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# The Poincaré Lemma and Local Embeddability.

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**Sunto.** – *Per varietà CR astratte pseudoconvesse la validità del Lemma di Poincaré per forme di tipo* $(0, 1)$ *implica l'immergibilità locale in* $\mathbb{C}^N$; *le due proprietà sono equivalenti per ipersuperfici di dimensione reale* $\geq 5$. *Come corollario si ottiene un criterio per la non validità del Lemma di Poincaré per forme di tipo* $(0, 1)$ *per una vasta classe di varietà CR astratte di codimensione CR maggiore di uno.*

**Summary.** – *For pseudocomplex abstract CR manifolds, the validity of the Poincaré Lemma for* $(0, 1)$ *forms implies local embeddability in* $\mathbb{C}^N$. *The two properties are equivalent for hypersurfaces of real dimension* $\geq 5$. *As a corollary we obtain a criterion for the non validity of the Poicaré Lemma for* $(0, 1)$ *forms for a large class of abstract CR manifolds of CR codimension larger than one.*

## 1. – Abstract *CR* manifolds.

An abstract *CR* manifold of type $(n, k)$ is a triple $(M, HM, J)$ where $M$ is a smooth real manifold of dimension $2n + k$, $HM$ is a subbundle of rank $2n$ of the tangent bundle $TM$, and $J : HM \to HM$ is a smooth fiber preserving bundle isomorphism with $J^2 = -I$. We also require that $J$ be formally integrable; i.e. that we have

$$[T^{0,1}, T^{0,1}] \subset T^{0,1}$$

where

$$T^{0,1} = \{X + iJX \mid X \in \Gamma(M, HM)\} \subset \Gamma(M, \mathbb{C}TM),$$

with $\Gamma$ denoting smooth sections.

We denote by $H^o M = \{\xi \in T^* M \mid \langle X, \xi \rangle = 0, \forall X \in H_{\pi(\xi)} M\}$ the *characteristic bundle* of $M$. To each $\xi \in H_p^o M$, we associate the Levi form at $\xi$:

$$\mathcal{L}_p(\xi, X) = \xi([J\tilde{X}, \tilde{X}]) = d\tilde{\xi}(X, JX) \qquad \text{for } X \in H_p M$$

which is Hermitian for the complex structure of $H_p M$ defined by $J$. Here $\tilde{\xi}$ is a section of $H^o M$ extending $\xi$ and $\tilde{X}$ a section of $HM$ extending $X$.

We denote by $\bar{\partial}_M$ the tangential Cauchy-Riemann operator on $M$. A smooth function $f$ is called a *CR* function on $M$ if $\bar{\partial}_M f = 0$.

We say that $(M, HM, J)$ is locally $CR$ embeddable at $p \in M$ if there exist $n+k$ smooth complex valued $CR$ functions on a neighborhood of $p$ whose differentials are linearly independent.

We say that the Poincaré lemma for $\overline{\partial}_M$ on $(0, 1)$ forms holds at $p \in M$ if for every open neighborhood $\Omega$ of $p$ and every smooth $(0, 1)$ form $f$ on $\Omega$ with $\overline{\partial}_M f = 0$, there exists an open neighborhood $\omega \subset \Omega$ of $p$ and a smooth function $u$ on $\omega$ satisfying $\overline{\partial}_M u = f$ on $\omega$.

## 2. – Codimension one.

Let $M$ be a smooth abstract $CR$ manifold of type $(n, 1)$ with $n \geqslant 2$, and consider a point $p \in M$.

THEOREM 1. – *Assume $M$ is strictly pseudoconvex at $p$. Then the Poincaré lemma for $\overline{\partial}_M$ on smooth forms of bidegree $(0, 1)$ is valid at $p$ if and only if $M$ is locally $CR$ embeddable at $p$.*

PROOF. – If $M$ is locally $CR$ embeddable at $p$, then since $M$ is strictly pseudoconvex and $\dim_{\mathbb{R}} M \geqslant 5$, the Poincaré lemma for smooth $(0, 1)$ forms is known to be valid by [AH].

For the proof in the other direction, we shall employ the trick of Boutet de Monvel [B] and use some a priori estimates from [AFN].

Choose local coordinates $x_1, x_2, \ldots, x_{2n+1}$ for $M$, centered at $p$, so that $p$ becomes the origin. By the formal Cauchy-Kowalewski procedure, we can find smooth complex valued functions $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{n+1})$ in an open neighborhood $U$ of $0$ with each $\varphi_j(0) = 0$, $d\varphi_1 \wedge d\varphi_2 \wedge \ldots \wedge d\varphi_{n+1} \neq 0$ in $U$, and such that $\overline{\partial}_M \varphi_j$ vanishes to infinite order at 0. Then $\varphi : U \to \mathbb{C}^{n+1}$ gives a smooth local embedding $\widetilde{M} = \varphi(U)$ of $M$ into $\mathbb{C}^{n+1}$. On $\widetilde{M}$ there is the $CR$ structure induced from $\mathbb{C}^{n+1}$; it agrees to infinite order at 0 with the original $CR$ structure on $M$. In particular $\widetilde{M}$ is a smooth real hypersurface in $\mathbb{C}^{n+1}$ which is strictly pseudoconvex with respect to the induced $CR$ structure. This means that after possibly shrinking $U$, there is a smooth complex valued function $h$ in $U$ such that $h(0) = 0$, $\overline{\partial}_M h$ vanishes to infinite order at 0 and

$$(*) \qquad \Re h(x) \geqslant c|x|^2$$

with a positive constant $c$. Indeed it is well known that after a suitable local biholomorphic mapping, $\widetilde{M}$ can be assumed to be strictly convex with $T_o M = \{\Re w_1 = 0\}$ and $H_o M = \{w_1 = 0\}$. It suffices to take $h = \varphi^* w_1$.

Set $w_j^\lambda = \overline{\partial}_M(\varphi_j e^{-\lambda h})$ for $j = 1, \ldots, n+1$ and $\lambda \in \mathbb{N}$. Then each $w_j^\lambda \to 0$ in the topology of $\mathcal{C}^\infty(U)$ as $\lambda \to \infty$. Indeed by $(*)$ the function $\exp\{-\lambda h\}$, and any derivative of it with respect to $x$, is rapidly decreasing as $\lambda \to \infty$, while all other terms, and their derivatives with respect to $x$, have only polynomial growth in $\lambda$.

By our assumption that the Poincaré lemma is valid at $0$, it follows from [AFN; p. 384] that there are open neighborhoods $W \subset V \subset\subset U$ of $0$, a positive integer $l$ and a positive constant $A$ such that:

For every $\lambda \in \mathbb{N}$ there exists $u_j^\lambda \in \mathcal{C}^\infty(W)$ satisfying $\overline{\partial}_M u_j^\lambda = w_j^\lambda$ in $W$ and

$$(**) \qquad \sup_{x \in W} \sup_{|\alpha| \leqslant 1} |D^\alpha u_j^\alpha| \leqslant A \sup_{x \in V} \sup_{|\beta| \leqslant l} \|D^\beta w_j^\lambda\| .$$

Here the $\| \ \|$ on the right hand side indicates a norm on $(0,1)$ forms.

Next we set $\psi_j^\lambda = \varphi_j e^{-\lambda h} - u_j^\lambda$. Then each $\psi_j^\lambda$ is a smooth *CR* function on $W$. Since $\varphi_j(0) = h(0) = 0$, it follows that $\varphi_j$ and $\varphi_j \exp\{-\lambda h\}$ have the same first derivatives at 0. Finally we take $\lambda$ sufficiently large. Then by $(**)$ we have that $d\psi_1^\lambda \wedge d\psi_2^\lambda \wedge \ldots \wedge d\psi_{n+1}^\lambda \neq 0$ on $W$. Thus $\psi = (\psi_1^\lambda, \ldots, \psi_{n+1}^\lambda) : W \to \mathbb{C}^{n+1}$ gives the desired *CR* embedding.

REMARKS. – In Theorem 1 we have $\dim_{\mathbb{R}} M \geqslant 5$. When $\dim_{\mathbb{R}} M \geqslant 7$ the local *CR* embeddability is known to be possible (see [K], [A], [W]). When $\dim_{\mathbb{R}} M \geqslant 5$, and $M$ is *CR* embedded, the Poincaré lemma for $(0,1)$ forms is also known to be valid (see [AH]). For abstract strictly pseudoconvex $M$ there are two open problems in dimension 5: the local *CR* embeddability, and the validity of the Poincaré lemma for $(0,1)$ forms. By Theorem 1, these two problems are equivalent. When $\dim_{\mathbb{R}} M = 3$, one cannot always locally *CR* embed by [Ni], and even if $M$ is *CR* embedded, the Poincaré lemma for $(0,1)$ forms fails by [AH].

We note that the two equivalent conditions in Theorem 1 are, in fact, equivalent to a third apparently weaker condition: namely that the range of $\overline{\partial}_M$ on functions is «closed». By this we mean that given any open neighborhood $\Omega$ of $p$, and any sequence $\{f_n\}_n$ of smooth functions on $\Omega$ such that $\overline{\partial}_M f_n$ converges to $g$ in $\mathcal{C}^\infty(\Omega)$, there exists an open neighborhood $\omega$ of $p$ such that $g$ is $\overline{\partial}_M$ exact on $\omega$. Indeed this condition is sufficient to prove the local *CR* embeddability at $p$, since one still has $(**)$ by [N].

Without strict pseudoconvexity, however, this «closed range property» is no longer sufficient to obtain the local *CR* embeddability. Indeed it follows from the subelliptic estimates for functions on abstract 1-concave *CR* manifolds proved in [HN1] that one has this closed range property. However, there are examples of 1-concave and even 2-concave *CR* manifolds of type $(n, 1)$ which cannot be locally *CR* embedded (see [JT], [R]).

## 3. – Higher codimension.

Let $M$ be a smooth abstract *CR* manifold of type $(n, k)$ with $n \geqslant 1$ and $k \geqslant 1$, and consider a point $p \in M$.

THEOREM 2. – *Assume there is a* $\xi \in H^o_p M$, $\xi \neq 0$, *with* $\mathcal{L}_p(\xi, \cdot)$ *positive definite. If the Poincaré lemma for* $\bar{\partial}_M$ *on forms of bidegree* $(0,1)$ *is valid at* $p$ *then* $M$ *is locally CR embeddable at* $p$.

PROOF. – The proof is essentially the same as for Theorem 1, with $\mathbb{C}^{n+1}$ replaced by $\mathbb{C}^{n+k}$, and with $j = 1, 2, \ldots, n+k$. The first crucial point is to have the estimate $(**)$; fortunately we have it again from [AFN; p. 384]. The second crucial point is to observe that the approximate $CR$ embedding $\widetilde{M}$ in $\mathbb{C}^{n+k}$, which now has real codimension $k$, is contained in a strictly pseudoconvex hypersurface. Indeed for $\xi \in H^o_p M = H^o_0 \widetilde{M}$ we have that the Levi form $\widetilde{\mathcal{L}}_0(\xi, \cdot)$ of $\widetilde{M}$ is positive definite, and we may choose smooth real local defining functions $\varrho_1, \varrho_2, \ldots, \varrho_k$ for $\widetilde{M}$ near the origin in $\mathbb{C}^{n+k}$, with $d\varrho_1(0), d\varrho_2(0), \ldots, d\varrho_k(0)$ orthonormal with respect to the standard Euclidean metric, such that locally

$$\widetilde{M} = \{\varrho_j(z) = 0 \,|\, j = 1, \ldots, k\}$$

and $d\varrho_1(0) = \tilde{J}^* \xi$, where $\tilde{J}^* = -\tilde{J}$ is the adjoint of the complex structure tensor in $\mathbb{C}^{n+k}$. Then replacing the $\varrho_j$ by

$$\tilde{\varrho}_1 = \varrho_1 \pm B \sum_{j=2}^{k} \varrho_j^2$$

$$\tilde{\varrho}_j = \varrho_j \quad \text{for } 2 \leq j \leq k$$

with a large positive constant $B$, and choosing the appropriate sign, we obtain a strictly pseudoconvex real hypersurface $\Sigma$, defined by $\Sigma = \{\tilde{\varrho}_1(z) = 0\}$, on which $\widetilde{M}$ lies. As before this gives us the existence of a smooth function $h$ such that $\bar{\partial}_M h$ vanishes to infinite order at the origin, and satisfies $(*)$, and the proof of Theorem 2 is complete.

COROLLARY. – *Suppose*

(a) *There exists* $\xi \in H^o_p M$ *with* $\mathcal{L}_p(\xi, \cdot)$ *having one negative eigenvalue, and all other eigenvalues greater than or equal to zero;*

*and*

(b) *There exists* $\eta \in H^o_p M$ *with* $\mathcal{L}_p(\eta, \cdot)$ *positive definite.*

*Then the Poincaré lemma for* $\bar{\partial}_M$ *fails for* $(0,1)$ *forms at* $p$.

PROOF. – Assume that the Poincaré lemma for $(0,1)$ forms is valid at $p$. Then by virtue of (b) and Theorem 2, $M$ is locally $CR$ embeddable at $p$. But then the Poincaré lemma for $(0,1)$ forms fails, using the results of [HN2]; this is a contradiction, completing the proof.

REMARKS. – Note that if $M$ is assumed a priori to be $CR$ embedded at $p$, we have the result of the Corollary dropping the hypothesis (b). If in this situation the remaining hypothesis (a) is strengthened to say that all the other eigenvalues are greater than zero, then the result already follows from [AFN].

Note that generically, when $k > 1$, we expect (a) to be a consequence of (b). Indeed, if $\mathcal{L}_p(\eta, \cdot)$ has $n$ positive eigenvalues, then $\mathcal{L}_p(-\eta, \cdot)$ has $n$ negative eigenvalues. Hence if we move along a continuous path $\gamma(t)$, $0 \leqslant t \leqslant 1$ from $\gamma(0) = \eta$ to $\gamma(1) = -\eta$ on the sphere $S^{k-1}$, all eigenvalues of the Levi form change sign. Since they are continuous functions of $t$, there are points where the determinant of the matrix of the Levi form vanishes. If at the first of these points, say $t_0$, the determinant of the Levi form has a simple zero, then at a nearby $\xi \in \gamma([0, 1])$, corresponding to a $t > t_0$ sufficiently close to $t_0$, (a) is satisfied.

Thus, generically we expect, by [HN2], that a locally embeddable $M$, of type $(n, k)$ for some $k \geqslant 2$, and satisfying (b), does not admit the Poincaré lemma for $(0, 1)$ forms. In the same situation, for an abstract $M$, the Poincaré lemma was known in general to fail for either $(0, 1)$ forms or for $(0, 2)$ forms (see [N]). Thus the Corollary above somewhat precises and improves, in the special case of the tangential Cauchy-Riemann complex, a result of [N].

When $M$ is abstract and of type $(1, k)$ then $\bar{\partial}_M u = f$ becomes a scalar equation $Lu = f$, and the hypotheses (a) and (b) in the Corollary are equivalent and both can be restated by saying that $L, \bar{L}, [L, \bar{L}]$ are linearly independent at $p$. Hence one obtains a more geometric explanation of (a special case of) Hörmander's nonsolvability result [Hö].

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