Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **6-B** (2003), n.2, p. 353–379.
Unione Matematica Italiana
<http://www.bdim.eu/item?id=BUMI_2003_8_6B_2_353_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2003.

Bollettino U. M. I.
(8) **6**-B (2003), 353-379

Restricting Cuspidal Representations of the Group of Automorphisms of a Homogeneous Tree.

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Sunto. – *Sia $\mathfrak{X}$ un albero omogeneo dove a ogni vertice si incontrano $q+1$ ($q \geqslant 2$) spigoli. Sia $\mathcal{A} = \mathrm{Aut}(\mathfrak{X})$ il gruppo di automorfismi di $\mathfrak{X}$ e H un sottogruppo chiuso isomorfo a $PGL(2, F)$ (F campo locale il cui campo residuo ha ordine q). Sia π una rappresentazione continua unitaria e irriducibile di $\mathcal{A}$ e si consideri π_H, la sua restrizione ad H. È noto che se π è una rappresentazione sferica o speciale π_H rimane irriducibile. In questo lavoro si mostra che quando π è cuspidale la situazione è molto più complessa. Si studia in dettaglio il caso in cui il sottoalbero minimale associato a π sia il più piccolo possibile, ottenendo una esplicita decomposizione di π_H.*

Summary. – *Let $\mathfrak{X}$ be a homogeneous tree in which every vertex lies on $q+1$ edges, where $q \geqslant 2$. Let $\mathcal{A} = \mathrm{Aut}(\mathfrak{X})$ be the group of automorphisms of $\mathfrak{X}$, and let H be the its subgroup $PGL(2, F)$, where F is a local field whose residual field has order q. We consider the restriction to H of a continuous irreducible unitary representation π of $\mathcal{A}$. When π is spherical or special, it was well known that π remains irreducible, but we show that when π is cuspidal, the situation is much more complicated. We then study in detail what happens when the minimal subtree of π is the smallest possible.*

1. – Introduction.

Continuing the notation in the abstract, $\mathcal{A}$ is a locally compact totally disconnected unimodular topological group with the topology of pointwise convergence. Fix a vertex o of $\mathfrak{X}$ and a vertex o' adjacent to o. A classification of the irreducible continuous unitary representations π of $\mathcal{A}$ was given by Ol'shanskii [11, 12], and is described in [4], the notation of which we shall basically be following. They are parametrized by (orbits of) finite complete subtrees $\mathfrak{x}$ of $\mathfrak{X}$ (a subtree $\mathfrak{x}$ is *complete* if for every vertex v of $\mathfrak{x}$ not in the boundary of $\mathfrak{x}$, all of the $q+1$ neighbours of v are also in $\mathfrak{x}$). For such a subtree, let $K(\mathfrak{x})$ denote the compact group of $g \in \mathcal{A}$ for which $gv = v$ for all vertices v of $\mathfrak{x}$, and let $\widetilde{K}(\mathfrak{x}) = \{g \in \mathcal{A} : g\mathfrak{x} = \mathfrak{x}\}$. We write $K_o = \{g \in \mathcal{A} : go = o\} = K(\{o\})$. If π has non-zero $K(\mathfrak{x})$-fixed vectors, but no non-zero $K(\mathfrak{y})$-fixed vectors for any finite complete subtree $\mathfrak{y}$ with fewer vertices than $\mathfrak{x}$, we call $\mathfrak{x}$ a *minimal sub-*

tree for π. If $\mathfrak{x}$ is a minimal subtree for π, then so is $g\mathfrak{x}$ for any $g \in \mathcal{A}$. If π has a minimal subtree with only one vertex, which we may assume is o, then π is called *spherical.* If π has a minimal subtree with exactly 2 vertices, which we may assume are o and o', then π is called *special.* If π has a larger minimal subtree $\mathfrak{x}$, i.e., $\operatorname{diam}(\mathfrak{x}) \geqslant 2$, then π is called *cuspidal.* These are obtained by induction from $\widetilde{K}(\mathfrak{x})$ to $\mathcal{A}$ of irreducible representations σ of $\widetilde{K}(\mathfrak{x})$ which are trivial on $K(\mathfrak{x})$ and which have no non-zero $K(\mathfrak{y})$-fixed vectors for any of the maximal proper complete subtrees $\mathfrak{y}$ of $\mathfrak{x}$ (note that $K(\mathfrak{x}) \subset K(\mathfrak{y}) \subset \widetilde{K}(\mathfrak{x})$ for such a $\mathfrak{y}$). The set of equivalence classes of these «standard» representations of $\widetilde{K}(\mathfrak{x})$ is denoted $(\widetilde{K}(\mathfrak{x}))\widehat{\ }_0$. Because any automorphism of $\mathfrak{x}$ can be extended to an automorphism of $\mathfrak{X}$, the map $g \mapsto g_{|\mathfrak{x}}$ induces an isomorphism $\widetilde{K}(\mathfrak{x})/K(\mathfrak{x}) \cong \operatorname{Aut}(\mathfrak{x})$, and so the representations of $\widetilde{K}(\mathfrak{x})$ satisfying the above conditions correspond to certain irreducible representations of $\operatorname{Aut}(\mathfrak{x})$, which we also refer to as *standard.*

Note that in Ol'shanskii's papers, the representations classified were the algebraically irreducible admissible ones. If π is a cuspidal irreducible continuous unitary representation on a Hilbert space $\mathcal{H}_\pi$, let V_π denote the space of vectors $\xi \in \mathcal{H}_\pi$ which are $K(\mathfrak{y})$-invariant for some finite complete subtree $\mathfrak{y}$. This a dense invariant subspace of $\mathcal{H}_\pi$. Let $\pi^\circ \colon \mathcal{A} \to GL(V_\pi)$ be the representation of $\mathcal{A}$ obtained from π. Then π° is admissible and algebraically irreducible [4, p. 115]. Conversely, if $\pi' \colon \mathcal{A} \to GL(V)$ is an admissible and algebraically irreducible representation of $\mathcal{A}$, which has minimal subtree of diameter at least 2, then π' is unitarizable [12, § 2.6], and extends to irreducible continuous unitary representation.

Let F be a commutative non-archimedean local field. Let $\operatorname{ord} \colon \mathrm{F} \to \mathbb{Z} \cup \{\infty\}$ be the valuation on F. Let $\mathfrak{O} = \{x \in F : \operatorname{ord}(x) \geqslant 0\}$ be the valuation ring of F, and let $\varpi \in \mathfrak{O}$ be an element of valuation 1. Let $\mathfrak{O}^\times = \{x \in \mathfrak{O} : \operatorname{ord}(x) = 0\}$ denote the group of invertible elements of the ring $\mathfrak{O}$. Let q be the order of the residual field $\mathfrak{O}/\varpi\mathfrak{O}$, which equals p^r for some prime p and some integer $r \geqslant 1$. Let $A \subset \mathfrak{O}$ be a set of q elements, one of them 0, such that the canonical map $\mathfrak{O} \to \mathfrak{O}/\varpi\mathfrak{O}$, restricted to A, is a bijection. Each element of $\mathfrak{O}$ is expressible uniquely as the sum of a series $a_0 + a_1\varpi + a_2\varpi^2 + \ldots$, where each a_i is in A.

Recall the construction of the Bruhat-Tits tree $\mathfrak{X}$ associated with $G = GL(2, F)$ [16, p. 69; 4, p. 127]. Let $V = F^2$ denote the space of all column vectors of length 2 with entries in F. A lattice in V is a subset of V of the form $\{t_1 v_1 + t_2 v_2 \colon t_1, t_2 \in \mathfrak{O}\}$, where $\{v_1, v_2\}$ is a basis of V over F. If $\{v_1, v_2\}$ is the usual basis of V, then the corresponding lattice is $\mathfrak{O}^2$, and is denoted L_0. If L is a lattice and if $g \in G$, then $g(L)$ is a lattice, and so G acts on the set of lattices. This action is clearly transitive, and the stabilizer of L_0 is the group $K = GL(2, \mathfrak{O})$ of matrices with entries in $\mathfrak{O}$ and having determinant in $\mathfrak{O}^\times$. Two lattices L, L' are called equivalent if $L' = \lambda L$ for some $\lambda \in F^\times$. Let $[L]$ denote the equivalence class of the lattice L. The Bruhat-Tits tree $\mathfrak{X}$ has as vertex set the set of equivalence classes of lattices. Two distinct lattice classes $[L]$ and $[L']$ are adjacent if representative lattices L and L' can be found such that $\varpi L \subsetneqq L' \subsetneqq L$. The tree $\mathfrak{X}$ is homogeneous of degree $q+1$.

The above action of G on $\mathfrak{X}$ gives a homomorphism $\varphi : G \to \mathcal{A}$ with kernel $Z = \{\lambda I : \lambda \in F^\times\}$. We write H for the image of φ. Thus $PGL(2, F) \cong H \leqslant \mathcal{A}$. It is natural to ask how the irreducible unitary representations π of $\mathcal{A}$ behave when restricted to H. When π is spherical or special, the restriction is known to remain irreducible [4, p. 117]. We are concerned here only with the cuspidal case.

We identify H and $PGL(2, F)$ throughout. The representations of H correspond to, and are here frequently identified with, representations of G which are trivial on Z. Everything we shall need about the representations of G is contained in Bump's book [1].

Let π be an irreducible unitary representation of $\mathcal{A}$ with minimal subtree $\mathfrak{x}$, where $\mathrm{diam}(\mathfrak{x}) \geqslant 2$. In Section 2 we prove some general results, showing in particular that the restriction of π to H is a direct sum of induced representations. Then in Sections 3 and 4 we discuss in detail the case when $\mathfrak{x}$ is as small as possible: o together with its $q+1$ neighbours. Except for the one example with $q = 2$, the restriction of π to H is then never irreducible, and we give for it an explicit decomposition as a direct integral of irreducible representations.

We thank Tim Steger for useful conversations on the subject of this paper.

2. – Restricting cuspidal representations to $PGL(2, F)$.

Let $\mathcal{A}$, G, K, Z, $\varphi : G \to \mathcal{A}$ and $H \cong G/Z = PGL(2, F)$ be as above.

Now let $\mathfrak{x}$ be a finite complete subtree of $\mathfrak{X}$, with $\mathrm{diam}(\mathfrak{x}) \geqslant 2$. Let $\sigma \in (\widetilde{K}(\mathfrak{x}))\hat{}_0$ have representation space $\mathcal{H}_\sigma$ (finite dimensional, of course). Let $\pi = Ind_{\widetilde{K}(\mathfrak{x})}^{\mathcal{A}} \sigma$. Because we are inducing from an open subgroup, the definition of an induced representation is particularly simple here. Counting measure on the discrete set $\mathcal{A}/\widetilde{K}(\mathfrak{x})$ is an invariant measure, and so the representation space of π is the space $\mathcal{H}_\pi$ of functions $f : \mathcal{A} \to \mathcal{H}_\sigma$ such that

$$(i) \quad f(kg) = \sigma(k)(f(g)) \quad \text{for all } g \in \mathcal{A} \text{ and } k \in \widetilde{K}(\mathfrak{x}), \text{ and}$$

$$(ii) \quad \sum_\alpha \|f(g_\alpha)\|^2 < \infty,$$

and we define $\|f\|$ to be the square root of the sum in (ii). Here $\{g_\alpha\}$ is any set of coset representatives for $\widetilde{K}(\mathfrak{x})$ in $\mathcal{A}$. Notice that we do not have to add measurability conditions, because any $f \in \mathcal{H}_\pi$ is left $K(\mathfrak{x})$-invariant, and therefore is locally constant. For $g \in \mathcal{A}$, the action of $\pi(g)$ on $f \in \mathcal{H}_\pi$ is right translation:

$(\pi(g)f)(g') = f(g'g)$. Because $\tilde{K}(\mathfrak{x})$ is also compact, if $f \in \mathcal{H}_\pi$, then the integral of $\|f(g)\|^2$ over $\mathcal{A}$ with respect to a Haar measure m is $m(\tilde{K}(\mathfrak{x}))$ times the sum in (ii) above.

Notice that in [4], the induced representation is defined so that $\mathcal{H}_\pi$ consists of functions satisfying $f(gk) = \sigma(k^{-1})(f(g))$ for all $g \in \mathcal{A}$ and $k \in \tilde{K}(\mathfrak{x})$, with $\pi(g)$ being left translation. The intertwining operator $f \mapsto \check{f}$, where $\check{f}(x) = f(x^{-1})$, shows that the two definitions give equivalent representations.

The algebraically irreducible admissible representation π° corresponding to π is just the representation obtained from σ by compact induction (see, for example, [1, p. 470]). To see this, let $f \in V_\pi$, the representation space of π°. Then $f \in \mathcal{H}_\pi$ is right $K(\mathfrak{y})$-invariant for some finite complete subtree $\mathfrak{y}$ of $\mathfrak{X}$. So for any $\xi \in \mathcal{H}_\sigma$, the function $g \mapsto \langle \check{f}(g), \xi \rangle$ is in $\mathcal{S}(\mathfrak{x})$ (see [4, p. 87]) and is left $K(\mathfrak{y})$-invariant, and so [4, Prop. III.3.2] is supported on the compact set $\{g \in \mathcal{A} : g\mathfrak{x} \subset \mathfrak{y}\}$, which is a finite union of cosets $g\tilde{K}(\mathfrak{x})$. Hence f is supported on a finite union of cosets $\tilde{K}(\mathfrak{x})\, g$. Conversely, if $f \in \mathcal{H}_\pi$ is supported on the union of cosets $\tilde{K}(\mathfrak{x})\, g_j$, $j = 1, \ldots, r$, choose a finite complete subtree $\mathfrak{y}$ containing the union of the trees $g_j^{-1}\mathfrak{x}$. If $k \in K(\mathfrak{y})$, then $g_j k g_j^{-1} \in K(\mathfrak{x})$ and so $\sigma(g_j g g_j^{-1}) = I$ for each j. It follows that f is right $K(\mathfrak{y})$-invariant, and so in V_π.

We start with two quite general results. In the first one, the hypotheses $\operatorname{diam}(\mathfrak{x}) \geq 2$ and $\sigma \in (\tilde{K}(\mathfrak{x}))\hat{}_0$ are not needed.

Proposition 2.1. – *Let $\mathfrak{x}$ be a finite complete subtree of $\mathfrak{X}$ satisfying* $\operatorname{diam}(\mathfrak{x}) \geq 2$. *Then $\mathcal{A}$ is a finite disjoint union*

$$\mathcal{A} = \bigcup_{j=1}^{r} \tilde{K}(\mathfrak{x})\, g_j H, \tag{2.1}$$

of double cosets $\tilde{K}(\mathfrak{x})\, gH$. Let $\sigma \in (\tilde{K}(\mathfrak{x}))\hat{}_0$, and let $\pi = \operatorname{Ind}_{\tilde{K}(\mathfrak{x})}^{\mathcal{A}} \sigma$. Then the restriction of π to H is unitarily equivalent to the representation

$$\bigoplus_{j=1}^{r} \operatorname{Ind}_{g_j^{-1}\tilde{K}(\mathfrak{x})\, g_j \cap H}^{H} \sigma_j, \tag{2.2}$$

where $\sigma_j(k) = \sigma(g_j k g_j^{-1})$ for $k \in g_j^{-1}\tilde{K}(\mathfrak{x})\, g_j \cap H$. In particular, if $r \geq 2$, then this restriction is reducible.

Proof. – Fix any vertex $v_0 \in \mathfrak{x}$. There are only finitely many subtrees of $\mathfrak{X}$ containing v_0 and of the form $g(\mathfrak{x})$ for some $g \in \mathcal{A}$. Write these $\gamma_j(\mathfrak{x})$, $j = 1, \ldots, m$. If $g \in \mathcal{A}$, then as H acts transitively on the vertices of $\mathfrak{X}$, there is an $h \in H$ such that $g^{-1}(v_0) = h^{-1}(v_0)$. Thus $hg^{-1}(v_0) = v_0$, so that $hg^{-1}(\mathfrak{x}) = \gamma_j(\mathfrak{x})$ for some j. Thus $g \in \tilde{K}(\mathfrak{x})\, \gamma_j^{-1} h$. Hence there are only finitely many distinct double cosets $\tilde{K}(\mathfrak{x})\, gH$, so that (2.1) holds for some $g_1, \ldots, g_r$.

We may write $\tilde{K}(\mathfrak{x})\, g_j H$ as a union of disjoint cosets $\tilde{K}(\mathfrak{x})\, g_j h_{j,\nu}$, where the $h_{j,\nu}$'s are in H. Hence, for each j, H is the union of the disjoint cosets

$(g_j^{-1}\widetilde{K}(\mathfrak{x})\, g_j \cap H)\, h_{j,\nu}$. Let $\mathcal{H}_\pi$ be defined as above, and let $\mathfrak{H}$ denote the representation space of the representation (2.2), i.e., the space of r-tuples $(f_1, \ldots, f_r)$ of functions $f_j \colon H \to \mathcal{H}_\sigma$ which satisfy

(i) $f_j(kh) = \sigma(g_j k g_j^{-1})(f_j(h))$ for all $h \in H$ and $k \in g_j^{-1}\widetilde{K}(\mathfrak{x})\, g_j \cap H$, and

(ii) $\sum_{j,\nu} \|f_j(h_{j,\nu})\|^2 < \infty$.

Given $F \in \mathcal{H}_\pi$, let $f_j(h) = F(g_j h)$ for $h \in H$. It is clear that the map $T : F \mapsto (f_1, \ldots, f_r)$ is an isometry $\mathcal{H}_\pi \to \mathfrak{H}$. Moreover, this map is surjective, because if $(f_1, \ldots, f_r) \in \mathfrak{H}$, then we may define $F \in \mathcal{H}_\pi$ by setting $F(kg_j h) = \sigma(k)(f_j(h))$ for all $h \in H$, $k \in \widetilde{K}(\mathfrak{x})$, and all j. It is routine to check that F is well-defined and that $(f_1, \ldots, f_r) = T(F)$. ■

Let π be as in Proposition 2.1. The following result, while not used in the sequel, is of interest because it guarantees that any irreducible subrepresentation of the restriction π_H of π to H occurs with only finite multiplicity. Since π_H is still square integrable as a representation of H, standard arguments show that it is a subrepresentation of the sum of infinitely many copies of the left regular representation λ_H of H. In fact, we can show more:

PROPOSITION 2.2. – *Let π be as in Proposition 2.1. Then for some $n < \infty$, the restriction to H of π is contained in the sum $n\lambda_H$ of n copies of the left regular representation of H.*

PROOF. – Let $\mathcal{H}_\pi$ be the representation space of π and let M be the space of $K(\mathfrak{x})$-fixed vectors in $\mathcal{H}_\pi$. Notice that if $f_1 \in M$ and $k \in \widetilde{K}(\mathfrak{x})$, then $\pi(k)\, f_1 \in M$ because $K(\mathfrak{x})$ is normal in $\widetilde{K}(\mathfrak{x})$. Let $g_1, \ldots, g_r$ be as in (2.1). Suppose that $\langle f, \pi(h)\, \pi(g_j^{-1})\, f_1 \rangle = 0$ for all $f_1 \in M$, all $h \in H$ and all j. Then $f = 0$. To see this, pick any $f_0 \in M \setminus \{0\}$. For if $g \in \mathcal{G}$, we can write $g = hg_j^{-1}k$ for some j, and some $h \in H$ and $k \in \widetilde{K}(\mathfrak{x})$. Then $\langle f, \pi(g)\, f_0 \rangle = \langle f, \pi(h)\, \pi(g_j^{-1})(\pi(k)\, f_0) \rangle = 0$. But f_0 is a cyclic vector for π, and so $f = 0$.

Now M is finite dimensional because $M \subset V_\pi$ and π° is admissible (cf. [4, p. 112]). Let M' be the sum of the subspaces $\pi(g_j^{-1})\, M$, $j = 1, \ldots, r$. Let $f_1, \ldots, f_n$ be any basis of M'. For each i, let $(T_i f)(h) = \langle f, \pi(h)\, f_i \rangle$. Then $T_i f \in L^2(H)$ by [4, Lemma 3.12]. Define $T : \mathcal{H}_\pi \to L^2(H) \oplus \ldots \oplus L^2(H)$ (n copies) by $Tf = (T_1 f, \ldots, T_n f)$. It is easily checked that T intertwines π and $n\lambda_H$. Moreover, T has kernel $\{0\}$ by the first paragraph of this proof. Now $T^* T : \mathcal{H}_\pi \to \mathcal{H}_\pi$ must intertwine π with itself, and so be cI for some $c \geqslant 0$. As T is injective, we have $c \neq 0$. Hence $c^{-1/2}T$ is an isometry embedding $\mathcal{H}_\pi$ in $L^2(H) \oplus \ldots \oplus L^2(H)$ and intertwining π and $n\lambda_H$. ■

We shall henceforth only be concerned with the case when the r in (2.1) is 1. In this case, Proposition 2.1 takes the following simpler form:

COROLLARY 2.3. – *Let* $\mathfrak{x}$ *be a finite complete subtree of* $\mathfrak{X}$ *satisfying* $\operatorname{diam}(\mathfrak{x}) \geq 2$ *for which*

$$\mathcal{A} = \widetilde{K}(\mathfrak{x})\, H\,. \tag{2.3}$$

Let $\sigma \in (\widetilde{K}(\mathfrak{x}))\widehat{}_0$, *and let* $\pi = \operatorname{Ind}_{\widetilde{K}(\mathfrak{x})}^{\mathcal{A}} \sigma$. *Then the restriction of* π *to* H *is unitarily equivalent to the representation*

$$\operatorname{Ind}_{\widetilde{K}(\mathfrak{x}) \cap H}^{H} \sigma_{\,|\,\widetilde{K}(\mathfrak{x}) \cap H} \tag{2.4}$$

obtained by inducing from $\widetilde{K}(\mathfrak{x}) \cap H$ *to* H *the restriction of* σ *to* $\widetilde{K}(\mathfrak{x}) \cap H$.

Notice that the hypothesis (2.3) is satisfied by $\mathfrak{x} = \mathfrak{x}_n = \{v \in \mathfrak{X} : d(v, o) \leq n\}$, because $\widetilde{K}(\mathfrak{x}_n) = K_o$, and (2.3) holds because H acts transitively on the set of vertices of $\mathfrak{X}$.

Another example in which (2.3) holds is $\mathfrak{x} = \mathfrak{x}_n'$, the subtree whose vertices are those at distance at most n from o or o' (recall that o' is a neighbour of o). Here $n \geq 1$. Clearly $\widetilde{K}(\mathfrak{x}_n') = \{g \in \mathcal{A} : g\{o, o'\} = \{o, o'\}\}$ for any n. Since G acts transitively on the vertices of $\mathfrak{X}$, (2.3) holds because $K = GL(2, \mathfrak{O})$ acts transitively on the set of neighbours of o. See the beginning of the next section.

Here is an example for which (2.3) is not true, i.e., $r > 1$ in (2.1). Assume that $q \geq 4$, and let $x_1, \ldots, x_5$ be 5 distinct neighbours of o. For each j, let v_j be a vertex at distance $j + 1$ from o such that x_j is on the geodesic from o to v_j. Let $\mathfrak{x}$ be the smallest complete subtree having all the vertices v_j as interior points. Choose a $g \in K_o$ which interchanges x_1 and x_2, but leaves the other neighbours of o fixed. Then any $h \in G$ which satisfies $g\mathfrak{x} = h\mathfrak{x}$ must interchange x_1 and x_2, and fix x_3, x_4 and x_5. But an $h \in G$ which fixes three neighbours of o must fix them all by Lemma 3.1 below.

In the context of Corollary 2.3, it is convenient to work with representations of $G = GL(2, F)$ instead of H, and so we transfer the last lemma to that setting:

LEMMA 2.4. – *With notation and hypotheses of Corollary* 2.3, *the representation of G obtained from the representation* (2.4) *of* H *by composing with* $\varphi : G \to H$ *is*

$$\operatorname{Ind}_{\widetilde{K}_G(\mathfrak{x})}^{G} \sigma'\,, \tag{2.5}$$

where $\widetilde{K}_G(\mathfrak{x}) = \{g \in G : \varphi(g) \in \widetilde{K}(\mathfrak{x})\}$ *and where* σ' *is the representation of*

$\widetilde{K}_G(\mathfrak{x})$ *obtained from* $\sigma_{|\widetilde{K}(\mathfrak{x})\cap H}$ *by composing with the restriction of* φ *to* $\widetilde{K}_G(\mathfrak{x})$.

PROOF. – Write G as a disjoint union of cosets $\widetilde{K}_G(\mathfrak{x})\, g_a$. Then H is the disjoint union of the cosets $(\widetilde{K}(\mathfrak{x}) \cap H)\, \varphi(g_a)$. It is easy to see that $f \mapsto f \circ \varphi$ is an isometric isomorphism from the representation space $\mathfrak{H}$ of the representation (2.4) to that of (2.5). ■

Let $K_G(\mathfrak{x}) = \{g \in G : \varphi(g) \in K(\mathfrak{x})\}$. Then φ induces an embedding

$$\widetilde{K}_G(\mathfrak{x})/K_G(\mathfrak{x}) \hookrightarrow \widetilde{K}(\mathfrak{x})/K(\mathfrak{x}) \cong \mathrm{Aut}(\mathfrak{x}), \tag{2.6}$$

and σ' corresponds to a representation of $\widetilde{K}_G(\mathfrak{x})/K_G(\mathfrak{x})$, obtained by restricting the irreducible standard representation σ of $\mathrm{Aut}(\mathfrak{x})$. So σ' will in general be a finite sum

$$\sigma' = \sigma_1' + \ldots + \sigma_m' \tag{2.7}$$

of irreducible representations of $\widetilde{K}_G(\mathfrak{x})/K_G(\mathfrak{x})$. Thus (2.5) will be the sum of the corresponding induced representations.

Obtaining the decomposition (2.7) is a non-trivial problem in the representation theory of the finite group $\mathrm{Aut}(\mathfrak{x})$, even for the simplest of $\mathfrak{x}$'s.

3. – The case $\mathfrak{x} = \mathfrak{x}_1$.

Recall that A denotes a set of q elements in $\mathfrak{O}$ containing 0 such that the map $a \mapsto a + \varpi\mathfrak{O}$ is a bijection $A \to \mathfrak{O}/\varpi\mathfrak{O}$. The neighbours of $o = [L_0]$ are the vertices $[g_\infty L_0]$ and $[g_a L_0]$, $a \in A$, where

$$g_\infty = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix} \quad \text{and} \quad g_a = \begin{bmatrix} \varpi & a \\ 0 & 1 \end{bmatrix}, \tag{3.1}$$

Clearly $\widetilde{K}_G(\mathfrak{x}_1) = ZK = Z \cdot GL(2, \mathfrak{O})$, and it is easy to see that $K_G(\mathfrak{x}_1)$ equals

$$\{\lambda(I + \varpi M) : \lambda \in F^\times \text{ and } M \in M_{2\times 2}(\mathfrak{O})\},$$

where $M_{2\times 2}(\mathfrak{O})$ is the space of 2×2 matrices with entries in $\mathfrak{O}$. Since $\{I + \varpi M : M \in M_{2\times 2}(\mathfrak{O})\}$ is the kernel of the natural map $GL(2, \mathfrak{O}) \to GL(2, \mathfrak{O}/\varpi\mathfrak{O})$, we see that $\widetilde{K}_G(\mathfrak{x}_1)/K_G(\mathfrak{x}_1) \cong PGL(2, \mathbb{F}_q)$, where $\mathbb{F}_q \cong \mathfrak{O}/\varpi\mathfrak{O}$ is the field with q elements. Thus the σ_j''s in (2.7) can be thought of as representations of $PGL(2, \mathbb{F}_q)$. The map $GL(2, \mathfrak{O}) \to GL(2, \mathbb{F}_q)$ induced by the surjection $\mathfrak{O} \to \mathfrak{O}/\varpi\mathfrak{O} \cong \mathbb{F}_q$ naturally gives rise to a surjection

$$ZK \mapsto PGL(2, \mathbb{F}_q) \tag{3.2}$$

which is trivial on Z. So the σ_j''s can be thought of as representations of ZK.

It is also clear that $\mathrm{Aut}(\mathfrak{x}_1) \cong \mathfrak{S}_{q+1}$, the symmetric group on $q+1$ letters. So in this case the embedding (2.6) gives us an embedding of the group $PGL(2, \mathbb{F}_q)$, which has order $(q+1)\,q(q-1)$, into $\mathfrak{S}_{q+1}$. This embedding is equivalent to the following well-known construction. Let $\mathbb{P}^1(\mathbb{F}_q)$ be the projective line over $\mathbb{F}_q$, i.e., the set of equivalence classes of non-zero vectors $v = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ in $\mathbb{F}_q^2$, where $v \sim v'$ if $v' = \lambda v$ for some $\lambda \in \mathbb{F}_q^\times$. Let $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ be the equivalence class of $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. The natural action of $PGL(2, \mathbb{F}_q)$ on $\mathbb{P}^1(\mathbb{F}_q)$, which has $q+1$ elements, is faithful. This gives an embedding of $PGL(2, \mathbb{F}_q)$ into $\mathfrak{S}_{q+1}$. We can define a bijection from $\mathbb{P}^1(\mathbb{F}_q)$ to the set of neighbours of o by mapping $\begin{bmatrix}a\\ 1\end{bmatrix}$ to $[g_a L_0]$, $a \in A$, and mapping $\begin{bmatrix}1\\ 0\end{bmatrix}$ to $[g_\infty L_0]$. One may check that this is an isomorphism of $PGL(2, \mathbb{F}_q)$-spaces. The following is a well-known and easily checked fact about the action of $PGL(2, \mathbb{F}_q)$ on $\mathbb{P}^1(\mathbb{F}_q)$ (see, for example, [15, Theorem 10.6.7]).

LEMMA 3.1. – *If u_1, u_2, u_3 are three distinct neighbours of o, and if also v_1, v_2, v_3 are three distinct neighbours of o, then there is a unique $g \in PGL(2, \mathbb{F}_q)$ such that $gu_j = v_j$ for each j.*

It is well-known that the irreducible representations of $\mathrm{Aut}(\mathfrak{x}_1) \cong \mathfrak{S}_{q+1}$ are in one to one correspondence with the partitions of $q+1$ [8, Theorem 2.1.11]. We next identify which of them are standard.

LEMMA 3.2. – *Of the irreducible representations of $\mathfrak{S}_{q+1}$, only two are non-standard, namely the trivial representation and the q dimensional representation of $\mathfrak{S}_{q+1}$ obtained from the natural action of $\mathfrak{S}_{q+1}$ on $V = \{(t_1, \ldots, t_{q+1}) : \sum_{i=1}^{q+1} t_i = 0\}$.*

PROOF. – Any maximal proper subtree of $\mathfrak{x}_1$ consists of o and a neighbour of o. So given any two maximal proper subtrees $\mathfrak{y}_1$ and $\mathfrak{y}_2$ of $\mathfrak{x}_1$, there is a $g \in \mathrm{Aut}(\mathfrak{x}_1)$ such that $g(\mathfrak{y}_1) = \mathfrak{y}_2$. Thus to check whether a representation of $\mathrm{Aut}(\mathfrak{x}_1)$ is non-standard, we need only check when it has a non-zero $K(\mathfrak{y})$-fixed vector for any particular maximal proper subtree $\mathfrak{y}$. The subgroup $K(\mathfrak{y})$ corresponds to the subgroup $\mathfrak{S}_q$ of $\mathfrak{S}_{q+1}$ which fixes a particular one of the letters $1, \ldots, q+1$. The irreducible representations of $\mathfrak{S}_{q+1}$ having a non-zero $\mathfrak{S}_q$-fixed vector are just the subrepresentations of the quasi-regular representation λ', say, of $\mathfrak{S}_{q+1}$ on $\mathfrak{S}_{q+1}/\mathfrak{S}_q$ (see [4, p. 104]). But it is easy to see that λ' is equivalent to the representation obtained from the natural representation of $\mathfrak{S}_{q+1}$ on $\mathbb{C}^{q+1}$, which is the sum of one copy of the trivial representation (because of the constant $q+1$-tuple $(1, 1, \dots, 1)$), and the above q-dimensional representation on the orthogonal complement V of $(1, 1, \dots, 1)$. ■

The two non-standard representations of $\mathfrak{S}_{q+1}$ appearing above correspond to the partitions $(q+1)$ and $(q, 1)$, respectively, of $q+1$ (see [8, Lemma 2.2.19(iii)]).

The irreducible representations of $PGL(2, \mathbb{F}_q)$ are also well-known. In [14] and [1, §4.1], for example, the irreducible representations of $G_0 = GL(2, \mathbb{F}_q)$ are described, and those of $PGL(2, \mathbb{F}_q)$ are just the ones which are trivial on the centre $Z_0 = \{\lambda I : \lambda \in \mathbb{F}_q^\times\}$ of G_0. If q is odd, there are 2 characters, 2 «special» representations of degree q, $(q-3)/2$ «principal series» representations (all of degree $q+1$), and $(q-1)/2$ «cuspidal» representations of degree $q-1$. If $q>2$ is even, there is only 1 character, and 1 special representation of degree q, and there are $(q-2)/2$ principal series representations (all of degree $q+1$), and $q/2$ cuspidal representations of degree $q-1$. If $q=2$, there are 2 characters and 1 representation of degree 2.

Thus for $\mathfrak{x} = \mathfrak{x}_1$, the problem of describing the representation (2.4), or equivalently, (2.5), becomes the following: Firstly, take an irreducible representation σ of $\mathfrak{S}_{q+1}$, not one of the two non-standard ones described in Lemma 3.2, and consider its restriction σ' to $PGL(2, \mathbb{F}_q)$, embedded in $\mathfrak{S}_{q+1}$ as described before Lemma 3.1.

(a) Decompose σ' into the sum (2.7) of irreducibles σ_j', $j = 1, \dots, m$.

(b) Regard each σ_j' as a representation on ZK via (3.2), and determine $\mathrm{Ind}_{ZK}^{G} \sigma_j'$.

We are able to perform step (a) explicitly for any particular small q. If $q \leqslant 3$, then $(q+1)\, q(q-1) = (q+1)!$, and so $PGL(2, \mathbb{F}_q) \cong \mathfrak{S}_{q+1}$. Thus m in (2.7) is 1. By Lemma 3.2, if $q=2$, then only the sign character ε is standard. If $q=3$, then $\mathfrak{S}_{q+1}$ has trivial character, the sign character ε, 1 representation of degree 2, and two of degree 3 (see, for example, [8, p. 349]). Thus the standard representations are ε, and one each of degrees 2 and 3. These must «restrict» to a non-trivial character, a cuspidal and a special representation, respectively, of $PGL(2, \mathbb{F}_3)$.

For somewhat larger q's, we first use [14, §1.5] to determine the conjugacy classes C_i in $PGL(2, \mathbb{F}_q)$. Then for each i, after choosing a representative g_i of C_i, it is easy to calculate the cycle type of the permutation of $\mathbb{P}^1(\mathbb{F}_q)$ induced by g_i. Then we use the character tables in [8, pp. 349-355], and routine calculations to find the decomposition into irreducibles of the restriction to $PGL(2, \mathbb{F}_q)$ of each irreducible representation of $\mathfrak{S}_{q+1}$. By way of example, the result for case $q=7$ is given in the table below. It is the smallest case in which multiplicities greater than 1 occur. The first row of the table gives the degree of each irreducible representation of S_{q+1}, in the order used in [8]. In the first column, χ_j is a character, and c_j, p_j and s_j refer to cuspidal, principal series, and special representations, respectively. The next two columns refer to the two non-standard representations of S_{q+1}, and so do not concern us here.

The case $q = 7$.

	1	7	20	21	28	64	35	14	70	56	90	35	42	56	70	64	21	14	28	20	7	1
χ_0	1	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0
χ_1	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1
p_1	0	0	0	1	0	2	1	0	2	1	2	0	1	2	1	2	0	0	1	1	0	0
p_2	0	0	1	0	1	2	0	0	1	2	2	1	1	1	2	2	1	0	0	0	0	0
s_0	0	1	0	0	1	1	1	0	2	1	2	1	1	0	2	1	1	0	1	0	0	0
s_1	0	0	0	1	1	1	1	0	2	0	2	1	1	1	2	1	0	0	1	0	1	0
c_1	0	0	0	1	1	1	0	0	1	0	3	0	2	0	1	1	1	0	1	0	0	0
c_2	0	0	1	0	0	1	1	1	1	2	1	1	0	2	1	1	0	1	0	1	0	0
c_3	0	0	1	0	0	1	1	1	1	2	1	1	0	2	1	1	0	1	0	1	0	0

We now turn to step (b) in the procedure for describing the representation (2.4): finding $\mathrm{Ind}_{ZK}^{G}\sigma_j'$ for each j. There are four cases, according to whether $\tau = \sigma_j'$ is cuspidal, a character, special or principal series.

PROPOSITION 3.3. – *When τ is cuspidal, then $\mathrm{Ind}_{ZK}^{G}\tau$ is an irreducible supercuspidal representation of G.*

PROOF. – This is a special case of a result of Kutzko [9], which is stated and proved in exactly our situation in [1, Theorem 4.8.1], with the central character being trivial in our case. A word is needed about the various types of induced representations used here and in [1]. Let us call the type defined at the beginning of Section 2 *unitary induction*. In [1], *ordinary induction* is defined as in our definition above, but without the condition (ii) there; *compact induction* is defined with (ii) replaced by the condition that $f(g_\alpha) \neq 0$ for only finitely many α's. If the representation spaces of $\mathrm{Ind}_{ZK}^{G}\tau$ are V_2, V' and V, respectively, for these three representations, then $V \subset V_2 \subset V'$. In the proof of irreducibility in Theorem 4.8.1 in [1], it is shown that $\mathrm{Hom}_G(V, V')$ is one-dimensional, and since there is a natural injection $\mathrm{Hom}_G(V_2, V_2) \to \mathrm{Hom}_G(V, V')$, the irreducibility of the representation on V_2 follows. The representation on V_2 is the completion of the representation on V, which is shown to be supercuspidal and admissible in [1]. ■

Before dealing with the case when τ is a character, we first need to give some properties of the spherical principal series representations π_s of $\mathcal{A}$ studied in [4], for example. Recall the boundary Ω of $\mathfrak{X}$ consists of equivalence classes of infinite geodesics in $\mathfrak{X}$. If $(x_0, x_1, \dots)$ and $(y_0, y_1, \dots)$ are both in the class ω, with $x_0 = x$ and $y_0 = y$, there is an $h \in \mathbb{Z}$ such that $y_n = x_{n+h}$ for all sufficiently large n. We write $h(x, y; \omega) = h$. There is a natural topology on Ω making it a totally disconnected compact space. Let $\mathcal{C}^\infty(\Omega)$ denote the space of locally constant functions $\Omega \to \mathbb{C}$. There is also a natural action of $\mathcal{A}$ on Ω. For non-zero $s \in \mathbb{C}$, we can define a representation of $\mathcal{A}$ on $\mathcal{C}^\infty(\Omega)$ by

$$(\pi_s(g)F)(\omega) = F(g^{-1}\omega)\left(\frac{s}{\sqrt{q}}\right)^{h(go,\, o;\, \omega)}.$$

The factor $\sqrt{q}$ on the right is a normalization so that, when $|s| = 1$, π_s is unitarizable with respect to the inner product $\langle F_1, F_2\rangle = \int_\Omega F_1(\omega)\,\overline{F_2(\omega)}\,d\nu_o(\omega)$ on $\mathcal{C}^\infty(\Omega)$. Here ν_o is the natural probability on Ω associated with the vertex o [4, p. 34]. The representations π_s are irreducible for $|s| = 1$, and make up the *spherical principal series* of representations of $\mathcal{A}$. They remain irreducible when restricted to H, and are also so named in that context.

Let $\chi_s\colon F^\times \to \mathbb{C}^\times$ be the quasi-character $a \mapsto s^{\mathrm{ord}(a)}$ of $F^\times$. Then it is routine to see that the restriction of π_s to H, regarded as a representation of G, is the principal series representation $\varrho_s = \mathcal{B}(\chi_s, \chi_{s^{-1}})$ defined in [1, p. 471]. Indeed, let ω_0 be the class of the geodesic $(g_0 o, g_1 o, \ldots)$, where $g_n = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^n \end{pmatrix}$ for $n \in \mathbb{N}$. The set of $g \in G$ such that $g\omega_0 = \omega_0$ is the set P of upper-triangular matrices in G. We define $T : \mathcal{C}^\infty(\Omega) \to V_s$, the representation space of ϱ_s by

$$(TF)(g) = F(g^{-1}\omega_0)\left(\frac{s}{\sqrt{q}}\right)^{h(go,\, o;\, \omega_0)}.$$

It is not hard to show that T is a bijection, intertwining π_s and ϱ_s on H.

The following is well-known. See [3]; cf. [4, Corollary II.6.5]. We include a proof for the convenience of the reader.

PROPOSITION 3.4. – *Let λ be the unitary representation of $\mathcal{A}$ on $l^2(\mathfrak{X})$ obtained from the natural action of $\mathcal{A}$ on $\mathfrak{X}$. Then λ is unitarily equivalent to* $\mathrm{Ind}_{K_o}^{\mathcal{A}} 1$, *and λ is the direct integral of the representations π_s, $|s| = 1$. The same is true when we restrict λ and the π_s's to H.*

PROOF. – Firstly, λ is unitarily equivalent to $\mathrm{Ind}_{K_o}^{\mathcal{A}} 1$. To see this, for each vertex $x \in \mathfrak{X}$, choose $g_x \in \mathcal{A}$ such that $g_x x = o$. Then $\mathcal{A}$ is the disjoint union of the cosets $K_o g_x$, $x \in \mathfrak{X}$. For $f \in l^2(\mathfrak{X})$, define $F : \mathcal{A} \to \mathbb{C}$ by $F(kg_x) = f(x)$ for all $k \in K_o$ and $x \in \mathfrak{X}$. Then F is in the representation space of $\mathrm{Ind}_{K_o}^{\mathcal{A}} 1$, and it is easy to check that this defines a unitary map intertwining λ and $\mathrm{Ind}_{K_o}^{\mathcal{A}} 1$.

The remaining statements are well-known, and implicit in [FN, Theorem 6.4], and we omit the proof. Proposition 4.7 below is a similar but somewhat less well-known fact, and we prove that for the convenience of the reader. ∎

PROPOSITION 3.5. – *When τ is a character, and $q > 2$, then* $\mathrm{Ind}_{ZK}^{G}\tau$, *as a representation of $H = PGL(2, F)$, is the product of a character of H and the direct integral of the spherical principal series representations of H. When τ is a character and $q = 2$, then* $\mathrm{Ind}_{ZK}^{G}\tau$ *is an irreducible supercuspidal representation of G.*

PROOF. – Our τ comes from a character of $PGL(2, \mathbb{F}_q)$, and hence a character of $G_0 = GL(2, \mathbb{F}_q)$ trivial on the centre Z_0 of G_0. So when $q \neq 2$, it is of the form $gZ_0 \mapsto \chi_0(\det(g))$, where χ_0 is a character of $\mathbb{F}_q^{\times}$ [14], [1, § 4.1]. For triviality on Z_0, χ_0 must take only values 1 and -1. Using $\mathfrak{O}/\varpi\mathfrak{O} \cong \mathbb{F}_q$, χ_0 lifts to a character of $\mathfrak{O}^{\times}$, and then to a character χ of $F^{\times}$ by setting $\chi(\varpi) = 1$. So τ is the restriction to ZK of the character $\tilde{\chi}: g \mapsto \chi(\det(g))$ of $GL(2, F)$, which is trivial on Z. Then

$$\mathrm{Ind}_{ZK}^{G}\tau = \mathrm{Ind}_{ZK}^{G}\tilde{\chi}_{|ZK} \cong \tilde{\chi}\cdot\mathrm{Ind}_{ZK}^{G}1 .$$

Now $\mathrm{Ind}_{ZK}^{G}1$ is clearly trivial on Z, and so factors through the representation $\mathrm{Ind}_{K_o \cap H}^{H}1$ of H, which is the restriction to H of the representation $\mathrm{Ind}_{K_o}^{\mathcal{A}}1$ of $\mathcal{A}$.

Let λ be as in Proposition 3.4. Then $\mathrm{Ind}_{K_o}^{\mathcal{A}}1$ is equivalent to λ. Hence by Proposition 3.4, $\mathrm{Ind}_{K_o \cap H}^{H}1$, regarded as a representation of G, is the direct integral of the representations $\mathcal{B}(\chi_s, \chi_{s^{-1}})$, $|s| = 1$.

The product of the character $\tilde{\chi}: g \mapsto \chi(\det(g))$ and $\mathcal{B}(\chi_s, \chi_{s^{-1}})$ is equivalent to $\mathcal{B}(\chi\chi_s, \chi\chi_{s^{-1}})$ [1, p. 490], and so $\mathrm{Ind}_{ZK}^{G}\tau$ is the direct integral of these principal series representations, which are not in the spherical series if χ_0 is non-trivial.

Finally, suppose that $q = 2$, and that τ is the non-trivial character of $PGL(2, \mathbb{F}_2) \cong \mathfrak{S}_3$. Then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ fixes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and interchanges $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So it is an odd permutation of $\mathbb{P}^1(\mathbb{F}_2)$, and the value of τ there is -1. Hence there is no non-zero linear functional $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi\left(\tau\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) v\right) = \phi(v)$ for all $v \in \mathbb{C}$. So τ satisfies the condition of being cuspidal (though it is usually not thought of as such), and the proof of Theorem 4.8.1 in [1] goes through without change, taking $V_0 = \mathbb{C}$ and $\pi_0 = \tau$. So $\mathrm{Ind}_{ZK}^{G}\tau$ is irreducible and supercuspidal. ∎

The case when τ is special.

When τ is special we are led to consider the representation π of $\mathcal{A}$ obtained from its natural action on the set $\mathcal{E}$ of (undirected) edges of $\mathfrak{X}$. We also consider the natural action on the set $\mathcal{E}^d$ of directed edges of $\mathfrak{X}$. If $e = (x, y)$ is a directed edge, let e' denote the edge (y, x). If $f : \mathcal{E}^d \to \mathbb{C}$ is a function, let $f' : \mathcal{E}^d \to \mathbb{C}$ be defined by $f'(e) = f(e')$. We call f *even* if $f' = f$, and *odd* if $f' = -f$. Let $l^2(\mathcal{E})$ and $l^2(\mathcal{E}^d)$ denote the spaces of square summable functions on $\mathcal{E}$ and $\mathcal{E}^d$, respectively. Let $l^2_{\mathrm{e}}(\mathcal{E}^d)$ and $l^2_{\mathrm{o}}(\mathcal{E}^d)$ denote the spaces of even and odd elements of $l^2(\mathcal{E}^d)$, respectively. Clearly, the map $f \mapsto ((f+f')/2, (f-f')/2)$ is an isomorphism $l^2(\mathcal{E}^d) \to l^2_{\mathrm{e}}(\mathcal{E}^d) \oplus l^2_{\mathrm{o}}(\mathcal{E}^d)$. Also, $(Tf)(\{x, y\}) = \sqrt{2} f((x, y))$ defines an isomorphism $T : l^2_{\mathrm{e}}(\mathcal{E}^d) \to l^2(\mathcal{E})$.

The group $\mathcal{A}$ acts on $\mathcal{E}$ and $\mathcal{E}^d$ in a natural way, and hence on each of the spaces $l^2(\mathcal{E})$, $l^2(\mathcal{E}^d)$, $l^2_{\mathrm{e}}(\mathcal{E}^d)$ and $l^2_{\mathrm{o}}(\mathcal{E}^d)$. Let π, π^d, π^d_{e} and π^d_{o} denote the corresponding representations of $\mathcal{A}$.

LEMMA 3.6. – *Let $\chi : \mathcal{A} \to \{-1, 1\}$ denote the non-trivial character $g \mapsto (-1)^{d(o, go)}$ of $\mathcal{A}$. Then we have the following unitary equivalences.*

(i) $\pi^d \cong \pi^d_{\mathrm{e}} \oplus \pi^d_{\mathrm{o}}$,

(ii) $\pi^d_{\mathrm{e}} \cong \pi$, *and*

(iii) $\pi^d_{\mathrm{o}} \cong \chi \otimes \pi^d_{\mathrm{e}}$.

PROOF. – The equivalences in (i) and (ii) are given by the bijections $l^2(\mathcal{E}^d) \to l^2_{\mathrm{e}}(\mathcal{E}^d) \oplus l^2_{\mathrm{o}}(\mathcal{E}^d)$ and $l^2_{\mathrm{e}}(\mathcal{E}^d) \to l^2(\mathcal{E})$ defined above. To see (iii), fix a vertex $o \in \mathcal{X}$, and define $S : l^2_{\mathrm{o}}(\mathcal{E}^d) \to l^2_{\mathrm{e}}(\mathcal{E}^d)$ by $(Sf)((x, y)) = (-1)^{d(o, x)} f((x, y))$. This is easily checked to be a well-defined map. For $g \in \mathcal{A}$,

$$\begin{aligned}
(S(\pi^d_{\mathrm{o}}(g) f))((x, y)) &= (-1)^{d(o, x)} (\pi^d_{\mathrm{o}}(g) f)((x, y)) \\
&= (-1)^{d(o, x)} f\big((g^{-1}x, g^{-1}y)\big) \\
&= (-1)^{d(o, go)} (-1)^{d(o, g^{-1}x)} f\big((g^{-1}x, g^{-1}y)\big) \\
&= \chi(g)(Sf)\big((g^{-1}x, g^{-1}y)\big) \\
&= \big(\chi(g)\, \pi^d_{\mathrm{e}}(g)(Sf)\big)((x, y)). \qquad \blacksquare
\end{aligned}$$

If $e = (x, y) \in \mathcal{E}^d$, let $i(e)$ denote the initial vertex x of e. The space V_{e} of $f \in l^2_{\mathrm{e}}(\mathcal{E}^d)$ which satisfy $\sum\limits_{e : i(e) = x} f(e) = 0$ for each $x \in \mathcal{X}$ is invariant under π^d_{e}, and so gives a subrepresentation sp_{e} of π^d_{e}. In the same way, we can define a subrepresentation sp_{o} of π^d_{o} on $V_{\mathrm{o}} \subset l^2_{\mathrm{o}}(\mathcal{E}^d)$. The representations sp_{e} and sp_{o} are known to be irreducible, and are called the *special* representations of $\mathcal{A}$ (see [4, § III.2]). By part (iii) of the above lemma, $\mathrm{sp}_{\mathrm{o}} \cong \chi \otimes \mathrm{sp}_{\mathrm{e}}$.

LEMMA 3.7. – *Let λ denote the unitary representation of $\mathcal{A}$ on $l^2(\mathcal{X})$ obtained by the natural action of $\mathcal{A}$ on $\mathcal{X}$. Then $\pi^d_{\mathrm{o}} \cong \mathrm{sp}_{\mathrm{o}} \oplus \lambda$, and so $\pi^d_{\mathrm{e}} \cong \mathrm{sp}_{\mathrm{e}} \oplus (\chi \otimes \lambda)$.*

PROOF. – Define $T : l^2(\mathfrak{X}) \to l_0^2(\mathcal{E}^d)$ by $(Tf)((x, y)) = f(y) - f(x)$. It is easy to check that T is continuous, with norm at most $2\sqrt{q+1}$, and intertwines λ and π_0^d. Let $T = UA$ be the polar decomposition of T. Thus A is a positive hermitian operator on $l^2(\mathfrak{X})$, and U is a partial isometry, inducing an isometric isomorphism of $M = \ker(T)^\perp$ onto $N = \ker(T^*)^\perp$ (cf. [13, Theorem 3.2.17]). From the construction of this decomposition, it is clear that U intertwines λ and π_0^d. Clearly T is injective, and so $M = l^2(\mathfrak{X})$, and thus the restriction of π_0^d to N is unitarily equivalent to λ. Also, for $F \in l_0^2(\mathcal{E}^d)$, $(T^* F)(x) = -2 \sum_{e \in \mathcal{E}^d : i(e) = x} F(e)$, and so $\ker(T^*) = V_0$. Hence $N = V_0^\perp$, and so $l_0^2(\mathcal{E}^d) = V_0 \oplus N$. The first statement in the lemma has now been proved, and the second one follows from Lemma 3.6, since $\chi^{-1} = \chi$. ■

Recall that o' is a vertex adjacent to o. Notice that π^d is the representation obtained by inducing to $\mathcal{A}$ the trivial character on $K(\{o, o'\}) = \{g \in \mathcal{A} : go = o$ and $go' = o'\}$. This is because $\mathcal{A}$ acts transitively on $\mathfrak{X}$ and K_o acts transitively on the set of neighbours of o, so that $\mathcal{A}$ acts transitively on $\mathcal{E}^d$.

If we take $o' = [g_1 L_0]$ for $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$, then the preimage in G of $K(\{o, o'\})$ is ZK', where K' is the set of all matrices

$$\begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix},$$

where $a, d \in \mathfrak{O}^\times$ and $b, c \in \mathfrak{O}$. Since G also acts transitively on $\mathfrak{X}$ and K acts transitively on the set of neighbours of o, the restriction of π^d to H, regarded as a representation of G, is $\mathrm{Ind}_{ZK'}^G 1$.

There is a special representation of $G_0 = GL(2, \mathbb{F}_q)$ corresponding to each character χ of $\mathbb{F}_q^\times$, obtained by inducing the character $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi(ad)$ of P_0 from P_0 to G_0, and taking a q-dimensional subrepresentation. For this to be trivial on the centre Z_0 of G_0, we need χ^2 to be trivial. When q is even, this forces χ to be trivial, but when q is odd, there is a unique character χ_1 of $\mathbb{F}_q^\times$ of order 2. Let τ_0 and τ_1 be the special representations of $PGL(2, \mathbb{F}_q)$ corresponding to the trivial character and to χ_1, respectively. We can lift χ_1 to a character $\tilde{\chi}_1$ of $F^\times$ by first lifting to $\mathfrak{O}^\times$ using the surjection $\mathfrak{O} \to \mathfrak{O}/\varpi\mathfrak{O} \cong \mathbb{F}_q$, then to $F^\times$ by mapping ϖ to 1.

PROPOSITION 3.8. – *Let τ_0 and τ_1 be the special representations of $PGL(2, \mathbb{F}_q)$, as above (the latter existing only when q is odd). Lift these to ZK using* (3.2). *Then $\mathrm{Ind}_{ZK}^G \tau_0$, regarded as a representation of H, is unitarily equivalent to the direct sum of the restrictions to H of $\chi \otimes \lambda$, sp_{e} and sp_{o}. For q odd, $\mathrm{Ind}_{ZK}^G \tau_1$ is equivalent to the product of $\mathrm{Ind}_{ZK}^G \tau_0$ by the character $g \mapsto \tilde{\chi}_1(\det(g))$.*

PROOF. – We have $\mathrm{Ind}_{P_0}^{G_0} 1 = 1 \oplus \tau_0$, where the 1's on the left and right denote the trivial characters of P_0 and G_0, respectively.

Next observe that when we lift $\mathrm{Ind}_{P_0}^{G_0} 1$ to ZK using (3.2), we get $\mathrm{Ind}_{ZK'}^{ZK} 1$, where K' is defined above. This is because K' is the preimage of P_0 in G_0 under the natural map $K \to GL(2, \mathbb{F}_q)$. Hence

$$\mathrm{Ind}_{ZK'}^{ZK} 1 \cong 1 \oplus \tau_0,$$

regarding the representations on the right as defined on ZK. Hence by transitivity of induction, we have

$$\mathrm{Ind}_{ZK}^{G} 1 \oplus \mathrm{Ind}_{ZK}^{G} \tau_0 \cong \mathrm{Ind}_{ZK'}^{G} 1.$$

Now $\mathrm{Ind}_{ZK}^{G} 1$ regarded as a representation of H, is the restriction to H of λ, as we saw in the proof of Proposition 3.5. Also, $\mathrm{Ind}_{ZK'}^{G} 1$ regarded as a representation of H, is the restriction to H of π^d, as we saw above. So by Lemma 3.7 and parts (i) and (iii) of Lemma 3.6 we have

$$\lambda \oplus \mathrm{Ind}_{ZK}^{G} \tau_0 \cong \lambda \oplus (\chi \otimes \lambda) \oplus \mathrm{sp}_{\mathrm{e}} \oplus \mathrm{sp}_{\mathrm{o}},$$

with the λ on the left and the representations on the right restricted to G. Since the representations on both sides are all finite in the sense of [10] (see pp. 33, 45 and 120-122 there), we can cancel λ from both sides, obtaining the stated decomposition of $\mathrm{Ind}_{ZK}^{G} \tau_0$. Starting from

$$\chi_1' \oplus \tau_1 = \mathrm{Ind}_{P_0}^{G_0} \chi_1 \cong \chi_1' \otimes \mathrm{Ind}_{P_0}^{G_0} 1,$$

where $\chi_1'(g) = \chi_1(\det(g))$, it is easy to prove the statement about $\mathrm{Ind}_{ZK} \tau_1$. ∎

4. – The case when τ is principal series.

There is a principal series representation $\mathcal{B}(\chi_1, \chi_2)$ of $G_0 = GL(2, \mathbb{F}_q)$ corresponding to each pair (χ_1, χ_2) of distinct characters of $\mathbb{F}_q^\times$, obtained by inducing the character $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$ of P_0 from P_0 to G_0 [14, § 8], [1, § 4.1]. Its dimension is $q+1$. The representations $\mathcal{B}(\chi_1, \chi_2)$ and $\mathcal{B}(\chi_2, \chi_1)$ are equivalent. For $\mathcal{B}(\chi_1, \chi_2)$ to be trivial on the centre Z_0 of G_0, we need $\chi_2 = \chi_1^{-1}$.

So we start with a character χ_0 of $\mathbb{F}_q^\times$ such that χ_0^2 is non-trivial. We define a character $\chi_0' : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_0(a/d)$ of P_0, then form $\tau_0 = \mathcal{B}(\chi_0, \chi_0^{-1}) = \mathrm{Ind}_{P_0}^{G_0} \chi_0'$. This lifts to a $q+1$-dimensional representation τ of ZK in the usual way: for $\lambda \in F^\times$ and $k \in K$, set $\tau(\lambda k) = \tau_0(\dot{k})$, where $\dot{k}$ denotes the image of k in G_0.

LEMMA 4.1. – *The above representation* τ *of* ZK *is unitarily equivalent to* $\mathrm{Ind}_{ZK'}^{ZK}\chi'$, *where* χ' *is the character* $\lambda k \mapsto \chi_0'(\dot{k})$ *of* ZK'. *Hence* $\mathrm{Ind}_{ZK}^{G}\tau \cong \mathrm{Ind}_{ZK'}^{G}\chi'$.

PROOF. – Let V_0 and V be the representation spaces of τ and $\mathrm{Ind}_{ZK'}^{ZK}\chi'$, respectively. If $f_0 \in V_0$, then $f_0 \colon G_0 \to \mathbb{C}$ is a function such that $f_0(pg) = \chi_0'(p)\, f_0(g)$ for all $p \in P_0$ and $g \in G_0$. We then define $f \in V$ by $f(\lambda k) = f_0(\dot{k})$. It is routine to check that $f_0 \mapsto f$ gives a unitary equivalence. The last statement follows by transitivity of induction. ■

Of course $\mathrm{Ind}_{ZK'}^{G}\chi'$ is trivial on Z, and it will be convenient to work with the corresponding representation $\mathrm{Ind}_{K''}^{H}\chi''$, where K'' is the image of K' in $H = PGL(2, F)$, and χ'' is the character $kZ \mapsto \chi'(k)$ of K''.

Studying $\mathrm{Ind}_{K''}^{H}\chi''$ leads us to consider the set $\widehat{H}_{\chi''}$ of equivalence classes of irreducible continuous unitary representations π of H for which $\mathcal{H}_{\pi, \chi''} = \{\xi \in \mathcal{H}_\pi \colon \pi(k)\, \xi = \chi''(k)\, \xi$ for all $k \in K''\}$ is non-zero. We also need to consider the space $\mathcal{H}'' = \mathcal{H}(H//K'', \overline{\chi''})$ consisting of compactly supported functions f on H for which

$$f(k_1 h k_2) = \overline{\chi''(k_1 k_2)}\, f(h) \tag{4.1}$$

for all $h \in H$ and $k_1, k_2 \in K''$. It is easy to see that if $f_1, f_2 \in \mathcal{H}''$, then $f_1 * f_2 \in \mathcal{H}''$ and $f_1^* \in \mathcal{H}''$, where $f_1^*(h) = \overline{f(h^{-1})}$. The algebra $\mathcal{H}''$ is an example of a τ-spherical Hecke algebra, described in [7, Appendix 1], for example.

To study $\mathcal{H}''$, it is convenient to work with the space $\mathcal{H}'$ of continuous functions $f \colon G \to \mathbb{C}$ of compact support such that

$$f(k_1' g k_2') = \overline{\chi'(k_1' k_2')}\, f(g) \tag{4.2}$$

for all $g \in G$ and $k_1', k_2' \in K'$. It is also an example of a τ-spherical Hecke algebra.

Define $\Lambda : \mathcal{C}_c(G) \to \mathcal{C}_c(H)$ by

$$(\Lambda f)(gZ) = \int_Z f(gz)\, dz = \int_{F^\times} f\left(g\begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix}\right) \frac{dx}{|x|},$$

where dz refers to Haar measure on Z. Then Λ is a linear surjection [1, Proposition 4.3.4]. It is clear that Λ maps $\mathcal{H}'$ into $\mathcal{H}''$. In fact, $\Lambda(\mathcal{H}') = \mathcal{H}''$, for if $f \in \mathcal{H}''$ and if $f_0 \in \mathcal{C}_c(G)$ satisfies $\Lambda(f_0) = f$, then setting

$$f_1(g) = \int_{K'}\int_{K'} \chi'(k_1' k_2')\, f_0(k_1 g k_2')\, dk_1'\, dk_2', \tag{4.3}$$

where dk' refers to normalized Haar measure on K', we have $f_1 \in \mathcal{H}'$ and $\Lambda(f_1) = f$ too.

It is easy to see that Λ is a $*$-algebra homomorphism.

Define matrices

$$g_{m,n} := \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} \quad (m, n \in \mathbb{Z}), \quad \text{and} \quad w_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

LEMMA 4.2. – *Let P be the group of upper-triangular matrices in G. Then we may write G as a disjoint union of double cosets in the following two ways: $G = PK' \cup Pw_0K'$, and*

$$G = \bigcup_{m,n \in \mathbb{Z}} K' g_{m,n} K' \cup \bigcup_{m,n \in \mathbb{Z}} K' w_0 g_{m,n} K'. \tag{4.3}$$

PROOF. – Suppose that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ has determinant D. If $\operatorname{ord}(c) > \operatorname{ord}(d)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} D/d & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix}$$

exhibits g as an element of PK'. If $\operatorname{ord}(c) \leqslant \operatorname{ord}(d)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -D/c & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$$

exhibits g as an element of Pw_0K'. Hence $G = PK' \cup Pw_0K'$. To see that these double cosets are disjoint, we must check that $w_0 \notin PK'$. But if $k = \begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix} \in K'$, then

$$w_0 k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix} = \begin{pmatrix} \varpi c & d \\ a & b \end{pmatrix} \notin P.$$

To show (4.3), it is enough to show that if $p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P$, then both p and pw_0 are in the union on the right in (4.3), which is easily seen to be disjoint. There are several cases:

(i) If either $\operatorname{ord}(b) \geqslant \operatorname{ord}(a)$ or $\operatorname{ord}(b) \geqslant \operatorname{ord}(d)$, let $m = \operatorname{ord}(a)$ and $n = \operatorname{ord}(d)$. Then $p \in K' g_{m,n} K'$ because

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a/\varpi^m & 0 \\ 0 & d/\varpi^n \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} \begin{pmatrix} a/\varpi^m & 0 \\ 0 & d/\varpi^n \end{pmatrix}.$$

(ii) If $\operatorname{ord}(b) < \operatorname{ord}(a), \operatorname{ord}(d)$, let $m = \operatorname{ord}(a) + \operatorname{ord}(d) - \operatorname{ord}(b)$ and $n =$

ord (b). Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d/b & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} \begin{pmatrix} -ad/b\varpi^m & 0 \\ a/\varpi^n & b/\varpi^n \end{pmatrix}$$

shows that $p \in K' w_0 g_{m,n} K'$.

(iii) If $\operatorname{ord}(b) > \operatorname{ord}(a)$, let $m = \operatorname{ord}(d)$ and $n = \operatorname{ord}(a)$. Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a/\varpi^n & 0 \\ 0 & d/\varpi^m \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b/a & 1 \end{pmatrix}$$

shows that $pw_0 \in K' w_0 g_{m,n} K'$.

(iv) If $\operatorname{ord}(b) \geqslant \operatorname{ord}(d)$, then again let $m = \operatorname{ord}(d)$ and $n = \operatorname{ord}(a)$. Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} \begin{pmatrix} d/\varpi^m & 0 \\ 0 & a/\varpi^n \end{pmatrix}$$

shows that $pw_0 \in K' w_0 g_{m,n} K'$.

(v) If $\operatorname{ord}(b) \leqslant \operatorname{ord}(a)$ and $\operatorname{ord}(b) < \operatorname{ord}(d)$, let $m = \operatorname{ord}(b)$ and $n = \operatorname{ord}(a) + \operatorname{ord}(d) - \operatorname{ord}(b)$. Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d/b & 1 \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} \begin{pmatrix} b/\varpi^m & a/\varpi^m \\ 0 & -ad/b\varpi^n \end{pmatrix}$$

shows that $pw_0 \in K' g_{m,n} K'$. ■

LEMMA 4.3. – *Any function f satisfying* (4.2) *must satisfy $f(w_0 g_{m,n}) = 0$ for all $m, n \in \mathbb{Z}$.*

PROOF. – Let $a \in \mathfrak{O}^\times$, let $\dot{a}$ denote its image in $\mathbb{F}_q^\times$, and evaluate f at

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

Then we must have $\chi_0(\dot{a}) f(w_0 g_{m,n}) = f(w_0 g_{m,n}) \chi_0(\dot{a}^{-1})$. Since $\chi_0^2 \neq 1$, we can choose a so that $\chi_0(\dot{a}) \neq \chi_0(\dot{a}^{-1})$. Hence $f(w_0 g_{m,n}) = 0$. ■

Thus $\mathcal{H}'$ is spanned by the functions $F_{m,n}$ defined by

$$F_{m,n}(g) = \begin{cases} \overline{\chi'(k_1' k_2')} & \text{if } g = k_1' g_{m,n} k_2' \in K' g_{m,n} K', \\ 0 & \text{if } g \notin K' g_{m,n} K'. \end{cases}$$

It is convenient to normalize these functions as follows:

(4.5) $$G_{m,n} = q^{\min\{m,n\}} F_{m,n} \quad \text{for } m, n \in \mathbb{Z}.$$

It is also convenient to work below with Haar measure on G normalized so that K' has measure 1.

PROPOSITION 4.4. – *For all $m, n, r, s \in \mathbb{Z}$,*

$$G_{m,n} * G_{r,s} = G_{m+r,n+s}. \tag{4.6}$$

Hence the convolution algebras $\mathcal{H}'$ and $\mathcal{C}_c(\mathbb{Z}^2)$ are isomorphic, as are $\mathcal{H}''$ and $\mathcal{C}_c(\mathbb{Z})$.

PROOF. – We first derive the formula

$$(F_{m,n} * F_{r,s})(g) = q^{|r-s|} \int_{K'} F_{m,n}(gk' g_{r,s}^{-1})\, \chi'(k')\, dk', \tag{4.7}$$

where dk' refers to normalized Haar measure on K'. By the unimodularity of G,

$$(F_{m,n} * F_{r,s})(g) = \int_G F_{m,n}(gx^{-1})\, F_{r,s}(x)\, dx = \int_{K' g_{r,s} K'} F_{m,n}(gx^{-1})\, F_{r,s}(x)\, dx\,.$$

Now $K' g_{r,s} K'$ is the union of N cosets $g_\alpha K'$, where N is the index of $K' \cap g_{r,s} K' g_{r,s}^{-1}$ in K'. It is easy to see that $N = q^{|r-s|}$. Writing $g_\alpha = k_1' g_{r,s} k_2'$,

$$\begin{aligned}\int_{g_\alpha K'} F_{m,n}(gx^{-1})\, F_{r,s}(x)\, dx &= \int_{K'} F_{m,n}(gx^{-1} g_\alpha^{-1})\, F_{r,s}(g_\alpha x)\, dx \\ &= \int_{K'} F_{m,n}(gk'^{-1} k_2'^{-1} g_{r,s}^{-1} k_1'^{-1})\, F_{r,s}(k_1' g_{r,s} k_2' k')\, dk' \\ &= \int_{K'} F_{m,n}(gkg_{r,s}^{-1}) \chi'(k)\, dk,\end{aligned}$$

using (4.2) and setting $k = k'^{-1} k_2'^{-1}$. As the integral is independent of α, (4.7) follows.

We can write $F_{m,n} * F_{r,s}$ as a linear combination

$$F_{m,n} * F_{r,s} = \sum_{\alpha,\beta \in \mathbb{Z}} c_{\alpha,\beta} F_{\alpha,\beta}$$

of $F_{\alpha,\beta}$'s, and the coefficient $c_{\alpha,\beta}$ equals $(F_{m,n} * F_{r,s})(g_{\alpha,\beta})$, which we calculate using the integral on the right in (4.7), with $g = g_{\alpha,\beta}$.

To evaluate this integral, we write a typical $k' \in K'$ as the product

$$k' = \begin{pmatrix} 1 & 0 \\ \varpi u' & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix},$$

where $u', v \in \mathfrak{O}$ and $t_1, t_2 \in \mathfrak{O}^\times$. According to [1, p. 466], the normalized Haar measure on K' is then $du' \, dv \, dt_1 \, dt_2$, where du' and dv are the normalized Haar measures on the compact additive group $\mathfrak{O}$, and dt_1 and dt_2 are the normalized Haar measures on the compact multiplicative group $\mathfrak{O}^\times$. Hence

$$g_{\alpha,\beta} k' g_{r,s}^{-1} = \begin{pmatrix} \varpi^{\alpha-r} t_1 & \varpi^{\alpha-s} t_1 v \\ \varpi^{\beta-r+1} t_1 u' & \varpi^{\beta-s}(t_2 + t_1 u' v \varpi) \end{pmatrix}$$

$$= \begin{pmatrix} t_1 & 0 \\ 0 & t_2(1+uv\varpi) \end{pmatrix} \begin{pmatrix} \varpi^{\alpha-r} & \varpi^{\alpha-s} v \\ \varpi^{\beta-r+1} \tilde{u} & \varpi^{\beta-s} \end{pmatrix},$$

where $u = t_1 t_2^{-1} u'$ and $\tilde{u} = u/(1+uv\varpi)$. So

$$F_{m,n}(g_{\alpha,\beta} k' g_{r,s}^{-1}) \, \chi'(k') = F_{m,n}\left(\begin{pmatrix} \varpi^{\alpha-r} & \varpi^{\alpha-s} v \\ \varpi^{\beta-r+1} \tilde{u} & \varpi^{\beta-s} \end{pmatrix} \right).$$

On making the change of variable $u' \mapsto u$, as the integrand is then independent of t_1 and t_2, we have

$$\int_{K'} F_{m,n}(g_{\alpha,\beta} k' g_{r,s}^{-1}) \, \chi'(k') \, dk' = \int_{\mathfrak{O}} \int_{\mathfrak{O}} F_{m,n}\left(\begin{pmatrix} \varpi^{\alpha-r} & \varpi^{\alpha-s} v \\ \varpi^{\beta-r+1} \tilde{u} & \varpi^{\beta-s} \end{pmatrix} \right) du \, dv. \tag{4.8}$$

Notice that $\operatorname{ord}(\tilde{u}) = \operatorname{ord}(u)$ for all $u \in \mathfrak{O}$. We now break the integral in (4.8) into integrals over six (non-disjoint) subsets $A_1, \ldots, A_6$, the first four covering the cases $C_u = \max\{\operatorname{ord}(u) + s - r, \operatorname{ord}(u) + \beta - \alpha\} \geqslant 0$ and $C_v = \max\{\operatorname{ord}(v) + r - s, \operatorname{ord}(v) + \alpha - \beta\} \geqslant 0$, and the last two sets covering the cases $C_u < 0$ and $C_v < 0$. In each case we express

$$M = M(u, v) = \begin{pmatrix} \varpi^{\alpha-r} & \varpi^{\alpha-s} v \\ \varpi^{\beta-r+1} \tilde{u} & \varpi^{\beta-s} \end{pmatrix}$$

as an element in a double K' coset. In the first four cases, (4.2) shows that the integrand in (4.8) is 1 or 0 according as $(\alpha, \beta) = (m+r, n+s)$ or not.

A_1: $\operatorname{ord}(v) + r - s \geqslant 0$ and $\operatorname{ord}(u) + \beta - \alpha \geqslant 0$. Then

$$M = \begin{pmatrix} 1 & 0 \\ \varpi^{\beta-\alpha+1} \tilde{u} & 1 \end{pmatrix} \begin{pmatrix} \varpi^{\alpha-r} & 0 \\ 0 & \varpi^{\beta-s} \end{pmatrix} \begin{pmatrix} 1 & \varpi^{r-s} v \\ 0 & 1 - \varpi \tilde{u} v \end{pmatrix}.$$

A_2: $\operatorname{ord}(v) + r - s \geqslant 0$ and $\operatorname{ord}(u) + s - r \geqslant 0$. Then

$$M = \begin{pmatrix} \varpi^{\alpha-r} & 0 \\ 0 & \varpi^{\beta-s} \end{pmatrix} \begin{pmatrix} 1 & \varpi^{r-s} v \\ \varpi^{s-r+1} \tilde{u} & 1 \end{pmatrix}.$$

A_3: $\operatorname{ord}(u) + s - r \geqslant 0$ and $\operatorname{ord}(v) + \alpha - \beta \geqslant 0$. Then

$$M = \begin{pmatrix} 1 & \varpi^{\alpha-\beta} v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi^{\alpha-r} & 0 \\ 0 & \varpi^{\beta-s} \end{pmatrix} \begin{pmatrix} 1 - \varpi \tilde{u} v & 0 \\ \varpi^{s-r+1} \tilde{u} & 1 \end{pmatrix}.$$

A_4: $\operatorname{ord}(u)+\beta-\alpha \geqslant 0$ and $\operatorname{ord}(v)+\alpha-\beta \geqslant 0$. Then

$$M=\begin{pmatrix} 1 & \varpi^{\alpha-\beta} v \\ \varpi^{\beta-\alpha+1}\tilde{u} & 1 \end{pmatrix}\begin{pmatrix} \varpi^{\alpha-r} & 0 \\ 0 & \varpi^{\beta-s} \end{pmatrix}.$$

In the remaining two cases, (4.2) show that the integrand in (4.8) is 0.

A_5: $\operatorname{ord}(u)+s-r<0$ and $\operatorname{ord}(u)+\beta-\alpha<0$. Let $i=\operatorname{ord}(u)$. Then

$$M=\begin{pmatrix} -\varpi^{i}\tilde{u}^{-1} & \varpi^{\alpha-\beta-i-1} \\ 0 & \varpi^{-i}\tilde{u} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\begin{pmatrix} \varpi^{\beta-r+i} & 0 \\ 0 & \varpi^{\alpha-s-i-1} \end{pmatrix}\begin{pmatrix} 1 & \varpi^{r-s-1}\tilde{u}^{-1} \\ 0 & 1-\varpi\tilde{u}v \end{pmatrix}.$$

A_6: $\operatorname{ord}(v)+r-s<0$ and $\operatorname{ord}(v)+\alpha-\beta<0$. Let $j=\operatorname{ord}(v)$. Then

$$M=\begin{pmatrix} 1 & 0 \\ \varpi^{\beta-\alpha}v^{-1} & 1-\varpi\tilde{u}v \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\begin{pmatrix} \varpi^{\beta-r-j-1} & 0 \\ 0 & \varpi^{j+\alpha-s} \end{pmatrix}\begin{pmatrix} -\varpi^{j}v^{-1} & 0 \\ \varpi^{s-r-j} & \varpi^{-j}v \end{pmatrix}.$$

Now $A_5 \neq \emptyset$ if and only if $s<r$ and $\beta<\alpha$, while $A_6 \neq \emptyset$ if and only if $r<s$ and $\alpha<\beta$. So at least one of the sets A_5 and A_6 is empty.

Also, the integrand on the right in (4.8) is 1 for all $u, v \in \mathfrak{O}\backslash(A_5 \cup A_6)$ if $(\alpha, \beta)=(m+r, n+s)$, and 0 for all $u, v \in \mathfrak{O}$ for any other (α, β). Hence $F_{m,n} * F_{r,s}=cF_{m+r,n+s}$, where $c=q^{|r-s|}(1-m(A_5)-m(A_6))$.

The Haar measure of the set of $u \in \mathfrak{O}$ such that $\operatorname{ord}(u)=i$ is $(q-1)/q^{i+1}$, and hence the measure of $\{u \in \mathfrak{O} : \operatorname{ord}(u)<l\}$ equals $1-1/q^{l}$ for all $l \geqslant 0$.

To complete the proof of Proposition 4.4, we again we need to consider cases. Firstly, if $r=s$, then $A_5=A_6=\emptyset$, and so $c=1$. Also, in this case, $\min\{m+r, n+s\}=\min\{m, n\}+\min\{r, s\}$, and (4.6) follows. We now consider the case $r \neq s$. Write $\alpha=m+r$ and $\beta=n+s$.

1. If $r>s$ and $m>n$, then $n+s<m+r$ and $\alpha-\beta=m-n+r-s>r-s$. So $m(A_5)=1-1/q^{r-s}$, $m(A_6)=0$ and $c=1$. Thus $G_{m,n} * G_{r,s}=q^{n+s}F_{m,n} * F_{r,s}=q^{n+s}F_{m+r,n+s}=G_{m+r,n+s}$.

2(a). If $r>s$, $m \leqslant n$ and $n+s<m+r$, then $0<\alpha-\beta=(r-s)-(n-m) \leqslant r-s$. So $m(A_5)=1-1/q^{\alpha-\beta}$, $m(A_6)=0$ and $c=q^{r-s}/q^{\alpha-\beta}=q^{n-m}$. Thus $G_{m,n} * G_{r,s}=q^{m+s}F_{m,n} * F_{r,s}=q^{m+s}q^{n-m}F_{m+r,n+s}=q^{n+s}F_{m+r,n+s}=G_{m+r,n+s}$.

2(b). If $r>s$, $m \leqslant n$ and $m+r \leqslant n+s$, then $\alpha-\beta \leqslant 0$. So $m(A_5)=m(A_6)=0$ and $c=q^{r-s}$. Thus $G_{m,n} * G_{r,s}=q^{m+s}F_{m,n} * F_{r,s}=q^{m+s}q^{r-s}F_{m+r,n+s}=q^{m+r}F_{m+r,n+s}=G_{m+r,n+s}$.

3. If $r<s$ and $m<n$, then $m+r<n+s$ and $\beta-\alpha=n-m+s-r>s-r$. So $m(A_5)=0$, $m(A_6)=1-1/q^{s-r}$ and $c=1$. Thus $G_{m,n} * G_{r,s}=q^{m+r}F_{m,n} * F_{r,s}=q^{m+r}F_{m+r,n+s}=G_{m+r,n+s}$.

4(a). If $r<s$, $m \geqslant n$ and $m+r<n+s$, then $0<\beta-\alpha=(s-r)-(m-n) \leqslant s-r$. So $m(A_5)=0$, $m(A_6)=1-1/q^{\beta-\alpha}$ and $c=q^{s-r}/q^{\beta-\alpha}=q^{m-n}$. Thus $G_{m,n} * G_{r,s}=q^{n+r}F_{m,n} * F_{r,s}=q^{n+r}q^{m-n}F_{m+r,n+s}=q^{m+r}F_{m+r,n+s}=G_{m+r,n+s}$.

4(b). If $r<s$, $m \geqslant n$ and $n+s \leqslant m+r$, then $\beta - \alpha \leqslant 0$. So $m(A_5) = m(A_6) = 0$ and $c = q^{s-r}$. Thus $G_{m,n} * G_{r,s} = q^{n+r} F_{m,n} * F_{r,s} = q^{n+r} q^{s-r} F_{m+r,n+s} = q^{n+s} F_{m+r,n+s} = G_{m+r,n+s}$. ■

COROLLARY 4.5. – *For any $\pi \in \widehat{H}$, the space $\mathcal{H}_{\pi,\chi''}$ is at most one-dimensional.*

PROOF. – If $f \in \mathcal{H}''$, then it is easy to see that $\pi(f)$ maps $\mathcal{H}_{\pi,\chi''}$ into itself. Hence we obtain a representation of the commutative algebra $\mathcal{H}''$ on $\mathcal{H}_{\pi,\chi''}$. If $\mathcal{H}_{\pi,\chi''}$ had dimension greater than 1, there would be a non-zero proper subspace W of $\mathcal{H}_{\pi,\chi''}$ invariant under $\pi(f)$ for all $f \in \mathcal{H}''$. Choose $\eta \in \mathcal{H}_{\pi,\chi''}$ of norm 1 such that $\eta \in W^\perp$. If $f \in \mathcal{C}_c(H)$, define $f_1 \colon H \to \mathbb{C}$ by

$$f_1(h) = \int_{K''} \int_{K''} \chi''(k_1 k_2)\, f(k_1 h k_2)\, dk_1\, dk_2,$$

where dk_1 and dk_2 refer to normalized Haar measure on K''. Then $f_1 \in \mathcal{H}''$, and for any $\xi \in W$ we have

$$\langle \pi(f)\,\eta,\, \xi \rangle = \langle \pi(f_1)\,\eta,\, \xi \rangle = \langle \eta,\, \pi(f_1^*)\,\xi \rangle = 0\,.$$

Hence $\{\pi(f)\,\eta : f \in \mathcal{C}_c(H)\}$ is a subset of $W^\perp$, and so its closure is a non-zero proper H-invariant subspace of $\mathcal{H}_\pi$, contradicting the irreducibility of π. ■

For each $z \in \mathbb{T}$, we get a character χ_z of $F^\times$ by setting

$$\chi_z(a\pi^r) = \chi_0(\dot{a})\, z^r \qquad \text{for } a \in \mathfrak{O}^\times \text{ and } r \in \mathbb{Z}\,,$$

where $\dot{a}$ is as usual the image of a in $\mathbb{F}_q$. Define a character χ_z' of P by setting

$$\chi_z'\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \chi_z(a/d)\,.$$

Let σ_z be the unitary representation of G obtained by unitarily inducing χ_z' from P to G. Thus the representation space $\mathcal{H}_z$ of σ_z consists of the completion of the space $\mathcal{H}_z^0$ of locally constant functions $f \colon G \to \mathbb{C}$ such that $f(pg) = \delta(p)^{1/2} \chi_z'(p)\, f(g)$ for all $p \in P$ and $g \in G$ with respect to the norm $\|f\| = \left(\int_K |f(k)|\, dk\right)^{1/2}$, and $(\sigma_z(g)\, f)(g') = f(g'g)$ for $f \in \mathcal{H}_z^0$ [1, pp. 469, 507]. Here δ is the modular quasi-character of P, defined by

$$\int_P f(gp)\, dg = \delta(p) \int_P f(g)\, dg \qquad \text{for any } f \in \mathcal{C}_c(P) \text{ and } p \in P,$$

where dg refers to left Haar measure on P. So $\delta(p) = q^{\operatorname{ord}(d/a)}$ if $p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ [1, p. 426]. Note that $\delta(p)$ is denoted $1/\Delta(p)$ in [5, p. 46].

PROPOSITION 4.6. – *The representations σ_z are irreducible, and trivial on Z. Regarding $\sigma_z \in \widehat{H}$, we have $\sigma_z \in \widehat{H}_{\chi''}$, and every $\pi \in \widehat{H}_{\chi''}$ is equivalent to exactly one of the σ_z.*

PROOF. – On the uncompleted space $\mathcal{H}_z^0$, σ_z is $\mathcal{B}(\chi_z, \chi_z^{-1})$, and so is (algebraically) irreducible [1, Theorem 4.5.1] and unitarizable [1, Proposition 4.6.11]. It follows that σ_z is irreducible on the completed space $\mathcal{H}_z$. For if T is a continuous linear operator which commutes with each $\sigma_z(g)$, then for each compact open subgroup K_0 of G, T commutes with $Q_{K_0} = \int_{K_0} \sigma_z(k)\, dk$, which is the orthogonal projection of the space $\mathcal{H}_z(K_0)$ of right K_0-invariant elements of $\mathcal{H}_z$. So T maps each $\mathcal{H}_z(K_0)$ into itself, and hence their union, $\mathcal{H}_z^0$, into itself. By algebraic irreducibility, T must be a multiple of the identity operator. So σ_z is irreducible.

By the first part of Lemma 4.2, and since $\delta(p) = 1$ and $\chi'(p) = \chi_z'(p)$ for all $p \in P \cap K'$,

$$f_z(g) = \begin{cases} \delta(p)^{1/2} \chi_z'(p)\, \chi'(k') & \text{if } g = pk' \in PK', \\ 0 & \text{if } g \in Pw_0 K'. \end{cases}$$

well-defines a function $f_z \in \mathcal{H}_z$ such that $\sigma_z(k') f_z = \chi'(k') f_z$ for all $k' \in K'$ and such that $f(1) = 1$. It follows that the representation of H corresponding to σ_z is in $\widehat{H}_{\chi''}$.

Any $f \in \mathcal{H}_z$ such that $\sigma_z(k') f = \chi'(k') f$ for all $k' \in K'$ must be a multiple of f_z. This is immediate from Corollary 4.5, but can easily be seen directly as follows: taking $p = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $p' = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, where $a \in \mathfrak{O}^\times$, we have $p \in P \cap K'$, $pw_0 = w_0 p'$, $\delta(p) = 1$ and $\chi_z'(p) = \chi'(p)$. Thus

$$\chi'(p)\, f(w_0) = f(pw_0) = f(w_0 p') = \chi'(p')\, f(w_0),$$

which means that $\chi_0(\dot{a}^{-1})\, f(w_0) = \chi_0(\dot{a})\, f(w_0)$. Since $\chi_0^2 \neq 1$, there is an $a \in \mathfrak{O}^\times$ such that $\chi_0(\dot{a}^{-1}) \neq \chi_0(\dot{a})$. Hence $f(w_0) = 0$. Since f is determined by $f(1)$ and $f(w_0)$, we must have $f = cf_z$ for $c = f(1)$.

For any $F \in \mathcal{H}'$, $f = \pi(F)(f_z)$ satisfies $\sigma_z(k') f = \chi'(k') f$ for all $k' \in K'$, and so $f = cf_z$ for $c = f(1)$. We next show that if $F = F_{m,n}$, then $c = q^{|m-n|/2} z^{m-n}$. Since $F_{m,n}^* = F_{-m,-n}$, we may assume that $m \leqslant n$. Now

$$c = (\sigma_z(F_{m,n})\, f_z)(1) = \int_G F_{m,n}(x)\, f_z(x)\, dx = (q+1) \int_P \left(\int_K F_{m,n}(pk)\, f_z(pk)\, dk \right) dp$$

by [1, Proposition 2.1.5(ii)]. Here dk denotes normalized Haar measure m_K on K and dp denotes left Haar measure on P, normalized so that $P \cap K$ has measure 1. The factor $q+1$ is to normalize the Haar measure dx on G so that K' has measure 1.

Now K is the union of the cosets $w_0 K'$ and $g_\alpha K'$, where $\alpha \in A$ and $g_\alpha = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$. Notice that

$$g_\alpha = \begin{pmatrix} -1/\alpha & 1/\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \in P w_0 K'$$

for all $\alpha \in A \backslash \{0\}$, so that $f_z(pk) = 0$ for $p \in P$ and $k \in K \backslash K'$. If $k \in K'$, then

$$F_{m,n}(pk)\, f_z(pk) = F_{m,n}(p)\, \overline{\chi'(k)} \chi'(k)\, f_z(p) = F_{m,n}(p)\, f_z(p).$$

Since $m_K(K') = 1/(q+1)$,

$$\int_K F_{m,n}(pk)\, f_z(pk)\, dk = \int_{K'} F_{m,n}(pk)\, f_z(pk)\, dk = F_{m,n}(p)\, f_z(p)/(q+1).$$

Hence $c = \int_P F_{m,n}(p)\, f_z(p)\, dp$.

Now P is the product of the two closed groups D and U, where D consists of the diagonal matrices $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, where $a_1, a_2 \in F^\times$ and U consists of the matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, where $x \in F$. So by [1, Proposition 2.1.5(ii)] again, for any $\varphi \in \mathcal{C}_c(P)$,

$$\int_P \varphi(p)\, dp = C \int_{F^\times} \int_{F^\times} \int_F \varphi\left(\begin{pmatrix} a_1 & a_1 x \\ 0 & a_2 \end{pmatrix} \right) \frac{da_1}{|a_1|} \frac{da_2}{|a_2|}\, dx, \tag{4.9}$$

for some $C > 0$, where da_1, da_2 and dx refer to additive Haar measure m_F on F, normalized so that $\mathfrak{O}$ has measure 1. The number C is determined by the condition that $P \cap K$ has measure 1. Taking φ to be the indicator function of $P \cap K$, and using the fact that $\begin{pmatrix} a_1 & a_1 x \\ 0 & a_2 \end{pmatrix} \in P \cap K$ if and only if $a_1, a_2 \in \mathfrak{O}^\times$ and $x \in \mathfrak{O}$, the right hand side of (4.9) is

$$C \int_{\mathfrak{O}^\times} \int_{\mathfrak{O}^\times} \int_{\mathfrak{O}} \varphi\left(\begin{pmatrix} a_1 & a_1 x \\ 0 & a_2 \end{pmatrix} \right) da_1\, da_2\, dx = C(q-1)^2/q^2.$$

Thus $C = q^2/(q-1)^2$.

Recall that we are assuming that $m \leqslant n$. For $a_1, a_2 \in F^\times$ and $x \in F$,

$$\begin{pmatrix} a_1 & a_1 x \\ 0 & a_2 \end{pmatrix} \in K' g_{m,n} K' \text{ if and only if } a_1/\varpi^m \in \mathfrak{O}^\times,\ a_2/\varpi^n \in \mathfrak{O}^\times \text{ and } x \in \mathfrak{O},$$

as is clear from the cases (i) and (ii) considered in the proof of Lemma 4.2. Hence

$$
\begin{aligned}
(4.10)\qquad c &= C\int_{F^\times}\int_{F^\times}\int_{F}(F_{m,n}\cdot f_z)\left(\begin{pmatrix} a_1 & a_1 x\\ 0 & a_2\end{pmatrix}\right)\frac{da_1}{|a_1|}\,\frac{da_2}{|a_2|}\,dx\\
&= C\int_{F^\times}\int_{F^\times}\int_{F}(F_{m,n}\cdot f_z)\left(\begin{pmatrix} \varpi^m a_1 & \varpi^m a_1 x\\ 0 & \varpi^n a_2\end{pmatrix}\right)\frac{da_1}{|a_1|}\,\frac{da_2}{|a_2|}\,dx\\
&= C\int_{\mathfrak{O}^\times}\int_{\mathfrak{O}^\times}\int_{\mathfrak{O}}(F_{m,n}\cdot f_z)\left(\begin{pmatrix} \varpi^m a_1 & \varpi^m a_1 x\\ 0 & \varpi^n a_2\end{pmatrix}\right)da_1\,da_2\,dx\,.
\end{aligned}
$$

If $a_1, a_2 \in \mathfrak{O}^\times$ and $x \in \mathfrak{O}$, then $p = \begin{pmatrix} a_1 & a_1 x\\ 0 & a_2\end{pmatrix} \in P \cap K'$, and so

$$
(F_{m,n}\cdot f_z)\left(\begin{pmatrix} \varpi^m a_1 & \varpi^m a_1 x\\ 0 & \varpi^n a_2\end{pmatrix}\right) = (F_{m,n}\cdot f_z)(g_{m,n}p) = (F_{m,n}\cdot f_z)(g_{m,n}),
$$

since $F_{m,n}(g_{m,n}p) = \overline{\chi'(p)}$ and $f_z(g_{m,n}p) = \chi'(p) f_z(g_{m,n})$. This equals

$$
f_z(g_{m,n}) = \delta(g_{m,n})^{1/2}\chi_z'(g_{m,n}) = q^{(n-m)/2}\chi_z(\varpi^{m-n}) = q^{(n-m)/2}z^{m-n}.
$$

Hence the integrand in (4.10) equals the constant $q^{(n-m)/2}z^{m-n}$, so that

$$
c = C m_F(\mathfrak{O}^\times)^2 m_F(\mathfrak{O})\, q^{(n-m)/2} z^{m-n} = q^{(n-m)/2} z^{m-n}.
$$

Let $\pi \in \widehat{H}_{\chi''}$. Since $\mathcal{H}_{\pi,\chi''}$ is 1-dimensional, if $f \in \mathcal{H}''$, then $\pi(f)(\xi)$ is a multiple $\lambda_\pi(f)\,\xi$ of ξ. Then $\lambda_\pi : \mathcal{H}'' \to \mathbb{C}$ is a $*$-algebra homomorphism. It does not depend on the choice of ξ, nor on the equivalence class of π. The map $\pi \mapsto \lambda_\pi$ is injective from the set $\widehat{H}_{\chi''}$ into the set of $*$-algebra homomorphisms on $\mathcal{H}''$ [7, Appendix 1].

Let $f_n = \Lambda(F_{0,n}) \in \mathcal{H}''$ for $n \in \mathbb{Z}$. Thus $f_n(gZ) = \overline{\chi'(k_1 k_2)}$ if $gZ = k_1 g_{0,n} k_2 Z$ for some $k_1, k_2 \in K'$, and $\mathcal{H}''$ is spanned by the f_n's. Then $f_n^* = \Lambda(F_{0,n}^*) = \Lambda(F_{0,-n}) = f_{-n}$, and by Proposition 4.4, f_n is the n-th convolution power of f_1 for all $n \geq 1$. Also, $f_0 * f_0 = f_0$, and $f_1 * f_1^* = f_1 * f_{-1} = q f_0$, since $F_{0,1} * F_{0,1}^* = F_{0,1} * F_{0,-1} = G_{0,1} * qG_{0,-1} = qG_{0,0} = qF_{0,0}$. Let λ be a $*$-algebra homomorphism on $\mathcal{H}''$. Then λ is determined by $\lambda(f_1)$, and we have $\lambda(f_0) = 1$, and $|\lambda(f_1)|^2 = q$. It follows that $\lambda = \lambda_{\sigma_z}$ for some $z \in \mathbb{T}$. Hence if $\pi \in \widehat{H}_{\chi''}$ then $\lambda_\pi = \lambda_{\sigma_z}$ for some z, and so π must be equivalent to this σ_z. ■

PROPOSITION 4.7. – *The representation* $\mathrm{Ind}_{K''}^{H}\chi''$ *is unitarily equivalent to the direct integral* $\int_{\mathbb{T}}^{\oplus}\sigma_z\,dz$ *of the representations* σ_z, $|z| = 1$.

PROOF. – Let π be an irreducible unitary representation of H, and let $\mathcal{HS}(\mathcal{H}_\pi)$ denote the space of Hilbert-Schmidt operators on the representation space $\mathcal{H}_\pi$ of π. It is a Hilbert space with inner product $\langle S, T\rangle = \mathrm{Trace}(T^* S)$, and π gives a unitary representation π' on $\mathcal{HS}(\mathcal{H}_\pi)$ by $\pi'(g)(T) = \pi(g)\,T$. If $f \in L^1(H) \cap L^2(H)$, let $\widehat{f}(\pi)$ denote the operator $\int_H f(x)\,\pi(x^{-1})\,dx$. Let $\widehat{H}$ denote the set of equivalence classes of irreducible representations of H. The Plancherel Theorem [5, p. 234], [2, p. 327] states that there is a measure μ on $\widehat{H}$ so that the map $f \mapsto (\widehat{f}(\pi))$ extends to an isometry of $L^2(H)$ onto $\int_{\widehat{H}}^{\oplus} \mathcal{HS}(\mathcal{H}_\pi)\,d\mu(\pi)$ which intertwines the right regular representation ϱ of H and the direct integral of the representations π'.

Let $f_0 \in \mathcal{H}''$ be as defined at the end of the last proof. It is easy to see that if $\pi \in \widehat{H}$, then $\widehat{f}_0(\pi)$ is the orthogonal projection $P_{\pi,\chi''}$ of $\mathcal{H}_\pi$ onto $\mathcal{H}_{\pi,\chi''}$.

Let V denote the representation space of $\mathrm{Ind}_{K''}^{H}\chi''$. Then $V = \{f_0 * f : f \in L^2(H)\}$. If $f \in L^1(H) \cap L^2(H)$ is in V, then $f = f_0 * f$, and so $\widehat{f}(\pi) = \widehat{f}(\pi)\,\widehat{f}_0(\pi) = \widehat{f}(\pi)\,P_{\pi,\chi''}$. Hence, considering the above unitary map $L^2(H) \to \int_{\widehat{H}}^{\oplus} \mathcal{HS}(\mathcal{H}_\pi)\,d\mu(\pi)$, the image in $\int^{\oplus} \mathcal{HS}(\mathcal{H}_\pi)\,d\mu(\pi)$ of $V \subset L^2(H)$ is the space of fields (S_π) of operators such that $S_\pi = S_\pi P_{\pi,\chi''}$ for all π. Hence $S_\pi = 0$ unless $\mathcal{H}_{\pi,\chi''} \neq \{0\}$. For each $\pi \in \widehat{H}_{\chi''}$, pick $\xi_\pi \in \mathcal{H}_{\pi,\chi''}$ of norm 1. An operator S_π on $\mathcal{H}_\pi$ such that $S_\pi = S_\pi P_{\pi,\chi''}$ is completely determined by $u_\pi = S_\pi(\xi_\pi)$. In fact, $S_\pi(t\xi_\pi + \eta) = tu_\pi$ if $\eta \in \{\xi_\pi\}^\perp$. Hence S_π is a Hilbert-Schmidt operator. If $S_\pi = S_\pi P_{\pi,\chi''}$ and $T_\pi = T_\pi P_{\pi,\chi''}$, let $u_\pi = S_\pi(\xi_\pi)$ and $v_\pi = T_\pi(\xi_\pi)$. Then $\mathrm{Trace}\,(T_\pi^* S_\pi) = \langle u_\pi, v_\pi \rangle$. Hence $S_\pi \mapsto S_\pi(\xi_\pi)$ defines an isometry of $\{S_\pi \in \mathcal{L}(\mathcal{H}_\pi) : S_\pi = S_\pi P_{\pi,\chi''}\}$ onto $\mathcal{H}_\pi$. Hence $f \mapsto (\pi(f)(\xi_\pi))$ is an isometry from the subspace V of $L^2(H)$ onto $\int_{\widehat{H}_{\chi''}}^{\oplus} \mathcal{H}_\pi\,d\mu(\pi)$ which intertwines the right translation on V, i.e., $\mathrm{Ind}_{K''}^{H}\chi''$, with $\int_{\widehat{H}_{\chi''}}^{\oplus} \pi\,d\mu(\pi)$.

By Proposition 4.6, any $\pi \in \widehat{H}_{\chi''}$ is equivalent to one of the representations σ_z, $|z| = 1$, and we can take $\xi_\pi = f_z$ if $\pi = \sigma_z$. Because $q^{|n|}\,\delta_{m,n} = \langle f_m, f_n \rangle$ equals

$$\begin{aligned}
\int_{\mathbb{T}} \langle \widehat{f}_m(\sigma_z)\,f_z, \widehat{F}_m(\sigma_z)\,f_z \rangle\,d\mu(\sigma_z) &= \int_{\mathbb{T}} \langle (\sigma_z)(f_{-m})\,f_z, (\sigma_z)(f_{-n})\,f_z \rangle\,d\mu(\sigma_z) \\
&= \int_{\mathbb{T}} \langle q^{|m|/2} z^m f_z,\, q^{|n|/2} z^n f_z \rangle\,d\mu(\sigma_z) \\
&= q^{(|m| + |n|)/2} \int_{\mathbb{T}} z^{m-n}\,d\mu(\sigma_z),
\end{aligned}$$

the Plancherel measure induces the Haar measure on $\mathbb{T}$ via the embedding $z \mapsto \sigma_z$. ∎

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Pervenuta in Redazione
il 21 gennaio 2002