consider $y \in C$ and V an open subset of X containing y; we have $C \in \widetilde{V}$ whence $q(x) \in \widetilde{V}$ (since $C \in \overline{\{q(x)\}}$). We thus get $\overline{\{x\}} \cap V \neq \emptyset$ and $x \in V$. This yields $y \in \overline{\{x\}} = q(x)$. Hence $C = q(x)$. Therefore, we get $\widetilde{U} \cap \overline{\{q(x)\}} = \{q(x)\}$, proving that $q(x) \in \mathrm{Gold}({}^sX)$. ■

REMARKS 2.4. – (1) Let $q : X \to Y$ be a quasi-homeomorphism and S an open subset of Y. Then the following properties hold:

(a) S is quasi-compact if and only if $q^{-1}(S)$ is quasi-compact.

(b) X is quasi-compact with a basis of quasi-compact open subsets, stable under finite intersections if and only if so is Y.

(2) The above remark (1) and Theorem 2.3 show that, if X is a topological space, then X is goldspectral if and only if X is a T_D-space and ${}^sX$ is a spectral space.

(3) Let $q : X \to Y$ be a quasi-homeomorphism. If X is goldspectral, then ${}^sY$ is spectral; this follows immediately from the above remark (2) and Theorem 1.8.

REMARK 2.5. – It is easy to check that an ordered set $(X, \leqslant)$ is goldspectral if and only if there exists a goldspectral topology on X which is compatible with the ordering $\leqslant$.

EXAMPLES 2.6. – (1) Every finite ordered set is goldspectral.

(2) A totally ordered set is goldspectral if and only if it has a greatest element. For this end, consider $(X, \leqslant)$ a totally ordered set with a maximal element, we equip X with the left topology; thus providing a goldspectral topology on X which is compatible with the ordering $\leqslant$. Therefore $(X, \leqslant)$ is a goldspectral set. Conversely, let $(X, \leqslant)$ be a goldspectral totally ordered set. Then there exists a ring R such that $(X, \leqslant)$ is order-isomorphic to $(\mathrm{Gold}(R), \subseteq)$. Since $\mathrm{Max}(R) \subseteq \mathrm{Gold}(R)$, $\mathrm{Max}(R)$ is a one point set and thus $(X, \leqslant)$ has a greatest element.

An ordered set is said to be *spectral* if it is order-isomorphic to the prime spectrum of some ring. Such spectral sets are of interest not only in (topological) ring and lattice theory, but also in computer science, in particular in domain theory (see for example [17], [18], [25]). In order that an ordered set $(X, \leqslant)$ be spectral it is necessary but not sufficient (see [22]) that it satisfies two conditions:

(K_1): Each nonempty totally eroded subset of $(X, \leqslant)$ has a supremum and an infimum (X is *up-complete* and *down-complete*).

(K_2): For each $a < b$ in X, there exist two adjacent element $a_1 < b_1$ such that $a \leqslant a_1 < b_1 \leqslant b$ (X is *weakly atomic*).

These properties were noted for a ring spectrum by Kaplansky (see [19, Theorems 9 and 11]), and then are called respectively the first condition and the second condition of Kaplansky.

According to Hochster [15, Proposition 8], if $(X, \leqslant)$ is spectral, then so is $(X, \geqslant)$.

It is worth noting that if $(X, \leqslant)$ is goldspectral, then it does not satisfy necessarily the conditions (K_1) and (K_2), and also $(X, \geqslant)$ is not necessarily goldspectral, as the following example shows.

EXAMPLE 2.7. – An ordered set $(X, \leqslant)$ satisfying the following properties:

(i) $(X, \leqslant)$ is goldspectral.

(ii) $(X, \geqslant)$ is not goldspectral.

(iii) $(X, \leqslant)$ does not satisfy the conditions (K_1) and (K_2).

It suffices to consider $\mathbb{R}^- = \{x \in \mathbb{R} : x \leqslant 0\}$ equipped with the natural order. Examples 2.6(2) show that $(\mathbb{R}^-, \leqslant)$ is a goldspectral set, but it does not satisfy neither the condition (K_1) nor the condition (K_2). Moreover, the ordered set $(\mathbb{R}^-, \geqslant)$ is not goldspectral, since it has no greatest element.

Kaplansky has observed that if $(p_i, i \in I)$ is a totally ordered set of prime ideals of a ring R, then $\bigcup_{i \in I} p_i$ and $\bigcap_{i \in I} p_i$ are also prime ideals. It is not the case for G-ideals, as the following example shows.

EXAMPLE 2.8. – We consider $\mathbb{R}^- - \{-1\}$ equipped with the natural order. The ordered set $(\mathbb{R}^- - \{-1\}, \leqslant)$ is goldspectral (see Examples 2.6(2)). Let $(p_i, i < -1)$ be the family of G-ideals of a ring R corresponding to the elements $i < -1$ in $(\mathbb{R}^- - \{-1\}, \leqslant)$. It is easily seen that both $\bigcup_{i < -1} p_i$ and $\bigcap_{i < -1} p_i$ are not G-ideals.

Particular goldspectral sets may be obtained when the left topology associated to the ordering is goldspectral (*L-goldspectral* for short).

The following result may be proved in much the same way as [9, Théorème 1] or [3, Théorème 3.10].

PROPOSITION 2.9. – *Let $(X, \leqslant)$ be an ordered set, then the following conditions are equivalent:*

(i) *$(X, \leqslant)$ is L-goldspectral.*

(ii) *X satisfies the following properties.*

(a) *There exist finitely many elements $x_1, x_2, \ldots, x_n \in X$, such that $X = \bigcup_{i=1}^{n} (\downarrow x_i)$.*

(b) *For all $x, y \in X$, if $(\downarrow x) \cap (\downarrow y) \neq \emptyset$, then there exist finitely many elements $z_1, \ldots, z_k \in X$, such that $(\downarrow x) \cap (\downarrow y) = \bigcup_{i=1} (\downarrow z_i)$, where $(\downarrow x) = \{y \in X : y \leqslant x\}$.*

Lewis and Ohm have proved in [22] that an ordered disjoint union of spectral sets is spectral. Moreover, Bouacida et al [3, Théorème 3.8] have proved this result in a more general setting using elementary topological facts.

By using the same type of proof that was used for [3, Théorème 3.8], we can state the following.

PROPOSITION 2.10. – *Let* $\{(X_\lambda, \leq_\lambda) : \lambda \in \Lambda\}$ *be a collection of ordered disjoint sets. Let* $X = \bigcup_{\lambda \in \Lambda} X$ *equipped with the ordering* $\leq$ *defined by:* $x \leq y$ *if and only if there exists* $\lambda \in \Lambda$ *such that:* $x, y \in X_\lambda$ *and* $x \leq_\lambda y$. *If, for each* $\lambda \in \Lambda$, $(X_\lambda, \leq_\lambda)$ *is goldspectral, then so is* $(X, \leq)$.

PROOF. – The proof consists in the construction of a goldspectral topology on X which is compatible with the ordering $\leq$. To do this, take $\mathcal{C}_\lambda$ a goldspectral topology on X_λ which is compatible with $\leq_\lambda$ and choose $m \in X_{\lambda_0}$. Consider the topology $\mathcal{C}$ on X, where the open sets are the subsets U of X such that;

– if $m \in U$, $U \cap X_\lambda$ is an open subset of $(X_\lambda, \mathcal{C}_\lambda)$ for finitely many $\lambda \in \Lambda$ and $U \cap X_\lambda = X_\lambda$, otherwise

– if $m \notin U$, $U \cap X_\lambda$ is an open subset of $(X_\lambda, \mathcal{C}_\lambda)$ for each $\lambda \in \Lambda$.

One may check that $\mathcal{C}$ is compatible with $\leq$, $(X, \mathcal{C})$ is quasi-compact and has a basis of quasi-compact open subsets which is stable under finite intersections (see the proof of [3, Théorème 3.8]).

To prove that $(X, \mathcal{C})$ is a T_D-space, it suffices to observe that X_λ is a clopen subset of X, for each $\lambda \neq \lambda_0$ and X_{λ_0} is a closed subset of X which is not open. ■

QUESTION 2.11. – *An analogous question of that of Lewis and Ohm* [22, *Question* 4.4] *is the following: «If* $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ *is an ordered disjoint union such that X is goldspectral, then are the* X_λ *also goldspectral?».*

3. – The Jacobson prime spectrum of a commutative ring.

Recall that a topological space X is said to be a *Jacobson space* if the set X_0 of its closed points is strongly dense in X [13, 0. Proposition 2.8.1].

The proofs of the following two propositions are straightforward.

PROPOSITION 3.1. – *Let X be a topological space and Y a subset of X containing* X_0. *Consider the subset* $\mathrm{Jac}(X) = \{x \in X : \overline{\{x\}} = \overline{\overline{\{x\}} \cap X_0}\}$. *If* X_0 *is a strongly dense subset of* Y, *then* $Y \subseteq \mathrm{Jac}(X)$.

PROPOSITION 3.2. – *Let X be a topological space, then the following properties hold:*

(1) X_0 *is strongly dense in* $\mathrm{Jac}(X)$.

(2) $(\mathrm{Jac}(X))_0 = X_0$, *consequently,* $\mathrm{Jac}(X)$ *is a Jacobson space.*

COROLLARY 3.3. – $\mathrm{Jac}(X)$ *is the largest subset of X in which* X_0 *is strongly dense.*

DEFINITION 3.4. – When X is a topological space, Jac(X) is called the *Jacobson subspace* of X. An element of Jac(X) is called a *Jacobson point* (or *J-point* for short).

PROPOSITION 3.5. – *Let X be a T_0-space which has a basis of quasi-compact open subsets. Then the following statements are equivalent:*

(*i*) *X is a Jacobson space.*

(*ii*) Gold$(X) = X_0$.

PROOF. – (*i*) $\Rightarrow$ (*ii*) On account of [6, Proposition 1.6], Gold(X) is the smallest strongly dense subset of X. On the other hand, X_0 is strongly dense in X, then Gold$(X) \subseteq X_0$. We thus get Gold$(X) = X_0$.

(*ii*) $\Rightarrow$ (*i*) Follows immediately from the fact that Gold(X) is strongly dense in X [6, Proposition 1.6]. ■

PROPOSITION 3.6. – *Let X be a topological space. Then* Gold$(X) \cap$ Jac$(X) = X_0$.

PROOF. – Let $x \in$ Gold$(X) \cap$ Jac(X). Then $\overline{\overline{\{x\}} \cap X_0} = \overline{\{x\}}$ and there exists an open subset U of X such that $\{x\} = \overline{\{x\}} \cap U$. Hence $\{x\} = \overline{\overline{\{x\}} \cap X_0} \cap U$. Consequently, $\overline{\{x\}} \cap X_0 \cap U \neq \emptyset$. Therefore $x \in X_0$. We conclude that Gold$(X) \cap$ Jac$(X) = X_0$. ■

Let R be a ring, we denote by Jac(R) the Jacobson subspace of Spec(R). Following [24], a prime ideal of R lies in Jac(R) if and only if it is the intersection of some maximal ideals of R. By a *jacspectral space*, we mean a topological space homeomorphic to the Jacobson subspace of Spec(R) for some ring R. Jacspectral spaces have been introduced and studied in [6], some examples of such spaces have been given by using foliation theory [6, Proposition 2.30]. The following is a complete characterization of jacspectral spaces.

THEOREM 3.7 [6, Theorem 2.29]. – *Let X be a topological space. Then the following statements are equivalent:*

(*i*) *X is a jacspectral space.*

(*ii*) *X is a quasi-compact Jacobson sober space.*

The proof of Theorem 3.7 is an important consequence of a weak version of the «second continuous extension theorem».

TERMINOLOGIES 3.8. – (1) Each prime ideal in Jac(R) is called a *Jacobson prime ideal* (or *J-ideal* for short).

(2) An ordered set $(X, \leqslant)$ is said to be *jacspectral* if there exists a ring R such that $(X, \leqslant)$ is order isomorphic to (Jac$(R), \subseteq$). Obviously, $(X, \leqslant)$ is jac-

spectral if and only if there exists a jacspectral topology on X which is compatible with the ordering $\leqslant$.

It is easy to check that, if p is a nonmaximal J-ideal of a ring R, then p is contained in infinitely many maximal ideals. An analogous fact may be proved, easily, for any J-point in a topological space.

PROPOSITION 3.9. – *Let X be a topological space and x a non closed point of X. Suppose that x is a J-point, then $\overline{\{x\}}$ contains infinitely many closed points of X.*

REMARKS 3.10. – (1) According to the previous proposition, we claim that if $(X, \leqslant)$ is a jacspectral set, then $(X, \geqslant)$ needs not be jacspectral. It suffices to take a Hilbert domain R with Krull dimension greater than or equal to 1, then $(\mathrm{Spec}(R), \subseteq)$ is jacspectral, however, $(\mathrm{Spec}(R), \supseteq)$ is not (since it has a unique maximal element).

(2) According to Examples 2.6(1) and Proposition 3.6, finite jacspectral ordered sets are exactly finite ordered sets with Krull dimension 0.

Our next concern will be the construction of jacspectral spaces from Jacobson quasi-compact spaces. First, we need some preliminary results.

PROPOSITION 3.11. – (1) *Let $q: X \to Y$ be an injective quasi-homeomorphism, then $Y_0 \subseteq q(X_0)$.*

(2) *Let X be a T_0-space. Then $q(X_0) = ({}^sX)_0$,* where $q: X \to {}^sX$ is the injection of X onto its sobrification ${}^sX$.

(3) *Let $q: X \to Y$ be a quasi-homeomorphism and S a subset of X. Then the following statements are equivalent:*

(i) S is strongly dense in X.

(ii) $q(S)$ is strongly dense in Y.

PROOF. – (1) Straightforward.

(2) Following (1), the proof will be complete, if we show that $q(X_0) \subseteq ({}^sX)_0$.

Let $x \in X_0$, we claim that $\overline{\{q(x)\}} = \{q(x)\}$:

Let $F \in \overline{\{q(x)\}}$, we must prove that $F = q(x) = \overline{\{x\}} = \{x\}$.

• First, we observe that $x \in F$. To see this, suppose that $x \notin F$, then $F \in \widetilde{U}$, where $U = X - \{x\}$. Since $F \in \overline{\{q(x)\}}$; $q(x) \in \widetilde{U}$; hence $q(x) \cap U \neq \emptyset$. This yields $x \in U$, a contradiction. It follows that $x \in F$.

• Let $y \in F$. Suppose that $y \neq x$, then $F \cap U \neq \emptyset$, where $U = X - \{x\}$. The rest of the proof runs as before, proving that $y = x$. Therefore $q(x) \in ({}^sX)_0$.

(3) Straightforward. ■

Next, we derive a useful tool for constructing jacspectral spaces.

COROLLARY 3.12. – *Let X be a T_0-space. Then the following statements are equivalent:*

(*i*) *X is a quasi-compact Jacobson space.*

(*ii*) *${}^sX$ is a jacspectral space.*

PROOF. – Let us start by the following two observations:

• Let $q : X \to {}^sX$ be the injection of X onto its sobrification ${}^sX$. Following Proposition 3.11(2), (3), X is a Jacobson space if and only if so is ${}^sX$.

• The equivalence; X is quasi-compact if and only if so is ${}^sX$, follows immediately from the fact that $q : X \to {}^sX$ is a quasi-homeomorphism.

Therefore, if X is a quasi-compact Jacobson space, then ${}^sX$ is a sober quasi-compact Jacobson space; and according to Theorem 3.7, ${}^sX$ is a jacspectral space. Conversely, if ${}^sX$ is jacspectral, then it is a quasi-compact Jacobson space and the above observations imply that so is X. ■

COROLLARY 3.13. – *Let X be a Noetherian T_0-space. Then ${}^s(\mathrm{Jac}(X))$ is a jacspectral space.*

Combining Theorem 1.8 and Corollary 3.12, we get the two following corollaries.

COROLLARY 3.14. – *Let X, Y be two T_0-spaces and $q : X \to Y$ a quasi-homeomorphism. Then X is a quasi-compact Jacobson space if and only if so is Y.*

COROLLARY 3.15. – *Let X be a T_0-space. Then the following statements are equivalent:*

(*i*) *X is a quasi-compact Jacobson space.*

(*ii*) *X is injected by a quasi-homeomorphism in a jacspectral space.*

REMARK 3.16. – The condition «T_0-space» is essential in Corollary 3.14 and Proposition 3.11(2). It suffices to consider a set $X = \{x, y\}$ with cardinality 2 and $Y = \{a\}$ a one point set. Equip X and Y with the trivial topologies $\mathcal{O}_X = \{\emptyset, X\}$ and $\mathcal{O}_Y = \{\emptyset, Y\}$. Consider the map $q : X \to Y$ defined by $q(x) = q(y) = a$. Then q is easily seen to be a quasi-homeomorphism. However, Y is Jacobson and X is not. We observe also that $X_0 = \emptyset$ and $q_X(X_0) \neq ({}^sX)_0$, where $q_X : X \to {}^sX$ is defined by $q_X(t) = \overline{\{t\}}$.

For the necessity of the condition «T_0-space» in Corollary 3.12, we are aiming to give a «good example».

EXAMPLE 3.17. – A topological space X such that ${}^sX$ is jacspectral and X is not Jacobson.

Let $Y = \mathrm{Spec}(\mathbb{Z})$ equipped with the Zariski topology. Since $\mathbb{Z}$ is a Hilbert domain, Y is jacspectral. Now consider the set $X = \{(0, 0)\} \cup \{(1, n) : n \in \mathbb{N}\} \cup \{(2, n) : n \in \mathbb{N}\}$ equipped with the topology $\mathcal{C}$, where the closed sets are $\emptyset$, X and $\bigcup_{n \in S} \cup \{(1, n), (2, n)\}$, the S are finite subsets of $\mathbb{N}$. Then $(X, \mathcal{C})$ is easily seen to be a Noetherian non T_0-space and $X_0 = \emptyset$. Thus $(X, \mathcal{C})$ is not a Jacobson space. It remains to prove that ${}^sX$ is jacspectral. It will be sufficient to construct a quasi-homeomorphism $q : X \to Y = \mathrm{Spec}(\mathbb{Z})$, and then use Theorem 1.8.

Let $(p_n, n \in \mathbb{N})$ be the increasing sequence of all nonnegative prime integers. Then $\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{p_n\mathbb{Z} : n \in \mathbb{N}\}$. Consider the map $q : X \to Y$ defined by: $q(0, 0) = (0)$, $q(1, n) = q(2, n) = p_n\mathbb{Z}$.

Clearly, q is a quasi-homeomorphism. We conclude, from Theorem 1.8, that ${}^sX$ is homeomorphic to ${}^sY$, hence ${}^sX$ is homeomorphic to Y, and finally that ${}^sX$ is jacspectral. ■

REMARK 3.18. – An analogous concept to that of L-goldspectral sets for jacspectral sets has no interest. For this purpose, let $(X, \leqslant)$ be an ordered set; we say that $(X, \leqslant)$ is left jacspectral (L -jacspectral) if the left topology on $(X, \leqslant)$ is jacspectral. Let $(X, \leqslant)$ be an L-jacspectral set. Then $(X, \leqslant)$ has finitely many maximal elements (since the left topology must be quasi-compact). Consequently, Proposition 3.9 shows that $(X, \leqslant)$ is reduced to $\mathrm{Max}(X, \leqslant)$, that is, the Krull dimension of $(X, \leqslant)$ is 0.

Let R be a ring and $(p_i, i \in I)$ a nonempty totally ordered collection of J-ideals of R. Then, clearly, the intersection $\bigcap_{i \in I} p_i$ is a J-ideal. However, the union $\bigcup_{i \in I} p_i$ needs not be a J-ideal as the following construction shows.

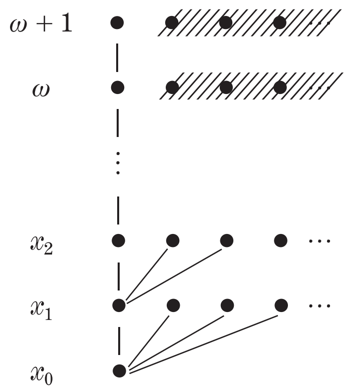
EXAMPLE 3.19. – Let $X = C \cup \left(\bigcup_{n \in \mathbb{N}^*} X_n\right)$, where

$$C = \left\{\left(0, 1 - \frac{1}{n}\right) : n \in \mathbb{N}^*\right\} \cup \{(0, 0), (1, 0)(2, 0)\} \text{ and } X_n = \left\{\left(p, \frac{1}{n}\right) : p \in \mathbb{N}^*\right\}.$$

We denote $x_0 = (0, 0)$, $\omega = (0, 1)$ and $\omega + 1 = (0, 2)$, $x_n = \left(0, 1 - \frac{1}{n}\right)$. Partially order X by declaring that:

- $x_0 \leqslant x$, $x \leqslant x$, $x \leqslant \omega$, $x \leqslant \omega + 1$, for each $x \in X$.
- $\omega \leqslant \omega + 1$
- $x_n \leqslant x_m$ if and only if $n \leqslant m$.
- For each $n < m$, $x_n \leqslant \left(p, \frac{1}{m}\right)$, for each $p \in \mathbb{N}^*$.

Then the ordered set $(X, \leqslant)$ looks like



Equip $(X, \leqslant)$ with the cop-topology $\mathcal{T}$ (that is, the topology which has the collection $\{(x \uparrow): x \in X\}$ as a subbase for closed sets [22], where $(x \uparrow) = \{y \in X : x \leqslant y\}$). Hence $\mathcal{T}$ is compatible with the ordering and $\mathcal{T}$ is easily seen to be quasi-compact.

Since $\{X - (x \uparrow): x \in X\}$ is a subbase of open sets of $(X, \mathcal{T})$, it follows immediately that the collection, $\mathcal{B} = \{\emptyset, X\} \cup \{B_n : n \in \mathbb{N}^*\}$ (where $B_n = \{x_0, x_1, \ldots, x_{n-1}\} \cup ((X_1 \cup X_2 \cup \ldots \cup X_n) - A_n)$, with A_n a finite subset of $X_1 \cup X_2 \cup \ldots \cup X_n$) is a basis of quasi-compact open subsets of $(X, \mathcal{T})$ which is stable under finite intersections.

It is, also, easily seen that each nonempty irreducible closed subset of $(X, \mathcal{T})$ has a generic point. We conclude that $(X, \mathcal{T})$ is a spectral space.

Clearly, $\mathrm{Jac}(X) = \{x \in X : \overline{\{x\}} = \overline{\overline{\{x\}} \cap X_0}\} = X - \{\omega\}$. The increasing sequence $(x_n, n \in \mathbb{N})$ of elements of $(X, \leqslant)$ corresponds to an increasing sequence of J-ideals $(p_n, n \in \mathbb{N})$ of some ring R. The element ω corresponds to a prime ideal p_ω of R such that $p_\omega = \bigcup_{n \in \mathbb{N}} p_n$ and p_ω is not a J-ideal. ∎

We close this section by pointing out that, since the problem of «ordered disjoint union of Lewis and Ohm» for a spectral set or a goldspectral set is still open, we were very surprised at finding the following result.

THEOREM 3.20. – *Let $\{(X_\lambda, \leqslant_\lambda): \lambda \in \Lambda\}$ be a collection of ordered disjoint sets. Let $X = \bigcup_{\lambda \in \Lambda} X$, equipped with the ordering $\leqslant$ defined by: $x \leqslant y$ if and only if there exists $\lambda \in \Lambda$ such that $x, y \in X_\lambda$ and $x \leqslant_\lambda y$. Then the following statements are equivalent.*

(*i*) *$(X, \leqslant)$ is a jacspectral set.*

(*ii*) *For each $\lambda \in \Lambda$, $(X_\lambda, \leqslant_\lambda)$ is a jacspectral set.*

PROOF. – (*i*) ⇒ (*ii*) The proof runs as in Proposition 2.10.

Let $\mathcal{T}_\lambda$ be a jacspectral topology on X_λ which is compatible with the ordering $\leq_\lambda$. We consider the topology $\mathcal{T}$ on X constructed in the proof of (2.10). Then $\mathcal{T}$ is compatible with the ordering $\leq$ and $(X, \mathcal{T})$ is quasi-compact. As in the proof of [3, Théorème 3.8], each nonempty irreducible closed subset of $(X, \mathcal{T})$ has a generic point. Hence $(X, \mathcal{T})$ is sober. What is left is to show that $(X, \mathcal{T})$ is a Jacobson space.

The set of closed points of $(X, \mathcal{T})$ is the set of maximal elements of $(X, \leq)$, and since X is a disjoint ordered union, $\mathrm{Max}(X, \leq) = \bigcup_{\lambda \in \Lambda} \mathrm{Max}(X_\lambda, \leq_\lambda)$. Thus $X_0 = \bigcup_{\lambda \in \Lambda} (X_\lambda)_0$. To see that X_0 is strongly dense in X, it suffices to remark that every nonempty locally closed subset L of $(X, \mathcal{T})$ is a union of locally closed subsets L_λ of X_λ.

$(ii) \Rightarrow (i)$ Let $\mathcal{T}$ be a jacspectral topology on X which is compatible with the ordering $\leq$. We define on X_λ the topology $\mathcal{T}_\lambda$, where the closed sets are: $\emptyset$, X_λ and the closed subsets of $(X, \mathcal{T})$ contained in X_λ. Of course, $\mathcal{T}_\lambda$ is compatible with $\leq_\lambda$ and $(X_\lambda, \mathcal{T}_\lambda)$ is a quasi-compact sober space. The proof will be complete, if we show that $(X_\lambda, \mathcal{T}_\lambda)$ is a Jacobson space.

First, observe that the set of closed points of X_λ is $X_0 \cap X_\lambda$. Let $L \neq X_\lambda$ be a nonempty locally closed subset of X_λ; there exist an open subset O of X_λ, and a closed subset C of X_λ such that $L = O \cap C$. Without loss of generality, we can suppose that $C \neq X_\lambda$. Since $X_\lambda - O$ and C are two closed subsets of X contained in X_λ, $U = X - (X_\lambda - O)$ is an open subset of X and $L = O \cap C = U \cap C$ is a locally closed subset of X. The fact that X_0 is strongly dense in X, yields $L \cap X_0 \neq \emptyset$. Therefore $L \cap (X_\lambda \cap X_0) \neq \emptyset$, that is, $L \cap (X_\lambda)_0 \neq \emptyset$, proving that $(X_\lambda, \mathcal{T}_\lambda)$ is a Jacobson space and that it is jacspectral. ■

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